
In tali nunquam lassat venatio sylva.
A.D.1884.

THE

# PRINCIPLES

OF

# PLANE TRIGONOMETRY, MENSURATION, NAVIGATION AND SURVEYING.

ADAPTED TO THE METHOD OF INSTRUCTION IN THE AMERICAN COLLEGES.

BY JEREMIAH DAY, D.D. LL. D.
PRESIDENT OF YALE COLLEGE.

THIRD EDITION.

NEW HAVEN:
PUBLISHED AND SOLD BY HEZEKIAH HOWE.
Sold also by COLLINS & HA[illegible]AY, *New York;* JOHN GRIGG, *Philadelphia;*
[illegible] *Boston;* and

James

# ADVERTISEMENT.

Of the course of Mathematics proposed by the author, Algebra, the first of the four parts which have been prepared, is published in a volume by itself. The other three parts, comprised in the present volume, being small, are sold either separately or together, as may best suit the convenience of the purchaser. On the remaining subjects originally contemplated as parts of the course, treatises by other authors have been substituted.

# HEZEKIAH HOWE,

## *Has published the following works which are used as Text Books at Yale College.*

An Introduction to Algebra, being the first part of a Course of Mathematics, adapted to the method of instruction in the American Colleges, by Jeremiah Day, D. D. LL. D. President of Yale College. 8vo.

A Treatise of Plane Trigonometry, to which is prefixed a summary view of the nature and use of Logarithms, being the second part of a Course of Mathematics, adapted to the method of instruction in the American Colleges, by the same. 8vo.

A Practical application of the principles of Geometry, to the Mensuration of superficies and solids, being the third part of a Course of Mathematics adapted to the method of instruction, &c. &c. by the same. 8vo.

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Part 1. Mathematical Elements of Mechanics, taken chiefly from Bridge's Mechanics.

Part 2. Practical application of Mechanics, compiled from various authorities.

By Denison Olmsted, Professor of Mathematics and Natural Philosophy in Yale College.

N. B. The first volume of the above work will be published in a few weeks, the first part being already printed, and the second nearly through the press. The second volume comprising the remaining topics of Natural Philosophy, will be commenced immediately afterwards.

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In press, a Treatise on the Construction, Properties and Analogies of the three Conic Sections, by the Rev. B. Bridge, B. D. F. R. S. with alterations and additions, adapted to the method of instruction in the American Colleges.

New Haven, May, 1831.

A

# TREATISE

OF

# PLANE TRIGONOMETRY.

TO WHICH IS PREFIXED

A SUMMARY VIEW OF THE NATURE AND USE OF

# LOGARITHMS;

BEING

THE SECOND PART

OF

A COURSE OF MATHEMATICS,

ADAPTED TO THE METHOD OF INSTRUCTION IN THE

AMERICAN COLLEGES.

BY JEREMIAH DAY, D.D. LL. D.

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1831.

The plan upon which this work was originally commenced, is continued in this second part of the course. As the single object is to provide for *a class in college*, such matter as is not embraced by this design is excluded. The mode of treating the subjects, for the reasons mentioned in the preface to Algebra, is, in a considerable degree, diffuse. It was thought better to err on this extreme, than on the other, especially in the early part of the course.

The section on right angled triangles will probably be considered as needlessly minute. The solutions might, in all cases, be effected by the theorems which are given for oblique angled triangles. But the applications of rectangular trigonometry are so numerous, in navigation, surveying, astronomy, &c. that it was deemed important, to render familiar the various methods of stating the relations of the sides and angles; and especially to bring distinctly into view the principle on which most trigonometrical calculations are founded, the proportion between the parts of the given triangle, and a similar one formed from the sines, tangents, &c. in the tables.

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TRIGONOMETRY.

# LOGARITHMS.

## SECTION I.

### NATURE OF LOGARITHMS.*

ART. 1. THE operations of Multiplication and Division, when they are to be often repeated, become so laborious, that it is an object of importance to substitute, in their stead, more simple methods of calculation, such as Addition and Subtraction. If these can be made to perform, in an expeditious manner, the office of multiplication and division, a great portion of the time and labor which the latter processes require, may be saved.

Now it has been shown, (Algebra, 233, 237,) that *powers* may be multiplied, by adding their *exponents*, and divided, by subtracting their exponents. In the same manner, *roots* may be multiplied and divided, by adding and subtracting their fractional exponents. (Alg. 280, 286.) When these exponents are arranged in tables, and applied to the general purposes of calculation, they are called *Logarithms*.

2. LOGARITHMS, THEN, ARE THE EXPONENTS OF A SERIES OF POWERS AND ROOTS.†

In forming a system of logarithms, some particular number is fixed upon, as the *base, radix,* or first power, whose logarithm is always 1. From this, a series of powers is raised, and the exponents of these are arranged in tables for use. To explain this, let the number which is chosen for the first

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* Maskelyne's Preface to Taylor's Logarithms. Introduction to Hutton's Tables. Keil on Logarithms. Maseres Scriptores Logarithmici. Briggs' Logarithms. Dodson's Anti-logarithmic Canon. Euler's Algebra.

† See note A.

power, be represented by $a$. Then taking a series of powers, both direct and reciprocal, as in Alg. 207;

$$a^4, a^3, a^2, a^1, a^0, a^{-1}, a^{-2}, a^{-3}, a^{-4}, \&c.$$

The logarithm of $a^3$ is 3, and the logarithm of $a^{-1}$ is $-1$,
of $a^1$ is 1, of $a^{-2}$ is $-2$,
of $a^0$ is 0, of $a^{-3}$ is $-3$, &c.
Universally, the logarithm of $a^x$ is $x$.

3. In the system of logarithms in common use, called *Briggs'* logarithms, the number which is taken for the radix or base is 10. The above series then, by substituting 10 for $a$, becomes

$$10^4, 10^3, 10^2, 10^1, 10^0, 10^{-1}, 10^{-2}, 10^{-3}, \&c.$$

Or 10000, 1000, 100, 10, 1, $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, &c.

Whose logarithms are

4, 3, 2, 1, 0, $-1$, $-2$, $-3$, &c.

4. The fractional exponents of *roots*, and of powers of roots, are converted into *decimals*, before they are inserted in the logarithmic tables. See Alg. 255.

The logarithm of $a^{\frac{1}{3}}$, or $a^{0.3333}$, is 0.3333,
of $a^{\frac{2}{3}}$, or $a^{0.6666}$, is 0.6666,
of $a^{\frac{3}{7}}$, or $a^{0.4285}$, is 0.4285,
of $a^{\frac{11}{3}}$, or $a^{3.6666}$, is 3.6666, &c.

These decimals are carried to a greater or less number of places, according to the degree of accuracy required.

5. In forming a system of logarithms, it is necessary to obtain the logarithm of each of the numbers in the natural series 1, 2, 3, 4, 5, &c.; so that the logarithm of any number may be found in the tables. For this purpose, the *radix* of the system must first be determined upon; and then every other number may be considered as some power or root of this. If the radix is 10, as in the common system, every other number is to be considered as some power of 10.

That a power or root of 10 may be found, which shall be equal to any other number whatever, or, at least, a very near approximation to it, is evident from this, that the *exponent* may be endlessly varied; and if this be increased or diminished, the *power* will be increased or diminished.

If the exponent is a fraction, and the *numerator* be increased, the power will be increased: but if the *denominator* be increased, the power will be diminished.

6. To obtain then the logarithm of any number, according to Briggs' system, we have to find a power or root of 10 which shall be equal to the proposed number. The *exponent* of that power or root is the logarithm required. Thus

| | | |
|---|---|---|
| $7=10^{0.8451}$ | therefore the logarithm | of 7 is 0.8451 |
| $20=10^{1.3010}$ | | of 20 is 1.3010 |
| $30=10^{1.4771}$ | | of 30 is 1.4771 |
| $400=10^{2.6020}$ | | of 400 is 2.6020, &c. |

7. A logarithm generally consists of two parts, an *integer* and a *decimal*. Thus the logarithm 2.60206, or, as it is sometimes written, 2+.60206, consists of the integer 2, and the decimal .60206. The integral part is called the *characteristic* or *index** of the logarithm; and is frequently omitted, in the common tables, because it can be easily supplied, whenever the logarithm is to be used in calculation.

By art. 3d, the logarithms of

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 10000, | 1000, | 100, | 10, | 1, | .1, | .01, | .001, | &c. |
| are | 4, | 3, | 2, | 1, | 0, | −1, | −2, | −3, | &c. |

As the logarithms of 1 and of 10 are 0 and 1, it is evident, that, if any given number be between 1 and 10, its logarithm will be between 0 and 1, that is, it will be greater than 0, but less than 1. It will therefore have 0 for its index, with a decimal annexed.

Thus the logarithm of 5 is 0.69897.

For the same reason, if the given number be between

| | | | |
|---|---|---|---|
| 10 and 100, | the log. will be between | 1 and 2, i. e. | 1+the dec. part. |
| 100 and 1000, | | 2 and 3, | 2+the dec. part. |
| 1000 and 10000, | | 3 and 4, | 3+the dec. part. |

We have, therefore, when the logarithm of an integer or mixed number is to be found, this general rule:

* The term *index*, as it is used here, may possibly lead to some confusion in the mind of the learner. For the logarithm itself is the index or exponent of a power. The characteristic, therefore, is the index of an index.

8. *The index of the logarithm is always one less, than the number of integral figures, in the natural number whose logarithm is sought:* or, the index shows how far the first figure of the natural number is removed from the place of units.

Thus the logarithm of 37 is 1.56820.

Here, the number of figures being *two*, the index of the logarithm is 1.

The logarithm of 253 is 2.40312.

Here, the proposed number 253 consists of *three* figures, the first of which is in the second place from the unit figure. The index of the logarithm is therefore 2.

The logarithm of 62.8 is 1.79796.

Here it is evident that the mixed number 62.8 is between 10 and 100. The index of its logarithm must, therefore, be 1.

9. As the logarithm of 1 is 0, the logarithm of a number *less* than 1, that is, of any proper *fraction*, must be *negative*.

Thus by art. 3d,
The logarithm of $\frac{1}{10}$ or .1 is $-1$,
of $\frac{1}{100}$ or .01 is $-2$,
of $\frac{1}{1000}$ or .001 is $-3$, &c.

10. If the proposed number is *between* $\frac{1}{100}$ and $\frac{1}{1000}$ its logarithm must be between $-2$ and $-3$. To obtain the logarithm, therefore, we must either *subtract* a certain fractional part from $-2$, or *add* a fractional part to $-3$; that is, we must either annex a *negative decimal* to $-2$, or a *positive one* to $-3$.

Thus the logarithm
of .008 is either $-2-.09691$, or $-3+.90309$.*

The latter is generally most convenient in practice, and is more commonly written $\bar{3}.90309$. The line over the index

* That these two expressions are of the same value will be evident, if we subtract the same quantity, +.90309 from each. The remainders will be equal, and therefore the quantities from which the subtraction is made must be equal. See note B.

denotes, that *that* is negative, while the *decimal* part of the logarithm is positive.

The logarithm of 0.3, is $\bar{1}.47712$,
of 0.06, is $\bar{2}.77815$,
of 0.009, is $\bar{3}.95424$,

And universally,

11. *The negative index of a logarithm shows how far the first significant figure of the natural number, is removed from the place of units, on the right;* in the same manner as a *positive* index shows how far the first figure of the natural number is removed from the place of units, on the left. (Art. 8.) Thus in the examples in the last article,

The decimal 3 is in the *first* place from that of units,
6 is in the *second* place,
9 is in the third place;

And the indices of the logarithms are $\bar{1}$, $\bar{2}$, and $\bar{3}$.

12. It is often more convenient, however, to make the *index* of the logarithm positive, as well as the decimal part. This is done by adding 10 to the index.

Thus, for $-1$, 9 is written; for $-2$, 8, &c.
Because $-1+10=9$, $-2+10=8$, &c.

With this alteration,

The logarithm $\bar{1}.90309$, $\bar{2}.90309$, $\bar{3}.90309$ becomes 9.90309, 8.90309, 7.90309, &c.

This is making the index of the logarithm 10 too great. But with proper caution, it will lead to no error in practice.

13. The *sum* of the logarithms of two numbers, is the logarithm of the *product* of those numbers; and the *difference* of the logarithms of two numbers, is the logarithm of the *quotient* of one of the numbers divided by the other. (Art. 2.) In Briggs' system, the logarithm of 10 is 1. (Art. 3.) If therefore any number be multiplied or divided by 10, its logarithm will be increased or diminished by 1: and as this is an integer, it will only change the *index* of the logarithm, without affecting the decimal part.

| | | |
|---|---|---|
| Thus the logarithm of | 4730 | is 3.67486, |
| And the logarithm of | 10 | is 1. |
| The logarithm of the product | 47300 | is 4.67486 |
| And the logarithm of the quotient | 473 | is 2.67486 |

Here the *index* only is altered, while the decimal part remains the same. We have then this important property,

14. *The* DECIMAL PART *of the logarithm of any number is the same, as that of the number multiplied or divided by* 10, 100, 1000, &c.

| | | | |
|---|---|---|---|
| Thus the log. of | 45670, | is 4.65963, | |
| | 4567, | 3.65963, | |
| | 456.7, | 2.65963, | |
| | 45.67, | 1.65963, | |
| | 4.567, | 0.65963, | |
| | .4567, | $\bar{1}.65963$, | or 9.65963, |
| | .04567, | $\bar{2}.65963$, | 8.65963, |
| | .004567, | $\bar{3}.65963$, | 7.65963. |

This property, which is peculiar to Briggs' system, is of great use in abridging the logarithmic tables. For when we have the logarithm of any number, we have only to change the index, to obtain the logarithm of every other number, whether integral, fractional, or mixed, consisting of the same significant figures. The decimal part of the logarithm of a fraction found in this way, is always *positive*. For it is the same as the decimal part of the logarithm of a whole number.

15. In a series of fractions *continually decreasing*, the negative indices of the logarithms *continually increase*. Thus,

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| In the series | 1, | .1, | .01, | .001, | .0001, | .00001, | &c. |
| The logarithms are | 0, | −1, | −2, | −3, | −4, | −5, | &c. |

If the progression be continued, till the fraction is reduced to 0, the negative logarithm will become greater than any assignable quantity. The logarithm of 0, therefore, is *infinite and negative*. (Alg. 447.)

16. It is evident also, that all *negative* logarithms belong to fractions which are between 1 and 0; while *positive* loga-

rithms belong to natural numbers which are greater than 1. As the whole range of numbers, both positive and negative, is thus exhausted in supplying the logarithms of integral and fractional positive quantities; there can be no other numbers to furnish logarithms for *negative* quantities. On this account the logarithm of a negative quantity is, by some writers, considered as *impossible*. But as there is no difference in the multiplication, division, involution, &c. of positive and negative quantities, except in applying the *signs*; they may be considered as all positive, while these operations are performing by means of logarithms; and the proper signs may be *afterwards* affixed.

17. *If a series of numbers be in* GEOMETRICAL *progression, their logarithms will be in* ARITHMETICAL *progression.* For, in a geometrical series ascending, the quantities increase by a common *multiplier;* (Alg. 436.) that is, each succeeding term is the *product* of the preceding term into the ratio. But the *logarithm* of this product is the *sum* of the logarithms of the preceding term and the ratio; that is, the logarithms increase by a common *addition*, and are, therefore, in arithmetical progression. (Alg. 422.) In a geometrical progression *descending*, the terms decrease by a common *divisor*, and their logarithms, by a common *difference.**

Thus the numbers 1, 10, 100, 1000, 10000, &c. are in geometrical progression.

And their logarithms 0, 1, 2, 3, 4, &c. are in arithmetical progression.

Universally, if in any geometrical series,

$a$=the least term, $r$=the ratio,

$L$=its logarithm, $l$=its logarithm;

Then the logarithm of $ar$ is $L+l$, (Art. 1.)

of $ar^2$ is $L+2l$,

of $ar^3$ is $L+3l$, &c.

Here, the quantities $a$, $ar^2$, $ar^3$, $ar^4$, &c. are in geometrical progression. (Alg. 436.)

And their logarithms $L$, $L+l$, $L+2l$, $L+3l$, &c. are in arithmetical progression. (Alg. 423.)

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* See note C.

## THE LOGARITHMIC CURVE.

19. The relations of logarithms, and their corresponding numbers, may be represented by the abscissas and ordinates of a curve. Let the line AC (Fig. 1.) be taken for unity. Let AF be divided into portions, each equal to AC, by the points 1, 2, 3, &c. Let the line $a$ represent the *radix* of a given system of logarithms, suppose it to be 1.3; and let $a^2$, $a^3$, &c. correspond, in length, with the different powers of $a$. Then the distances from A to 1, 2, 3, &c. will represent the *logarithms* of $a$, $a^2$, $a^3$, &c. (Art. 2.) The line CH is called the *logarithmic curve*, because its *abscissas* are proportioned to the logarithms of numbers represented by its *ordinates*. (Alg. 527.)

20. As the abscissas are the distances from AC, on the line AF, it is evident, that the abscissa of the point C is 0, which is the logarithm of 1=AC. (Art. 2.) The distance from A to 1 is the logarithm of the ordinate $a$, which is the *radix* of the system. For Briggs' logarithms, this ought to be ten times AC. The distance from A to 2 is the logarithm of the ordinate $a^2$; from A to 3 is the logarithm of $a^3$, &c.

21. The logarithms of numbers less than a unit are *negative*. (Art. 9.) These may be represented by portions of the line AN, on the *opposite side* of AC. (Alg. 507.) The ordinates $a^{-1}$, $a^{-2}$, $a^{-3}$, &c. are less than AC, which is taken for unity; and the abscissas, which are the distances from A to −1, −2, −3, &c. are negative.

22. If the curve be continued ever so far, it will never meet the axis AN. For, as the ordinates are in geometrical progression decreasing, each is a certain portion of the preceding one. They will be diminished more and more, the farther they are carried, but can never be reduced absolutely to nothing. The axis AN is, therefore, an *asymptote* of the curve. (Alg. 545.) As the ordinate decreases, the abscissa increases; so that, when one becomes infinitely small, the other becomes infinitely great. This corresponds with what has been stated, (Art. 15.) that the logarithm of 0 is *infinite* and *negative*.

23. To find the *equation* of this curve,

Let $a$=the *radix* of the system,
$x$=any one of the abscissas,
$y$=the corresponding ordinate.

Then, by the nature of the curve, (Art. 19.) the *ordinate* to any point, is that power of $a$ whose exponent is equal to the *abscissa* of the same point; that is, (Alg. 528.)

$$y=a^x.*$$

* For other properties of the logarithmic curve, see Fluxions.

## SECTION II.

### DIRECTIONS FOR TAKING LOGARITHMS AND THEIR NUMBERS FROM THE TABLES.*

Art. 24. The purpose which logarithms are intended to answer, is to enable us to perform arithmetical operations with *greater expedition*, than by the common methods. Before any one can avail himself of this advantage, he must become so familiar with the tables, that he can readily find the logarithm of any number; and, on the other hand, the number to which any logarithm belongs.

In the common tables, the *indices* to the logarithms of the first 100 numbers, are inserted. But, for all other numbers, the *decimal* part only of the logarithm is given; while the index is left to be supplied, according to the principles in arts. 8 and 11.

25. *To find the logarithm of any number between* 1 *and* 100;

Look for the proposed number, on the left; and against it, in the next column, will be the logarithm, with its index. Thus

The log. of 18 is 1.25527. The log. of 73 is 1.86332.

26. *To find the logarithm of any number between* 100 *and* 1000; or of any number consisting of not more than three significant figures, with ciphers annexed.

In the smaller tables, the three first figures of each number, are generally placed in the left hand column; and the fourth figure is placed at the head of the other columns.

Any number, therefore, between 100 and 1000, may be found on the left hand; and directly opposite, in the next column, is the decimal part of its logarithm. To this the *index* must be prefixed, according to the rule in art. 8.

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* The best English Tables are Hutton's in 8vo. and Taylor's in 4to. In these, the logarithms are carried to seven places of decimals, and proportional parts are placed in the margin. The smaller tables are numerous; and, when accurately printed, are sufficient for common calculations.

| | |
|---|---|
| The log. of 458 is 2.66087, | The log. of 935 is 2.97081, |
| of 796 2.90091, | of 386 2.58659. |

If there are *ciphers* annexed to the significant figures, the logarithm may be found in a similar manner. For, by art. 14, the *decimal* part of the logarithm of any number is the same, as that of the number multiplied into 10, 100, &c. All the difference will be in the *index*; and this may be supplied by the same general rule.

| | |
|---|---|
| The log. of 4580 is 3.66087, | The log. of 326000 is 5.51322, |
| of 79600 4.90091, | of 8010000 6.90363. |

27. *To find the logarithm of any number consisting of* FOUR *figures, either with, or without, ciphers annexed.*

Look for the three first figures, on the left hand, and for the fourth figure, at the head of one of the columns. The logarithm will be found, opposite the three first figures, and in the column which, at the head, is marked with the fourth figure.*

| | |
|---|---|
| The log. of 6234 is 3.79477, | The log. of 783400 is 5.89398, |
| of 5231 3.71858, | of 6281000 6.79803. |

28. *To find the logarithm of a number containing* MORE *than* FOUR *significant figures.*

By turning to the tables, it will be seen, that if the *differences* between several numbers be small, in comparison with the numbers themselves; the differences of the *logarithms* will be nearly proportioned to the differences of the *numbers*. Thus

| | |
|---|---|
| The log. of 1000 is 3.00000, | Here the differences in the |
| of 1001 3.00043, | numbers are, 1, 2, 3, 4, &c. |
| of 1002 3.00087, | and the corresponding dif- |
| of 1003 3.00130, | ferences in the logarithms, |
| of 1004 3.00173, &c. | are 43, 87, 130, 173, &c. |

Now 43 is nearly half of 87, one third of 130, one fourth of 173, &c.

Upon this principle, we may find the logarithm of a number which is between two other numbers whose logarithms

* In Taylor's, Hutton's and other tables, *four* figures are placed in the left hand column, and the *fifth* at the top of the page.

The log. of 2.36 is 0.37291, The log. of 364.2 is 2.56134,
of 27.8 1.44404, of 69.42 1.84148.

31. *To find the logarithm of a* VULGAR FRACTION.

From the nature of a vulgar fraction, the numerator may be considered as a *dividend*, and the denominator as a *divisor;* in other words, the value of the fraction is equal to the quotient, of the numerator divided by the denominator. (Alg. 135.) But in logarithms, division is performed by *subtraction;* that is, the *difference* of the logarithms of two numbers, is the logarithm of the *quotient* of those numbers. (Art. 1.) To find then the logarithm of a vulgar fraction, *subtract the logarithm of the denominator from that of the numerator.* The difference will be the logarithm of the fraction. Or the logarithm may be found, by first reducing the vulgar fraction to a *decimal.* If the numerator is less than the denominator, the index of the logarithm must be *negative*, because the value of the fraction is less than a unit. (Art. 9.)

Required the logarithm of $\frac{34}{87}$.

The log. of the numerator is 1.53148
of the denominator 1.93952

of the fraction $\bar{1}.59196$, or 9.59196.

The logarithm of $\frac{362}{7854}$ is $\bar{2}.66362$, or 8.66362.
of $\frac{7}{6329}$ $\bar{3}.04376$, or 7.04376.

32. If the logarithm of a *mixed number* is required, reduce it to an improper fraction, and then proceed as before.

The logarithm of $3\frac{7}{9}=\frac{34}{9}$ is 0.57724.

33. *To find the* NATURAL NUMBER *belonging to any logarithm.*

In computing by logarithms, it is necessary, in the first place, to take from the tables the logarithms of the numbers which enter into the calculation; and, on the other hand, at the close of the operation, to find the number belonging to the logarithm obtained in the result. This is evidently done, by *reversing* the methods in the preceding articles.

Where great accuracy is not required, look in the tables for the logarithm which is *nearest* to the given one ; and directly opposite, on the left hand, will be found the *three first* figures, and at the top, over the logarithm, the *fourth* figure, of the number required. This number, by pointing off decimals, or by adding ciphers, if necessary, must be made to correspond with the *index* of the given logarithm, according to arts. 8 and 11.

The natural number belonging

| | |
|---|---|
| to 3.86493 is 7327, | to 1.62572 is 42.24, |
| to 2.90140 796.9, | to $\bar{2}$.89115 0.07783. |

In the last example, the index requires that the first significant figure should be in the *second* place from units, and therefore a cipher must be prefixed. In other instances, it is necessary to annex ciphers on *the right*, so as to make the number of figures exceed the index by 1.

The natural number belonging

| | |
|---|---|
| to 6.71567 is 5196000, | to $\bar{3}$.65677 is 0.004537, |
| to 4.67062 46840, | to $\bar{4}$.59802 0.0003963. |

34. When great accuracy is required, and the given logarithm is not exactly, or very nearly, found in the tables, it will be necessary to reverse the rule in art. 28.

Take from the tables two logarithms, one the next greater, the other the next less than the given logarithm. Find the difference of the two logarithms, and the difference of their natural numbers ; also the difference between the least of the two logarithms, and the given logarithm. Then say,

As the difference of the two logarithms,
To the difference of their numbers ;
So is the difference between the given
logarithm and the least of the other two,
To the proportional part to be added to
the least of the two numbers.

Required the number belonging to the logarithm 2.67325.

| | | | |
|---|---|---|---|
| Next great. log. 2.67330. | Its numb. 471.3. | Given log. 2.67325. | |
| Next less 2.67321. | Its numb. 471.2. | Next less 2.67321. | |
| Differences 9 | 0.1 | 4 | |

Then 9 : 0.1 : : 4 : 0.044, which is to be added to the number 471.2

The number required is 471.244.

The natural number belonging
to 4.37627 is 23783.45, to 1.73698 is 54.57357,
to 3.69479 4952.08, to $\bar{1}.09214$ 0.123635.

35. *Correction of the tables.*—The tables of logarithms have been so carefully and so repeatedly calculated, by the ablest computers, that there is no room left to question their general correctness. They are not, however, exempt from the common imperfections of the press. But an error of this kind is easily corrected, by comparing the logarithm with any two others to whose *sum* or *difference* it ought to be equal. (Art. 1.)

Thus $48=24\times2=16\times3=12\times4=8\times6$. Therefore, the logarithm of 48 is equal to the *sum* of the logarithms of 24 and 2, of 16 and 3, &c.

And $3=\frac{6}{2}=\frac{12}{4}=\frac{15}{5}=\frac{18}{6}=\frac{21}{7}$, &c. Therefore, the logarithm of 3 is equal to the *difference* of the logarithms of 6 and 2, of 12 and 4, &c.

## SECTION III.

### METHODS OF CALCULATING BY LOGARITHMS.

ART. 36. THE arithmetical operations for which logarithms were originally contrived, and on which their great utility depends, are chiefly multiplication, division, involution, evolution, and finding the term required in single and compound proportion. The principle on which all these calculations are conducted, is this;

*If the logarithms of two numbers be added, the* SUM *will be the logarithm of the* PRODUCT *of the numbers;* and

*If the logarithm of one number be subtracted from that of another, the* DIFFERENCE *will be the logarithm of the* QUOTIENT *of one of the numbers divided by the other.*

In proof of this, we have only to call to mind, that logarithms are the EXPONENTS *of a series of powers and roots.* (Arts. 2, 5.) And it has been shown, that powers and roots are *multiplied* by *adding* their exponents; and *divided*, by *subtracting* their exponents. (Alg. 233, 237, 280, 286.)

### MULTIPLICATION BY LOGARITHMS.

37. ADD THE LOGARITHMS OF THE FACTORS: THE SUM WILL BE THE LOGARITHM OF THE PRODUCT.

In making the addition, 1 is to be carried, for every 10, from the decimal part of the logarithm, to the index. (Art. 7.)

| | Numbers. | Logarithms. | | Numbers. | Logarithms. |
|---|---|---|---|---|---|
| Mult. | 36.2 (Art. 30.) | 1.55871. | Mult. | 640 | 2.80618 |
| Into | 7.84 | 0.89432. | Into | 2.316 | 0.36474 |
| Prod. | 283.8 | 2.45303 | Prod. | 1482 | 3.17092 |

The logarithms of the two factors are taken from the tables. The product is obtained, by finding, in the tables, the natural number belonging to the sum. (Art. 33.)

| | | | | | |
|---|---|---|---|---|---|
| Mult. | 89.24 | 1.95056 | Mult. | 134. | 2.12710 |
| Into | 3.687 | 0.56667 | Into | 25.6 | 1.40824 |
| Prod. | 329. | 2.51723 | Prod. | 3430 | 3.53534 |

38. When any or all of the indices of the logarithms are *negative*, they are to be added according to the rules for the addition of positive and negative quantities in algebra. But it must be kept in mind, that the decimal part of the logarithm is *positive*. (Art. 10.) Therefore, that which is carried from the decimal part to the index, must be considered positive also.

| | | | | | |
|---|---|---|---|---|---|
| Mult. | 62.84 | 1.79824 | Mult. | 0.0294 | $\bar{2}.46835$ |
| Into | 0.682 | $\bar{1}.83378$ | Into | 0.8372 | $\bar{1}.92283$ |
| Prod. | 42.86 | 1.63202 | Prod. | 0.0246 | $\bar{2}.39118$ |

In each of these examples, +1 is to be carried from the decimal part of the logarithm. This added to −1, the lower index, makes it 0; so that there is nothing to be added to the upper index.

If any perplexity is occasioned, by the addition of positive and negative quantities, it may be avoided, by borrowing 10 to the index. (Art. 12.)

| | | | | | |
|---|---|---|---|---|---|
| Mult. | 62.84 | 1.79824 | Mult. | 0.0294 | 8.46835 |
| Into | 0.682 | 9.83378 | Into | 0.8372 | 9.92283 |
| Prod. | 42.86 | 1.63202 | Prod. | 0.0246 | 8.39118 |

Here 10 is added to the negative indices, and afterwards rejected from the index of the sum of the logarithms.

| | | | | |
|---|---|---|---|---|
| Multiply | 26.83 | 1.42862 | | 1.42862 |
| Into | 0.00069 | $\bar{4}.83885$ | or | 6.83885 |
| Product | 0.01851 | $\bar{2}.26747$ | | 8.26747 |

Here +1 carried to −4 makes it −3, which added to the upper index +1, gives −2 for the index of the sum.

| | | | | |
|---|---|---|---|---|
| Multiply | .00845 | $\bar{3}$.92686 | or | 7.92686 |
| Into | 1068. | 3.02857 | | 3.02857 |
| Product | 9.0246 | 0.95543 | | 0.95543 |

The product of 0.0362 into 25.38 is 0.9188
of 0.00467 into 348.1 is 1.626
of 0.0861 into 0.00843 is 0.0007258

39. *Any number of factors* may be multiplied together, by adding their logarithms. If there are several *positive*, and several *negative* indices, these are to be reduced to one, as in algebra, by taking the difference between the sum of those which are negative, and the sum of those which are positive, increased by what is carried from the decimal part of the logarithms. (Alg. 78.)

| | | | | |
|---|---|---|---|---|
| Multiply | 6832 | 3.83455 | | 3.83455 |
| Into | 0.00863 | $\bar{3}$.93601 | or | 7.93601 |
| And | 0.651 | $\bar{1}$.81358 | | 9.81358 |
| And | 0.0231 | $\bar{2}$.36361 | or | 8.36361 |
| And | 62.87 | 1.79844 | | 1.79844 |
| Prod. | 55.74 | 1.74619 | | 1.74619 |

Ex. 2. The prod. of 36.4 × 7.82 × 68.91 × 0.3846 is 7544.

3. The prod. of 0.00629 × 2.647 × 0.082 × 278.8 × 0.00063 is 0.0002398.

40. *Negative* quantities are multiplied, by means of logarithms, in the same manner as those which are positive. (Art. 16.) But, after the operation is ended, the proper sign must be applied to the natural number expressing the product, according to the rules for the multiplication of positive and negative quantities in algebra. The negative index of a *log-*

*arithm*, must not be confounded with the sign which denotes that the *natural number* is negative. That which the index of the logarithm is intended to show, is not whether the natural number is *positive or negative*, but whether it is *greater or less than a unit.* (Art. 16.)

| | | | | | |
|---|---|---|---|---|---|
| Mult. | +36.42 | 1.56134 | Mult. | −2.681 | 0.42830 |
| Into | −67.31 | 1.82808 | Into | +37.24 | 1.57101 |
| Prod. | −2451 | 3.38942 | Prod. | −99.84 | 1.99931 |

In these examples, the logarithms are taken from the tables, and added, in the same manner, as if both factors were positive. But after the product is found, the negative sign is prefixed to it, because + is multiplied into −. (Alg. 105.)

| | | | | | |
|---|---|---|---|---|---|
| Mult. | 0.263 | $\bar{1}.41996$ | Mult. | 0.065 | $\bar{2}.81291$ |
| Into | 0.00894 | $\bar{3}.95134$ | Into | 0.693 | $\bar{1}.84073$ |
| Prod. | 0.002351 | $\bar{3}.37130$ | Prod. | 0.04504 | $\bar{2}.65364$ |

Here, the indices of the logarithms are negative, but the product is positive, because the factors are both positive.

| | | | | | |
|---|---|---|---|---|---|
| Mult. | −62.59 | 1.79650 | Mult. | −68.3 | 1.83442 |
| Into | −0.00863 | $\bar{3}.93601$ | Into | −0.0096 | $\bar{3}.98227$ |
| Prod. | +0.5402 | $\bar{1}.73251$ | Prod. | +0.6557 | $\bar{1}.81669$ |

## DIVISION BY LOGARITHMS.

41. FROM THE LOGARITHM OF THE DIVIDEND, SUBTRACT THE LOGARITHM OF THE DIVISOR; THE DIFFERENCE WILL BE THE LOGARITHM OF THE QUOTIENT. (Art. 36.)

| | Numbers. | Logarithms. | | Numbers. | Logarithms. |
|---|---|---|---|---|---|
| Divide | 6238 | 3.79505 | Divide | 896.3 | 2.95245 |
| By | 2982 | 3.47451 | By | 9.847 | 0.99330 |
| Quot. | 2.092 | 0.32054 | Quot. | 91.02 | 1.95915 |

42. The *decimal* part of the logarithm may be subtracted as in common arithmetic. But for the *indices*, when either of them is negative, or the lower one is greater than the upper one, it will be necessary to make use of the general rule for subtraction in algebra; that is, to change the signs of the subtrahend, and then proceed as in addition. (Alg. 82.) When 1 is carried from the decimal part, this is to be considered affirmative, and applied to the index, before the sign is changed.

| | | | | |
|---|---|---|---|---|
| Divide | 0.8697 | $\bar{1}.93937$ | or | 9.93937 |
| By | 98.65 | 1.99410 | | 1.99410 |
| Quot. | 0.008816 | $\bar{3}.94527$ | | 7.94527 |

In this example, the upper logarithm being less than the lower one, it is necessary to borrow 10, as in other cases of subtraction; and therefore to carry 1 to the lower index, which then becomes $+2$. This changed to $-2$, and added to $-1$ above it, makes the index of the difference of the logarithms $-3$.

| | | | | |
|---|---|---|---|---|
| Divide | 29.76 | 1.47363 | | 1.47363 |
| By | 6254 | 3.79616 | | 3.79616 |
| Quot. | 0.00476 | $\bar{3}.67747$ | or | 7.67747 |

Here, 1 carried to the lower index, makes it +4. This changed to −4, and added to 1 above it, gives −3 for the index of the difference of the logarithms.

| | | | | | |
|---|---|---|---|---|---|
| Divide | 6.832 | 0.83455 | Divide | 0.00634 | $\bar{3}$.80209 |
| By | .0362 | $\bar{2}$.55871 | By | 62.18 | 1.79365 |
| Quot. | 188.73 | 2.27584 | Quot. | 0.000102 | $\bar{4}$.00844 |

The quotient of 0.0985 divided by 0.007241, is 13.6.

The quotient of 0.0621 divided by 3.68, is 0.01687.

43. To divide *negative* quantities, proceed in the same manner as if they were positive, (Art. 40.) and prefix to the quotient, the sign which is required by the rules for division in algebra.

| | | | | | |
|---|---|---|---|---|---|
| Divide | +3642 | 3.56134 | Divide | −0.657 | $\bar{1}$.81757 |
| By | −23.68 | 1.37438 | By | +0.0793 | $\bar{2}$.89927 |
| Quot. | −153.8 | 2.18696 | Quot. | −8.285 | 0.91830 |

In these examples, the sign of the divisor being different from that of the dividend, the sign of the quotient must be negative. (Alg. 123.)

| | | | | | |
|---|---|---|---|---|---|
| Divide | −0.364 | $\bar{1}$.56110 | Divide | −68.5 | 1.83569 |
| By | −2.56 | 0.40824 | By | +0.094 | $\bar{2}$.97313 |
| Quot. | +0.1422 | $\bar{1}$.15286 | Quot. | −728.7 | 2.86256 |

## INVOLUTION BY LOGARITHMS.

44. Involving a quantity is multiplying it into itself. By means of logarithms, multiplication is performed by addition. If, then, the logarithm of any quantity be *added to itself*, the

logarithm of a *power* of that quantity will be obtained. But adding a logarithm, or any other quantity, to itself, is *multiplication*. The involution of quantities, by means of logarithms, is therefore performed, by multiplying the logarithms.

Thus the logarithm

| | | | |
|---|---|---|---|
| of 100 | | is 2 | |
| of $100\times100$, that is, of | $\overline{100}^2$ | is $2+2$ | $=2\times2$. |
| of $100\times100\times100$, | $\overline{100}^3$ | is $2+2+2$ | $=2\times3$. |
| of $100\times100\times100\times100$, | $\overline{100}^4$ | is $2+2+2+2$ | $=2\times4$. |

On the same principle, the logarithm of $\overline{100}^n$ is $2\times n$.
And the logarithm of $x^n$, is (log. $x$)$\times n$. Hence,

45. To involve a quantity by logarithms. MULTIPLY THE LOGARITHM OF THE QUANTITY, BY THE INDEX OF THE POWER REQUIRED.

The reason of the rule is also evident, from the consideration, that logarithms are the exponents of powers and roots, and a power or root is involved, by *multiplying* its index into the index of the power required. (Alg. 220, 288.)

Ex. 1. What is the cube of 6.296?

| | | |
|---|---|---|
| Root 6.296, | its log. | 0.79906 |
| | Index of the power | 3 |
| Power 249.6 | | 2.39718 |

2. Required the 4th power of 21.32

| | | |
|---|---|---|
| Root 21.32 | log. | 1.32879 |
| | Index | 4 |
| Power 206614 | | 5.31516 |

3. Required the 6th power of 1.689

| | | |
|---|---|---|
| Root 1.689 | log. | 0.22763 |
| | Index | 6 |
| Power 23.215 | | 1.36578 |

4. Required the 144th power of 1.003

| | | | |
|---|---|---|---|
| Root | 1.003 | log. | 0.00130 |
| | | Index | 144 |
| Power | 1.539 | | 0.18720 |

46. It must be observed, as in the case of multiplication, (Art. 38.) that what is carried from the *decimal* part of the logarithm is *positive*, whether the index itself is positive or negative. Or, if 10 be added to a negative index, to render it positive, (Art. 12.) this will be multiplied, as well as the other figures, so that the logarithm of the square, will be 20 too great; of the cube, 30 too great, &c.

Ex. 1. Required the cube of 0.0649

| | | | | | |
|---|---|---|---|---|---|
| Root | 0.0649 | log. | $\bar{2}.81224$ | or | 8.81224 |
| | | Index | 3 | | 3 |
| Power | 0.0002733 | | $\bar{4}.43672$ | | 6.43672 |

2. Required the 4th power of 0.1234

| | | | | | |
|---|---|---|---|---|---|
| Root | 0.1234 | log. | $\bar{1}.09132$ | or | 9.09132 |
| | | Index | 4 | | 4 |
| Power | 0.0002319 | | $\bar{4}.36528$ | | 6.36528 |

3. Required the 6th power of 0.9977

| | | | | | |
|---|---|---|---|---|---|
| Root | 0.9977 | log. | $\bar{1}.99900$ | or | 9.99900 |
| | | Index | 6 | | 6 |
| Power | 0.9863 | | $\bar{1}.99400$ | | 9.99400 |

4. Required the cube of 0.08762

| | | | | | |
|---|---|---|---|---|---|
| Root | 0.08762 | log. | $\bar{2}.94260$ | or | 8.94260 |
| | | Index | 3 | | 3 |
| Power | 0.0006727 | | $\bar{4}.82780$ | | 6.82780 |

5. The 7th power of 0.9061 is 0.5015.
6. The 5th power of 0.9344 is 0.7123.

## EVOLUTION BY LOGARITHMS.

47. Evolution is the opposite of involution. Therefore, as quantities are involved, by the *multiplication* of logarithms, roots are extracted by the *division* of logarithms; that is,

To extract the root of a quantity by logarithms, DIVIDE THE LOGARITHM OF THE QUANTITY, BY THE NUMBER EXPRESSING THE ROOT REQUIRED.

The reason of the rule is evident also, from the fact, that logarithms are the exponents of powers and roots, and evolution is performed, by dividing the exponent, by the number expressing the root required. (Alg. 257.)

1. Required the square root of 648.3.

| | Numbers. | Logarithms. |
|---|---|---|
| Power | 648.3 | 2)2.81178 |
| Root | 25.46 | 1.40589 |

2. Required the cube root of 897.1.

| | | |
|---|---|---|
| Power | 897.1 | 3)2.95284 |
| Root | 9.645 | 0.98428 |

In the first of these examples, the logarithm of the given number is divided by 2; in the other, by 3.

3. Required the 10th root of 6948.

| | | |
|---|---|---|
| Power | 6948 | 10)3.84186 |
| Root | 2.422 | 0.38418 |

4. Required the 100th root of 983.

| | | |
|---|---|---|
| Power | 983 | 100)2.99255 |
| Root | 1.071 | 0.02992 |

The division is performed here, as in other cases of decimals, by removing the decimal point to the left.

5. What is the ten thousandth root of 49680000?

| | | |
|---|---|---|
| Power | 49680000 | 10000)7.69618 |
| Root | 1.00179 | 0.00077 |

We have, here, an example of the great rapidity with which arithmetical operations are performed by logarithms.

48. If the index of the logarithm is *negative*, and is *not divisible* by the given divisor, without a remainder, a difficulty will occur, unless the index be altered.

Suppose the cube root of 0.0000892 is required. The logarithm of this is $\bar{5}.95036$. If we divide the index by 3, the quotient will be $-1$, with $-2$ remainder. This remainder, if it were positive, might, as in other cases of division, be prefixed to the next figure. But the remainder is *negative*, while the decimal part of the logarithm is positive; so that, when the former is prefixed to the latter, it will make neither $+2.9$ nor $-2.9$, but $-2+.9$. This embarrassing intermixture of positives and negatives may be avoided, by adding to the index another negative number, to make it exactly divisible by the divisor. Thus, if to the index $-5$ there be added $-1$, the sum $-6$ will be divisible by 3. But this addition of a negative number must be *compensated*, by the addition of an equal positive number, which may be prefixed to the decimal part of the logarithm. The division may then be continued, without difficulty, through the whole.

Thus, if the logarithm $\bar{5}.95036$ be altered to $\bar{6}+1.95036$ it may be divided by 3, and the quotient will be $\bar{2}.65012$. We have then this rule,

49. *Add to the index*, if necessary, *such a negative number as will make it exactly divisible by the divisor, and prefix an equal positive number to the decimal part of the logarithm.*

1. Required the 5th root of 0.009642.

| | | | |
|---|---|---|---|
| Power | 0.009642 | log. | $\bar{3}.98417$ |
| | | or | $\bar{5}+2.98417$ |
| Root | 0.3952 | | $\bar{1}.59683$ |

2. Required the 7th root of 0.0004935.

| | | | |
|---|---|---|---|
| Power | 0.0004935 | log. | $\bar{4}.69329$ |
| | | or | $7)\bar{7}+3.69329$ |
| Root | 0.337 | | $\bar{1}.52761$ |

50. If, for the sake of performing the division conveniently, the negative index be rendered *positive*, it will be expedient to borrow as many tens, as there are units in the number denoting the root.

What is the fourth root of 0.03698 ?

| | | | | |
|---|---|---|---|---|
| Power | 0.03698 | 4)$\bar{2}$.56797 | or | 4)38.56797 |
| Root | 0.4385 | $\bar{1}$.64199 | | 9.64199 |

Here the index, by borrowing, is made 40 too great, that is, +38 instead of −2. When, therefore, it is divided by 4, it is still 10 too great, +9 instead of −1.

What is the 5th root of 0.003926 ?

| | | | | |
|---|---|---|---|---|
| Power | 0.003926 | 5)$\bar{3}$.95066 | or | 5)47.95066 |
| Root | 0.38916 | $\bar{1}$.59013 | | 9.59013 |

51. A *power of a root* may be found by first *multiplying* the logarithm of the given quantity into the index of the power, (Art. 45.) and then *dividing* the product by the number expressing the root. (Art. 47.)

1. What is the value of $(53)^{\frac{6}{7}}$, that is, the 6th power of the 7th root of 53 ?

| | | |
|---|---|---|
| Given number 53 | log. | 1.72428 |
| Multiplying by | | 6 |
| Dividing by | | 7)10.34568 |
| Power required 30.06 | | 1.47795 |

2. What is the 8th power of the 9th root of 654 ?

## PROPORTION BY LOGARITHMS.

52. In a proportion, when three terms are given, the fourth is found, in common arithmetic, by multiplying together the second and third, and dividing by the first. But when logarithms are used, *addition* takes the place of multiplication, and *subtraction*, of division.

To find then, by logarithms, the fourth term in a proportion, ADD THE LOGARITHMS OF THE SECOND AND THIRD TERMS, AND *from the sum* SUBTRACT THE LOGARITHM OF

THE FIRST TERM. The remainder will be the logarithm of the term required.

Ex. 1. Find a fourth proportional to 7964, 378, and 27960.

| | Numbers. | Logarithms. |
|---|---|---|
| Second term | 378 | 2.57749 |
| Third term | 27960 | 4.44654 |
| | | 7.02403 |
| First term | 7964 | 3.90113 |
| Fourth term | 1327 | 3.12290 |

2. Find a 4th proportional to 768, 381, and 9780.

| | | |
|---|---|---|
| Second term | 381 | 2.58092 |
| Third term | 9780 | 3.99034 |
| | | 6.57126 |
| First term | 768 | 2.88536 |
| Fourth term | 4852 | 3.68590 |

## ARITHMETICAL COMPLEMENT.

53. When one number is to be subtracted from another, it is often convenient, first to subtract it from 10, then to *add the difference* to the other number, and afterwards to reject the 10.

Thus, instead of $a-b$, we may put $10-b+a-10$.

In the first of these expressions, $b$ is subtracted from $a$. In the other, $b$ is subtracted from 10, the difference is added to $a$, and 10 is afterwards taken from the sum. The two expressions are equivalent, because they consist of the same terms, with the addition, in one of them, of $10-10=0$. The alteration is, in fact, nothing more than borrowing 10, for the sake of convenience, and then rejecting it in the result.

Instead of 10, we may borrow, as occasion requires, 100, 1000, &c.

Thus $a-b=100-b+a-100=1000-b+a-1000$, &c.

54. The DIFFERENCE *between a given number and* 10, *or* 100, *or* 1000, *&c. is called the* ARITHMETICAL COMPLEMENT *of that number.*

The arithmetical complement of a number consisting of *one* integral figure, either with or without decimals, is found, by subtracting the number from 10. If there are *two* integral figures, they are subtracted from 100; if *three*, from 1000, &c.

Thus the arithmetical compl't of 3.46 is 10 − 3.46 = 6.54
of 34.6 is 100 − 34.6 = 65.4
of 346. is 1000 − 346. = 654. &c.

According to the rule for subtraction in arithmetic, any number is subtracted from 10, 100, 1000, &c. by beginning on the right hand, and taking each figure from 10, after *increasing* all except the first, by *carrying* 1.

| | |
|---|---|
| Thus, if from | 10.00000 |
| We subtract | 7.63125 |

The difference, or arith'l compl't. is 2.36875, which is obtained, by taking 5 from 10, 3 from 10, 2 from 10, 4 from 10, 7 from 10, and 8 from 10; we may take it *without being increased*, from 9.

Thus 2 from 9 is the same as 3 from 10,
3 from 9, the same as 4 from 10, &c. Hence,

55. *To obtain the* ARITHMETICAL COMPLEMENT *of a number, subtract the right hand significant figure from* 10, *and each of the other figures from* 9. If, however, there are *ciphers* on the right hand of all the significant figures, they are to be set down without alteration.

In taking the arithmetical complement of a logarithm, if the index is *negative*, it must be *added* to 9; for adding a negative quantity is the same as subtracting a positive one. (Alg. 81.) The difference between −3 and +9, is not 6, but 12.

The arithmetical complement

| | | | |
|---|---|---|---|
| of 6.24897 is | 3.75103 | of $\bar{2}$.70649 is | 11.29351 |
| of 2.98643 | 7.01357 | of 3.64200 | 6.35800 |
| of 0.62430 | 9.37570 | of 9.35001 | 0.64999 |

56. The principal use of the arithmetical complement, is in working proportions by logarithms; where some of the terms are to be *added*, and one or more to be *subtracted*. In the Rule of Three or simple proportion, two terms are to be added, and from the sum, the first term is to be subtracted. But if, instead of the logarithm of the first term, we substitute its arithmetical complement, this may be *added* to the sum of the other two, or more simply, all three may be added together, by one operation. After the index is diminished by 10, the result will be the same as by the common method. For subtracting a number is the same, as adding its arithmetical complement, and then rejecting 10, 100, or 1000, from the sum. (Art. 53.)

It will generally be expedient, to place the terms in the same order, in which they are arranged in the statement of the proportion.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1. | As | 6273 *a. c.* | 6.20252 | 2. | As | 253 *a. c.* | 7.59688 |
| | Is to | 769.4 | 2.88615 | | Is to | 672.5 | 2.82769 |
| | So is | 37.61 | 1.57530 | | So is | 497 | 2.69636 |
| | To | 4.613 | 0.66397 | | To | 1321.1 | 3.12093 |
| 3. | As | 46.34 *a. c.* | 8.33404 | 4. | As | 9.85 *a. c.* | 9.00656 |
| | Is to | 892.1 | 2.95041 | | Is to | 643 | 2.80821 |
| | So is | 7.638 | 0.88298 | | So is | 76.3 | 1.88252 |
| | To | 147 | 2.16743 | | To | 4981 | 3.69729 |

## COMPOUND PROPORTION.

57. In compound, as in single proportion, the term required may be found by logarithms, if we substitute addition for multiplication, and subtraction for division.

Ex. 1. If the interest of \$365, for 3 years and 9 months, be \$82.13; what will be the interest of \$8940, for 2 years and 6 months?

In common arithmetic, the statement of the question is made in this manner,

$\left.\begin{matrix}\text{365 dollars}\\ \text{3.75 years}\end{matrix}\right\} : \text{82.13 dollars} :: \left\{\begin{matrix}\text{8940 dollars}\\ \text{2.5 years}\end{matrix}\right\} :$

And the method of calculation is, to *divide* the *product* of the third, fourth, and fifth terms, by the *product* of the two first.* This, if logarithms are used, will be to *subtract* the *sum* of the logarithms of the two first terms, from the *sum* of the logarithms of the other three.

| | | |
|---|---|---|
| Two first terms | 365 log. | 2.56229 |
| | 3.75 | 0.57403 |
| Sum of the logarithms | | 3.13632 |
| Third term | 82.13 | 1.91450 |
| Fourth and fifth terms | 8940 | 3.95134 |
| | 2.5 | 0.39794 |
| Sum of the logs. of the 3d, 4th, and 5th | | 6.26378 |
| Do. 1st and 2d | | 3.13632 |
| Term required | 1341 | 3.12746 |

58. The calculation will be more simple, if, instead of *subtracting* the logarithms of the two first terms, we *add* their *arithmetical complements*. But it must be observed, that *each* arithmetical complement increases the index of the logarithm by 10. If the arithmetical complement be introduced into *two* of the terms, the index of the sum of the logarithms will be 20 too great; if it be in *three* terms, the index will be 30 too great, &c.

| | | |
|---|---|---|
| Two first terms | 365 *a. c.* | 7.43771 |
| | 3.75 *a. c.* | 9.42597 |
| Third term | 82.13 | 1.91450 |
| Fourth and fifth terms | 8940 | 3.95134 |
| | 2.5 | 0.39794 |
| Term required | 1341 | 23.12746 |

The result is the same as before, except that the index of the logarithm is 20 too great.

* See Arithmetic.

Ex. 2. If the wages of 53 men for 42 days be 2200 dollars; what will be the wages of 87 men for 34 days?

$$\left.\begin{matrix}\text{53 men}\\ \text{42 days}\end{matrix}\right\} : 2200 :: \left\{\begin{matrix}\text{87 men}\\ \text{34 days}\end{matrix}\right\} :$$

| | | |
|---|---|---|
| Two first terms | 53 *a. c.* | 8.27572 |
| | 42 *a. c.* | 8.37675 |
| Third term | 2200 | 3.34242 |
| Fourth and fifth terms | 87 | 1.93952 |
| | 34 | 1.53148 |
| Term required | 2923.5 | 3.46589 |

59. In the same manner, if the product of *any number* of quantities, is to be divided, by the product of several others; we may add together the logarithms of the quantities to be divided, and the arithmetical complements of the logarithms of the divisors.

Ex. If $29.67 \times 346.2$ be divided by $69.24 \times 7.862 \times 497$; what will be the quotient?

| | | |
|---|---|---|
| Numbers to be divided | 29.67 | 1.47232 |
| | 346.2 | 2.53933 |
| Divisors | 69.24 *a. c.* | 8.15964 |
| | 7.862 *a. c.* | 9.10447 |
| | 497 *a. c.* | 7.30364 |
| Quotient | 0.03797 | 8.5794 |

In this way, the calculations in *Conjoined Proportion* may be expeditiously performed.

## COMPOUND INTEREST.

60. In calculating compound interest, the amount for the first year, is made the principal for the second year; the amount for the second year, the principal for the third year, &c. Now the amount at the end of each year, must be proportioned to the principal at the beginning of the year. If

In the right angled triangle BCP, (Trig. 134.)
R : BC : : sin B : : PC = 18.79.
And the solidity of the pyramid is 225.48 feet.

3. What is the solidity of a pyramid whose perpendicular height is 72, and the sides of whose base are 67, 54, and 40?
Ans. 25920.

PROBLEM IV.

*To find the* LATERAL SURFACE *of a* REGULAR PYRAMID.

49. MULTIPLY HALF THE SLANT-HEIGHT INTO THE PERIMETER OF THE BASE.

Let the triangle ABC (Fig. 18.) be one of the sides of a regular pyramid. As the sides AC and BC are equal, the angles A and B are equal. Therefore a line drawn from the vertex C to the middle of AB is *perpendicular* to AB. The area of the triangle is equal to the product of half this perpendicular into AB. (Art. 8.) The perimeter of the base is the sum of its sides, each of which is equal to AB. And the areas of all the equal triangles which constitute the lateral surface of the pyramid, are together equal to the product of the perimeter into half the slant-height CP.

The *slant-height* is the hypothenuse of a right angled triangle, whose legs are the axis of the pyramid, and the distance from the center of the base to the middle of one of the sides. See Def. 10.

Ex. 1. What is the lateral surface of a regular hexagonal pyramid, whose axis is 20 feet, and the sides of whose base are each 8 feet?

The square of the distance from the center of the base to one of the sides (Art. 16.) $=48$.
The slant-height (Euc. 47. 1.) $=\sqrt{48+(20)^2}=21.16$.
And the lateral surface $=21.16\times4\times6=507.84$ sq. feet.

2. What is the whole surface of a regular triangular pyramid whose axis is 8, and the sides of whose base are each 20.78?

| | |
|---|---|
| The lateral surface is | 312 |
| The area of the base is | 187 |
| And the whole surface is | 499 |

3. What is the lateral surface of a regular pyramid whose axis is 12 feet, and whose base is 18 feet square?

Ans. 540 square feet.

The lateral surface of an *oblique* pyramid may be found, by taking the sum of the areas of the unequal triangles which form its sides.

PROBLEM V.

*To find the* SOLIDITY *of a* FRUSTUM *of a pyramid.*

50. ADD TOGETHER THE AREAS OF THE TWO ENDS, AND THE SQUARE ROOT OF THE PRODUCT OF THESE AREAS; AND MULTIPLY THE SUM BY $\frac{1}{3}$ OF THE PERPENDICULAR HEIGHT OF THE SOLID.

Let CDGL (Fig. 17.) be a vertical section, through the middle of a frustum of a right pyramid CDV whose base is a *square.*

Let $CD=a$, $LG=b$, $RN=h$.

By similar triangles, $LG : CD :: RV : NV$.

Subtracting the antecedents, (Alg. 389.)

$LG : CD-LG :: RV : NV-RV=RN$.

$$\text{Therefore } RV=\frac{RN\times LG}{CD-LG}=\frac{hb}{a-b}$$

The square of CD is the base of the pyramid CDV;
And the square of LG is the base of the small pyramid LGV.
Therefore, the solidity of the larger pyramid (Art. 48.) is

$$\overline{CD}^2\times\tfrac{1}{3}(RN+RV)=a^2\times\tfrac{1}{3}\left(h+\frac{hb}{a-b}\right)=\frac{ha^3}{3a-3b}$$

And the solidity of the smaller pyramid is equal to

$$\overline{LG}^2\times\tfrac{1}{3}RV=b^2\times\frac{hb}{3a-3b}=\frac{hb^3}{3a-3b}.$$

If the smaller pyramid be taken from the larger, there will remain the frustum CDLG, whose solidity is equal to

$$\frac{ha^3-hb^3}{3a-3b}=\tfrac{1}{3}h\times\frac{a^3-b^3}{a-b}=\tfrac{1}{3}h\times(a^2+ab+b^2)\qquad\text{(Alg. 466.)}$$

Or, because $\sqrt{a^2b^2}=ab$, (Alg. 259.)

$$\tfrac{1}{3}h\times(a^2+b^2+\sqrt{a^2b^2})$$

Here $h$, the height of the frustum, is multiplied into $a^2$ and $b^2$, the areas of the two ends, and into $\sqrt{a^2b^2}$ the square root of the products of these areas.

In this demonstration, the pyramid is supposed to be *square*. But the rule is equally applicable to a pyramid of any other form. For the solid contents of pyramids are equal, when they have equal heights and bases, whatever be the *figure* of their bases. (Sup. Euc. 14. 3.) And the sections parallel to the bases, and at equal distances, are equal to one another. (Sup. Euc. 12. 3. Cor. 2.)*

Ex. 1. If one end of the frustum of a pyramid be 9 feet square, the other end 6 feet square, and the height 36 feet, what is the solidity?

The areas of the two ends are 81 and 36.

The square root of their product is 54.

And the solidity of the frustum$=(81+36+54)\times 12=2052$.

2. If the height of a frustum of a pyramid be 24, and the areas of the two ends 441 and 121; what is the solidity?
Ans. 6344.

3. If the height of a frustum of a hexagonal pyramid be 48, each side of one end 26, and each side of the other end 16; what is the solidity? Ans. 56034.

### PROBLEM VI.

*To find the* LATERAL SURFACE *of a* FRUSTUM *of a regular pyramid.*

51. MULTIPLY HALF THE SLANT-HEIGHT BY THE SUM OF THE PERIMETERS OF THE TWO ENDS.

Each side of a frustum of a regular pyramid is a *trapezoid*, as ABCD. (Fig. 19.) The slant-height HP, (Def. 11.) though it is oblique to the base of the solid, is perpendicular to the line AB. The area of the trapezoid is equal to the product of half this perpendicular into the sum of the parallel sides AB and DC. (Art. 12.) Therefore the area of all the equal trapezoids which form the lateral surface of

* See note F.

the frustum, is equal to the product of half the slant-height into the sum of the perimeters of the ends.

Ex. If the slant-height of a frustum of a regular octagonal pyramid be 42 feet, the sides of one end 5 feet each, and the sides of the other end 3 feet each; what is the lateral surface?, Ans. 1344 square feet.

52. If the slant-height be not given, it may be obtained from the perpendicular height, and the dimensions of the two ends. Let GD (Fig. 17.) be the slant-height of the frustum CDGL, RN or GP the perpendicular height, ND and RG the radii of the circles inscribed in the perimeters of the two ends. Then PD is the difference of the two radii:

And the slant-height GD $=\sqrt{(\overline{GP}^2+\overline{PD}^2)}$.

Ex. If the perpendicular height of a frustum of a regular hexagonal pyramid be 24, the sides of one end 13 each, and the sides of the other end 8 each; what is the whole surface?

$\sqrt{(\overline{BC}^2-\overline{BP}^2)}=CP$,(Fig. 7.) that is, $\sqrt{(\overline{13}^2-\overline{6.5}^2)}=11.258$

And $\sqrt{8^2-4^2} = 6.928$

The difference of the two radii is, therefore, 4.33

The slant-height $=\sqrt{(\overline{24}^2+\overline{4.33}^2)}=24.3875$

The lateral surface is 1536.4

And the whole surface, 2141.75.

53. The height of the *whole pyramid* may be calculated from the dimensions of the frustum. Let VN (Fig. 17.) be the height of the pyramid, RN or GP the height of the frustum, ND and RG the radii of the circles inscribed in the perimeters of the ends of the frustum.

Then, in the similar triangles GPD and VND,

DP : GP :: DN : VN.

The height of the frustum subtracted from VN, gives VR the height of the small pyramid VLG. The *solidity* and *lateral surface* of the frustum may then be found, by subtracting from the whole pyramid, the part which is above the cutting plane. This method may serve to verify the calculations which are made by the rules in arts. 50 and 51.

Ex. If one end of the frustum CDGL (Fig. 17.) be 90 feet square, the other end 60 feet square, and the height RN 36 feet; what is the height of the whole pyramid VCD: and what are the solidity and lateral surface of the frustum?

DP=DN−GR=45−30=15. And GP=RN=36.

Then 15 : 36 : : 45 : 108=VN, the height of the whole pyramid.

And 108−36=72=VR, the height of the part VLG.

| | | |
|---|---|---|
| The solidity of the large pyramid is | 291600 | (Art. 48.) |
| of the small pyramid | 86400 | |
| of the frustum CDGL | 205200 | |
| The lateral surface of the large pyramid is | 21060 | (Art. 49.) |
| of the small pyramid | 9360 | |
| of the frustum | 11700 | |

### PROBLEM VII.

*To find the* SOLIDITY *of a* WEDGE.

54. ADD THE LENGTH OF THE EDGE TO TWICE THE LENGTH OF THE BASE, AND MULTIPLY THE SUM BY $\frac{1}{6}$ OF THE PRODUCT OF THE HEIGHT OF THE WEDGE AND THE BREADTH OF THE BASE.

Let L=AB the length of the base. (Fig. 20.)
$l$=GH the length of the edge.
$b$=BC the breadth of the base.
$h$=PG the height of the wedge.
Then L − $l$=AB−GH=AM.

If the length of the base and the edge be *equal*, as BM and GH, (Fig 20.) the wedge MBHG is half a parallelopiped of the same base and height. And the solidity (Art. 43.) is equal to half the product of the height, into the length and breadth of the base; that is to $\frac{1}{2}$ *bhl.*

If the length of the base be *greater* than that of the edge, as ABGH; let a section be made by the plane GMN, parallel to HBC. This will divide the whole wedge into two parts MBHG and AMG. The latter is a pyramid, whose solidity (Art. 48.) is $\frac{1}{3}bh \times (L-l)$

The solidity of the parts together, is, therefore,
$\frac{1}{2}bhl+\frac{1}{3}bh\times(L-l)=\frac{1}{6}bh3l+\frac{1}{6}bh2L-\frac{1}{6}bh2l=\frac{1}{6}bh\times(2L+l)$

If the length of the base be *less* than that of the edge, it is evident that the pyramid is to be *subtracted* from half the parallelopiped, which is equal in height and breadth to the wedge, and equal in length to the edge.

The solidity of the wedge is, therefore,
$\frac{1}{2}bhl-\frac{1}{3}bh\times(l-L)=\frac{1}{6}bh3l-\frac{1}{6}bh2l+\frac{1}{6}bh2L=\frac{1}{6}bh\times(2L+l)$

Ex. 1. If the base of a wedge be 35 by 15, the edge 55, and the perpendicular height 12.4; what is the solidity?

$$\text{Ans. } (70+55)\times\frac{15\times12.4}{6}=3875.$$

2. If the base of a wedge be 27 by 8, the edge 36, and the perpendicular height 42; what is the solidity?

Ans. 5040.

### PROBLEM VIII.

*To find the* SOLIDITY *of a rectangular* PRISMOID.

55. TO THE AREAS OF THE TWO ENDS, ADD FOUR TIMES THE AREA OF A PARALLEL SECTION EQUALLY DISTANT FROM THE ENDS, AND MULTIPLY THE SUM BY $\frac{1}{6}$ OF THE HEIGHT.

Let L and B (Fig. 21.) be the length and breadth of one end,
$l$ and $b$ the length and breadth of the other end,
M and $m$ the length and breadth of the section in the middle.
and $h$ the height of the prismoid.

The solid may be divided into two wedges, whose bases are the ends of the prismoid, and whose edges are L and $l$. The solidity of the whole, by the preceding article, is

$$\tfrac{1}{6}Bh\times(2L+l)+\tfrac{1}{6}bh\times(2l+L)=\tfrac{1}{6}h(2BL+Bl+2bl+bL)$$

As M is equally distant from L and $l$,

$$2M=L+l, 2m=B+b, \text{ and } 4Mm=(L+l)(B+b)=BL+Bl+bL+lb.$$

Substituting $4Mm$ for its value, in the preceding expression for the solidity, we have

$$\tfrac{1}{6}h(BL+bl+4Mm)$$

That is, the solidity of the prismoid is equal to $\frac{1}{6}$ of the height, multiplied into the areas of the two ends, and 4 times the area of the section in the middle.

This rule may be applied to prismoids of other forms. For, whatever be the figure of the two ends, there may be drawn in each, such a number of small rectangles, that the sum of them shall differ less, than by any given quantity, from the figure in which they are contained. And the solids between these rectangles will be rectangular prismoids.

Ex. 1. If one end of a rectangular prismoid be 44 feet by 23, the other end 36 by 21, and the perpendicular height 72; what is the solidity?

The area of the larger end $=44\times23=1012$
of the smaller end $=36\times21=\ 756$
of the middle section $=40\times22=\ 880$

And the solidity $=(1012+756+4\times880)\times12=63456$ feet.

2. What is the solidity of a stick of hewn timber, whose ends are 30 inches by 27, and 24 by 18, and whose length is 48 feet? Ans. 204 feet.

Other solids not treated of in this section, if they be bounded by plane surfaces, may be measured by supposing them to be divided into prisms, pyramids, and wedges. And, indeed, every such solid may be considered as made up of triangular pyramids.

## THE FIVE REGULAR SOLIDS.

56. **A SOLID IS SAID TO BE REGULAR, WHEN ALL ITS SOLID ANGLES ARE EQUAL, AND ALL ITS SIDES ARE EQUAL AND REGULAR POLYGONS.**

The following figures are of this description;

| | | |
|---|---|---|
| 1. The *Tetraedron*, | whose sides are | four triangles; |
| 2. The *Hexaedron or cube*, | | six squares; |
| 3. The *Octaedron*, | | eight triangles; |
| 4. The *Dodecaedron*, | | twelve pentagons; |
| 5. The *Icosaedron*, | | twenty triangles.* |

Besides these five, there can be no other regular solids. The only plane figures which can form such solids, are triangles, squares, and pentagons. For the plane angles which contain any solid angle, are together less than four right angles or 360°. (Sup. Euc. 21. 2.) And the least number which can form a solid angle is three. (Sup. Euc. Def. 8. 2.) If they are angles of equilateral *triangles*, each is 60°. The sum of *three* of them is 180°, of *four* 240°, of *five* 300°, and of *six* 360°. The latter number is too great for a solid angle.

The angles of *squares* are 90° each. The sum of *three* of these is 270°, of four 360°, and of any other greater number, still more.

The angles of regular *pentagons* are 108° each. The sum of *three* of them is 324°; of four, or any other greater number, more than 360°. The angles of all other regular polygons are still greater.

In a regular solid, then, each solid angle must be contained by three, four, or five equilateral triangles, by three squares, or by three regular pentagons.

57. As the sides of a regular solid are similar and equal, and the angles are also alike; it is evident that the sides are all equally distant from a central point in the solid. If then, planes be supposed to proceed from the several edges to the center, they will divide the solid into as many equal *pyramids*, as it has sides. The base of each pyramid will be one of the sides; their common vertex will be the central point; and their height will be a perpendicular from the center to one of the sides.

---

* For the geometrical construction of these solids, see Legendre's Geometry; Appendix to Books VI and VII.

64. The exponent of a power may be itself a power, as in the equation

$$a^{m^x}=b\,;$$

where $x$ is the exponent of the power $m^x$, which is the exponent of the power $a^{m^x}$.

Ex. 4. Find the value of $x$, in the equation $9^{3^x}=1000$.

$3^x \times (\log.\ 9)=\log.\ 1000$. Therefore $3^x=\frac{\log.\ 1000}{\log.\ 9}=3.14$.

Then as $3^x=3.14$. $x(\log.\ 3)=\log.\ 3.14$.

Therefore $x=\frac{\log.\ 3.14}{\log.\ 3}=\frac{4969296}{4771213}=1.04$.

In cases like this, where the factors, divisors, &c. are logarithms, the calculation may be facilitated, by taking the *logarithms of the logarithms*. Thus the value of the fraction $\frac{4969296}{4771213}$ is most easily found, by subtracting the logarithm of the logarithm which constitutes the denominator, from the logarithm of that which forms the numerator.

5. Find the value of $x$, in the equation $\frac{ba^x+d}{c}=m$.

Ans. $x=\frac{\log.\ (cm-d)-\log.\ b.}{\log.\ a.}$

## SECTION IV.

### DIFFERENT SYSTEMS OF LOGARITHMS, AND COMPUTATION OF THE TABLES.

65. For the common purposes of numerical computation, Briggs' system of logarithms has a decided advantage over every other. But the theory of logarithms is an important instrument of investigation, in the higher departments of mathematical science. In its numerous applications, there is frequent occasion to compare the common system with others; especially with that which was adopted, by the celebrated inventor of logarithms, Lord Napier. In conducting these investigations, it is often expedient to express the logarithm of a number, in the form of a *series*.

If $a^x = \mathrm{N}$, then $x$ is the logarithm of N. (Art. 2.)

To find the value of $x$, in a series, let the quantities $a$ and N be put into the form of a binomial, by making $a = 1+b$, and $\mathrm{N} = 1+n$. Then $(1+b)^x = 1+n$, and extracting the root $y$ of both sides, we have

$$(1+b)^{\frac{x}{y}} = (1+n)^{\frac{1}{y}}$$

By the binomial theorem

$$(1+b)^{\frac{x}{y}} = 1 + \frac{x}{y}(b) + \frac{x}{y}\left(\frac{x}{y}-1\right)\left(\frac{b^2}{2}\right) + \frac{x}{y}\left(\frac{x}{y}-1\right)\left(\frac{x}{y}-2\right)\left(\frac{b^3}{2.3}\right) + \text{\&c.}$$

$$(1+n)^{\frac{1}{y}} = 1 + \frac{1}{y}(n) + \frac{1}{y}\left(\frac{1}{y}-1\right)\left(\frac{n^2}{2}\right) + \frac{1}{y}\left(\frac{1}{y}-1\right)\left(\frac{1}{y}-2\right)\left(\frac{n^3}{2.3}\right) + \text{\&c.}$$

As these expressions will be the same, whatever be the value of $y$, let $y$ be taken indefinitely great; then $\frac{x}{y}$ and $\frac{1}{y}$ being indefinitely small, in comparison with the numbers $-1$, $-2$, &c. with which they are connected, may be cancelled from the factors $\left(\frac{x}{y}-1\right)$, $\left(\frac{x}{y}-2\right)$, &c. $\left(\frac{1}{y}-1\right)$, $\left(\frac{1}{y}-2\right)$, &c. (Alg. 456.) leaving $1 + \frac{x}{y}b - \frac{x}{y}\left(\frac{b^2}{2}\right) + \frac{x}{y}\left(\frac{b^3}{3}\right) - \frac{x}{y}\left(\frac{b^4}{4}\right)$, &c.

$$= 1 + \frac{1}{y}n - \frac{1}{y}\left(\frac{n^2}{2}\right) + \frac{1}{y}\left(\frac{n^3}{3}\right) - \frac{1}{y}\left(\frac{n^4}{4}\right), \text{\&c.}$$

Rejecting 1 from each side of the equation, multiplying by $y$, (Alg. 159.) and dividing by the compound factor into which $x$ is multiplied, we have

$$x=\text{Log. N}=\frac{n-\frac{1}{2}n^2+\frac{1}{3}n^3-\frac{1}{4}n^4+\text{\&c.}}{b-\frac{1}{2}b^2+\frac{1}{3}b^3-\frac{1}{4}b^4+\text{\&c.}} \quad \text{A}$$

Or, as $n=N-1$, and $b=a-1$,

$$\text{Log. N}=\frac{(N-1)-\frac{1}{2}(N-1)^2+\frac{1}{3}(N-1)^3-\frac{1}{4}(N-1)^4+\text{\&c.}}{(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\frac{1}{4}(a-1)^4+\text{\&c.}}$$

Which is a general expression, for the logarithm of any number N, in any system in which the base is $a$. The numerator is expressed in terms of N only; and the denominator in terms of $a$ only: So that, whatever be the number, the denominator will remain the same, unless the base is changed. The reciprocal of this constant denominator, viz.

$$\frac{1}{(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\frac{1}{4}(a-1)^4+\text{\&c.}}$$

is called the *Modulus* of the system of which $a$ is the base. If this be denoted by M, then

$$\text{Log. N}=\text{M}\times\Big((N-1)-\tfrac{1}{2}(N-1)^2+\tfrac{1}{3}(N-1)^3-\tfrac{1}{4}(N-1)^4+\text{\&c.}\Big)$$

66. The foundation of Napier's system of Logarithms is laid, by making the modulus equal to *unity*. From this condition the *base* is determined. Taking the equation above marked A. and making the denominator equal to 1, we have

$$x=n-\tfrac{1}{2}n^2+\tfrac{1}{3}n^3-\tfrac{1}{4}n^4+\tfrac{1}{5}n^5-\text{\&c.}$$

By reverting this equation*

$$n=x+\frac{x^2}{2}+\frac{x^3}{2.3}+\frac{x^4}{2.3.4}+\frac{x^5}{2.3.4.5}+\text{\&c.}$$

Or, as by the notation, $n+1=N=a^x$,

$$a^x=1+x+\frac{x^2}{2}+\frac{x^3}{2.3}+\frac{x^4}{2.3.4}+\frac{x^5}{2.3.4.5}+\text{\&c.}$$

If then $x$ be taken equal to 1, we have

$$a=1+1+\frac{1}{2}+\frac{1}{2.3}+\frac{1}{2.3.4}+\frac{1}{2.3.4.5}+\text{\&c.}$$

Adding the first fifteen terms, we have

2.7182818284

Which is the base of Napier's system, correct to ten places of decimals.

* See note D.

Napier's logarithms are also called *hyperbolic* logarithms, from certain relations which they have to the spaces between the asymptotes and the curve of an hyperbola; although these relations are not, in fact, peculiar to Napier's system.

67. The logarithms of *different* systems are compared with each other, by means of the modulus. As in the series

$$\frac{(N-1)-\frac{1}{2}(N-1)^2+\frac{1}{3}(N-1)^3-\frac{1}{4}(N-1)^4+\&c.}{(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\frac{1}{4}(a-1)^4+\&c.}$$

which expresses the logarithm of N, the *denominator* only is affected by a change of the base $a$; and as the value of fractions, whose numerators are given, are reciprocally as their denominators: (Alg. 360. cor. 2.)

*The logarithm of a given number, in one system,*
*Is to the logarithm of the same number in another system;*
*As the modulus of one system,*
*To the modulus of the other.*

So that, if the modulus of each of the systems be given, and the logarithm of any number be calculated in one of the systems; the logarithm of the same number in the other system may be calculated by a simple proportion. Thus if M be the modulus in Briggs' system, and M′ the modulus in Napier's; $l$ the logarithm of a number in the former, and $l'$ the logarithm of the same number in the latter; then,

$$M : M' :: l : l',$$
Or, as $M'=1$,
$$M : 1 :: l : l'$$

Therefore, $l=l'\times M$; that is, the common logarithm of a number, is equal to Napier's logarithm of the same, multiplied into the modulus of the common system.

To find this modulus, let $a$ be the base of Briggs' system, and $e$ the base of Napier's; and let $l.a$ denote the common logarithm of $a$, and $l'.a$ denote Napier's logarithm of $a$.

Then $M : 1 :: l.a : l'.a$. Therefore $M=\frac{l.a}{l'.a}$

But in the common system, $a=10$, and $l.a=1$.

So that, $M=\frac{1}{l'.10}$, that is, the modulus of Briggs' system, is equal to 1 divided by Napier's logarithm of 10.

Again $M : 1 :: l.e : l'.e$

But as $e$ denotes Napier's base, $l'.e=1$.

So that $M=l.e$, that is, the modulus of the common system, is equal to the common logarithm of Napier's base.

Therefore, either of the expressions, $l.e$, or $\frac{1}{l'.a}$ may be used, to convert the logarithms of one of the systems into those of the other.

*The ratio of the logarithms of two numbers to each other, is the same in one system as in another.* If N and $n$ be the two numbers;

Then, $l.N : l'.N :: M : M'$

$l.n : l'.n :: M : M'$

Therefore, $l.N : l.n :: l'.N : l'.n.$

COMPUTATION OF LOGARITHMS.

68. The logarithms of most numbers can be calculated by approximation only, by finding the sum of a sufficient number of terms, in the series which expresses the value of the logarithms. According to art. 65.

$$\text{Log. } N=M\times((N-1)-\tfrac{1}{2}(N-1)^2+\tfrac{1}{3}(N-1)^3, \text{\&c.})$$

Or, putting as before, $n=N-1$,

$$\text{Log. }(1+n)=M(n-\tfrac{1}{2}n^2+\tfrac{1}{3}n^3-\tfrac{1}{4}n^4+\tfrac{1}{5}n^5-\text{\&c.})$$

But this series will not converge, when $n$ is a whole number, greater than unity. To convert it into another which will converge, let $(1-n)$ be expanded in the same manner as $(1+n)$, (Art. 65.) The formula will be the same, except that the odd powers of $n$ will be negative instead of positive. We shall then have,

$$\text{Log. }(1+n)=M(n-\tfrac{1}{2}n^2+\tfrac{1}{3}n^3-\tfrac{1}{4}n^4+\tfrac{1}{5}n^5-\text{\&c.})$$
$$\text{Log. }(1-n)=M(-n-\tfrac{1}{2}n^2-\tfrac{1}{3}n^3-\tfrac{1}{4}n^4-\tfrac{1}{5}n^5-\text{\&c.})$$

Subtracting the one from the other the even powers of $n$ disappear, and we have

$$M(2n+\tfrac{2}{3}n^3+\tfrac{2}{5}n^5+\tfrac{2}{7}n^7+\text{\&c.})$$

or

$$2M(n+\tfrac{1}{3}n^3+\tfrac{1}{5}n^5+\tfrac{1}{7}n^7+\text{\&c.})$$

But this, which is the *difference* of the logarithms of $(1+n)$ and $(1-n)$ is the logarithm of the *quotient* of the one divided by the other. (Art. 36.)

That is, Log. $\frac{1+n}{1-n}=2M(n+\frac{1}{3}n^3+\frac{1}{5}n^5+\frac{1}{7}n^7+\&c.)$

Now put $n=\frac{1}{z-1}$

$$\text{Then,}\quad \frac{1+n}{1-n}=\frac{1+\frac{1}{z-1}}{1-\frac{1}{z-1}}=\frac{\frac{z}{z-1}}{\frac{z-2}{z-1}}=\frac{z}{z-2}$$

Therefore, substituting $\frac{z}{z-2}$ for $\frac{1+n}{1-n}$, and $\frac{1}{z-1}$ for $n$, we have

$$\text{Log. } \frac{z}{z-2}=2M\left(\frac{1}{(z-1)}+\frac{1}{3(z-1)^3}+\frac{1}{5(z-1)^5}+\&c.\right)$$

Or, (Art. 36.)

$$\text{Log.}z-\text{log.}(z-2)=2M\left(\frac{1}{(z-1)}+\frac{1}{3(z-1)^3}+\frac{1}{5(z-1)^5}+\&c.\right)$$

Therefore,

$$\text{Log.}z=\text{log.}(z-2)+2M\left(\frac{1}{(z-1)}+\frac{1}{3(z-1)^3}+\frac{1}{5(z-1)^5}+\&c.\right)$$

This series may be applied to the computation of any number greater than 2.

To find the logarithm of 2, let $z=4$,

Then $(z-1)=3$, and the preceding series, after transposing log. $(z-2)$ becomes

$$\text{Log. }4-\text{log. }2=2M\left(\frac{1}{3}+\frac{1}{3.3^3}+\frac{1}{5.3^5}+\frac{1}{7.3^7}, \&c.\right)$$

But as 4 is the square of 2; log. 4=2 log. 2. (Alg. 44.) So that log. 4 − log. 2=log. 2. We have then

$$\text{Log.}\,2=2\text{M}\left(\frac{1}{3}+\frac{1}{3.3^3}+\frac{1}{5.3^5}+\frac{1}{7.3^7}+\frac{1}{9.3^9}+,\&c.\right)$$

When the logarithms of the *prime* numbers are computed, the logarithms of all other numbers may be found, by simply adding the logarithms of the factors of which the numbers are composed. (Art. 36.)

69. In Napier's system, where M=1, the logarithms may be computed, as in the following table.

NAPIER'S OR HYPERBOLIC LOGARITHMS.

| | |
|---|---|
| $\text{Log.}\,2=2\left(\frac{1}{3}+\frac{1}{3.3^3}+\frac{1}{5.3^5}+\frac{1}{7.3^7},\&c.\right)$ | =0.693147 |
| $\text{Log.}\,3=2\left(\frac{1}{2}+\frac{1}{3.2^3}+\frac{1}{5.2^5}+\frac{1}{7.2^7},\&c.\right)$ | =1.098612 |
| Log. 4=2 log. 2. | =1.386294 |
| $\text{Log.}\,5=\text{log.}\,3+2\left(\frac{1}{4}+\frac{1}{3.4^3}+\frac{1}{5.4^5}+\frac{1}{7.4^7},\&c.\right)$ | =1.609438 |
| Log. 6=log. 3+log. 2. | =1.791759 |
| $\text{Log.}\,7=\text{log.}\,5+2\left(\frac{1}{6}+\frac{1}{3.6^3}+\frac{1}{5.6^5}+\frac{1}{7.6^7},\&c.\right)$ | =1.955900 |
| Log. 8=log. 4+log. 2. | =2.079441 |
| Log. 9=2 log. 3. | =2.197224 |
| Log. 10=log. 5+log. 2. | =2.302585 |
| &c. &c. | &c. |

70. To compute the logarithms of the common system, it will be necessary to find the value of the *modulus*. This is equal to 1 divided by Napier's logarithm of 10, (Art. 67.) that is,

$$\frac{1}{2.302585}=.43429448.$$

This number substituted for M, or twice the number, viz. .86858896 substituted for 2 M, in the series in art. 68. will enable us to calculate the common logarithm of any number.

## COMMON OR BRIGGS' LOGARITHMS.

$$\text{Log. } 2 = .86858896\left(\frac{1}{3}+\frac{1}{3.3^3}+\frac{1}{5.3^5}+\frac{1}{7.3^7}\&c.\right) = 0.301030$$

$$\text{Log. } 3 = .86858896\left(\frac{1}{2}+\frac{1}{3.2^3}+\frac{1}{5.2^5}+\frac{1}{7.2^7}\&c.\right) = 0.477121$$

Log. 4 = 2 log. 2. = 0.602060

Log. 5 = log. 10 − log. 2 = 1 − log. 2. = 0.698970

Log. 6 = log. 3 + log. 2. = 0.778151

$$\text{Log. } 7 = .86858896\left(\frac{1}{6}+\frac{1}{3.6^3}+\frac{1}{5.6^5}+\frac{1}{7.6^7}\&c.\right) + \text{log. } 5. = 0.845098$$

Log. 8 = 3 log. 2. = 0.903090

Log. 9 = 2 log. 3. = 0.954243

Log. 10 = 1.000000

&c. &c.

# TRIGONOMETRY.

## SECTION I.

### SINES, TANGENTS, SECANTS, &c.

ART. 71. TRIGONOMETRY *treats of the relations of the sides and angles of* TRIANGLES. Its first object is, to determine the length of the sides, and the quantity of the angles. In addition to this, from its principles are derived many interesting methods of investigation in the higher branches of analysis, particularly in physical astronomy. Scarcely any department of mathematics is more important, or more extensive in its applications. By trigonometry, the mariner traces his path on the ocean; the geographer determines the latitude and longitude of places, the dimensions and positions of countries, the altitude of mountains, the courses of rivers, &c. and the astronomer calculates the distances and magnitudes of the heavenly bodies, predicts the eclipses of the sun and moon, and measures the progress of light from the stars.

72. Trigonometry is either *plane* or *spherical*. The former treats of triangles bounded by *right lines;* the latter, of triangles bounded by *arcs of circles.*

### *Divisions of the Circle.*

73. In a triangle there are two classes of quantities which are the subjects of inquiry, the *sides* and the *angles.* For the purpose of measuring the latter, a *circle* is introduced.

The periphery of every circle, whether great or small, is supposed to be divided into 360 equal parts called *degrees*, each degree into 60 *minutes*, each minute into 60 *seconds*, each second into 60 *thirds*, &c, marked with the characters $^\circ$, $'$, $''$, $'''$, &c. Thus, $32^\circ\ 24'\ 13''\ 22'''$ is 32 degrees 24 minutes 13 seconds 22 thirds.*

A degree, then, is not a magnitude of a given *length*; but a certain *portion* of the whole circumference of any circle. It is evident, that the 360th part of a large circle is greater, than the same part of a small one. On the other hand, the *number* of degrees, in a small circle, is the same as in a large one.

The fourth part of a circle is called a *quadrant*, and contains 90 degrees.

74. To *measure* an angle, a circle is so described that its center shall be the angular point, and its periphery shall cut the two lines which include the angle. The *arc* between the two lines is considered a *measure of the angle*, because, by Euc. 33. 6, angles at the center of a given circle, have the same ratio to each other, as the arcs on which they stand. Thus the arc AB, (Fig. 2.) is a measure of the angle ACB.

It is immaterial what is the size of the circle, provided it cuts the lines which include the angle. Thus the angle ACD (Fig. 4.) is measured by either of the arcs AG, *ag*. For ACD is to ACH, as AG to AH, or as *ag* to *ah*. (Euc. 33. 6.)

75. In the circle ADGH, (Fig. 2.) let the two diameters AG and DH be perpendicular to each other. The angles ACD, DCG, GCH, and HCA will be right angles; and the periphery of the circle will be divided into four equal parts, each containing 90 degrees. As a right angle is subtended by an arc of $90^\circ$, the angle itself is said to contain $90^\circ$. Hence, in two right angles, there are $180^\circ$, in four right angles $360^\circ$; and in any other angle, as many degrees, as in the arc by which it is subtended.

76. The sum of the three angles of any triangle being equal to two right angles, (Euc. 32. 1.) is equal to $180^\circ$. Hence, there can never be more than one obtuse angle in a triangle. For the sum of two obtuse angles is more than $180^\circ$.

77. *The* COMPLEMENT *of an arc or an angle, is the difference between the arc or angle and* 90 *degrees.*

The complement of the arc AB (Fig. 2.) is DB; and the complement of the angle ACB is DCB. The complement of the arc BDG is also DB.

---

* See note E.

| | | |
|---|---|---|
| The complement | of 10° is 80°, | of 60° is 30°, |
| | of 20° is 70°, | of 120° is 30°, |
| | of 50° is 40°, | of 170° is 80°, &c. |

Hence, an acute angle and its complement are always equal to 90°. The angles ACB and DCB are together equal to a right angle. The two acute angles of a right angled triangle are equal to 90°: therefore each is the complement of the other.

78. *The* SUPPLEMENT *of an arc or an angle is the difference between the arc or angle and* 180 *degrees.*

The supplement of the arc BDG (Fig. 2.) is AB; and the supplement of the angle BCG is BCA.

| | | |
|---|---|---|
| The supplement | of 10° is 170°, | of 120° is 60°, |
| | of 80° is 100°, | of 150° is 30°, &c. |

Hence an angle and its supplement are always equal to 180°. The angles BCA and BCG are together equal to two right angles.

79. Cor. As the three angles of a plane triangle are equal to two right angles, that is, to 180° (Euc. 32. 1.) the sum of any two of them is the supplement of the other. So that the third angle may be found, by subtracting the sum of the other two from 180°. Or the sum of any two may be found, by subtracting the third from 180°.

80. A straight line drawn from the center of a circle to any part of the periphery, is called a *radius* of the circle. In many calculations, it is convenient to consider the radius, whatever be its length, as *a unit.* (Alg. 510.) To this must be referred the numbers expressing the lengths of other lines. Thus 20 will be twenty times the radius, and 0.75, three fourths of the radius.

### *Definitions of Sines, Tangents, Secants, &c.*

81. To facilitate the calculations in trigonometry, there are drawn, within and about the circle, a number of straight lines, called *Sines, Tangents, Secants, &c.* With these the learner should make himself perfectly familiar.

82. *The* SINE *of an arc is a straight line drawn from one end of the arc, perpendicular to a diameter which passes through the other end.*

Thus BG (Fig. 3.) is the sine of the arc AG. For BG is a line drawn from the end G of the arc, perpendicular to the diameter AM which passes through the other end A of the arc.

Cor. The sine is *half the chord* of *double the arc.* The sine BG is half PG, which is the chord of the arc PAG, double the arc AG.

83. *The* VERSED SINE *of an arc is that part of the diameter which is between the sine and the arc.*

Thus BA is the versed sine of the arc AG.

84. *The* TANGENT *of an arc, is a straight line drawn perpendicular from the extremity of the diameter which passes through one end of the arc, and extended till it meets a line drawn from the center through the other end.*

Thus AD (Fig. 3.) is the tangent of the arc AG.

85. *The* SECANT *of an arc, is a straight line drawn from the center, through one end of the arc, and extended to the tangent which is drawn from the other end.*

Thus CD (Fig. 3.) is the secant of the arc AG.

86. In Trigonometry, the terms *tangent* and *secant* have a more limited meaning, than in Geometry. In both, indeed, the tangent *touches* the circle, and the secant *cuts* it. But in Geometry, these lines are of no determinate length; whereas, in Trigonometry, they extend from the diameter to the point in which they intersect each other.

87. The lines just defined are sines, tangents and secants of *arcs.* BG (Fig. 3.) is the sine of the arc AG. But this arc subtends the *angle* GCA. BG is then the sine of the arc which subtends the angle GCA. This is more concisely expressed, by saying that BG is the sine of the angle GCA. And universally, the sine, tangent, and secant of an *arc*, are said to be the sine, tangent, and secant of the *angle* which stands at the center of the circle, and is subtended by the arc. Whenever, therefore, the sine, tangent, or secant of an angle is spoken of; we are to suppose a circle to be drawn whose center is the angular point; and that the lines mentioned belong to that arc of the periphery which subtends the angle.

88. The *sine*, and *tangent* of an acute angle, are *opposite* to the angle. But the *secant* is one of the lines which *include* the angle. Thus the sine BG, and the tangent AD (Fig. 3.) are opposite to the angle DCA. But the secant CD is one of the lines which include the angle.

89. *The sine complement or* COSINE *of an angle, is the sine of the* COMPLEMENT *of that angle.* Thus, if the diameter HO (Fig. 3.) be perpendicular to MA, the angle HCG is the complement of ACG; (Art. 77.) and LG, or its equal CB, is the sine of HCG. (Art. 82.) It is, therefore, the *cosine* of GCA. On the other hand GB is the sine of GCA, and the cosine of GCH.

So also the *cotangent* of an angle is the tangent of the *complement* of the angle. Thus HF is the cotangent of GCA. And the *cosecant* of an angle is the secant of the *complement* of the angle. Thus CF is the cosecant of GCA.

Hence, as in a right angled triangle, one of the acute angles is the complement of the other; (Art. 77.) the sine, tangent, and secant of one of these angles, are the cosine, cotangent, and cosecant of the other.

90. The sine, tangent, and secant of the *supplement* of an angle, are each equal to the sine, tangent, and secant of the angle itself. It will be seen, by applying the definition (Art. 82.) to the figure, that the sine of the obtuse angle GCM is BG, which is also the sine of the acute angle GCA. It should be observed, however, that the sine of an acute angle is *opposite* to it; while the sine of an obtuse angle *falls without* the angle, and is opposite to its supplement. Thus BG, the sine of the angle MCG, is not opposite to MCG, but to its supplement ACG.

The *tangent* of the obtuse angle MCG is MT, or its equal AD, which is also the tangent of ACG. And the *secant* of MCG is CD, which is also the secant of ACG.

91. But the *versed sine* of an angle *is not the same*, as that of its *supplement*. The versed sine of an *acute* angle is equal to the *difference* between the cosine and radius. But the versed sine of an *obtuse* angle is equal to the *sum* of the cosine and radius. Thus the versed sine of ACG is AB=AC−BC. (Art. 83.) But the versed sine of MCG is MB=MC+BC.

*Relations of Sines, Tangents, Secants, &c. to each other.*

92. The relations of the sine, tangent, secant, cosine, &c. to each other, are easily derived from the proportions of the sides of similar triangles. (Euc. 4. 6.) In the quadrant ACH, (Fig. 3.) these lines form three similar triangles, viz. ACD, BCG or LCG, and HCF. For, in each of these, there is one

right angle, because the sines and tangents are, by definition, perpendicular to AC; as the cosine and cotangent are to CH. The lines CH, BG, and AD are parallel, because CA makes a right angle with each. (Euc. 27. 1.) For the same reason, CA, LG, and HF are parallel. The alternate angles GCL, BGC, and the opposite angle CDA are equal; (Euc. 29. 1.) as are also the angles GCB, LGC, and HFC. The triangles ACD, BCG, and HCF are therefore similar.

It should also be observed, that the line BC, between the sine and the center of the circle, is parallel and equal to the cosine; and that LC, between the cosine and center, is parallel and equal to the sine; (Euc. 34. 1.) so that one may be taken for the other, in any calculation.

93. From these similar triangles, are derived the following proportions; in which R is put for radius,

| | |
|---|---|
| *sin* for sine, | *cos* for cosine, |
| *tan* for tangent, | *cot* for cotangent, |
| *sec* for secant, | *cosec* for cosecant. |

By comparing the triangles CBG and CAD,

1. AC : BC : : AD : BG, that is, R : cos : : tan : sin.
2. CG : CD : : BG : AD      R : sec : : sin : tan.
3. CB : CA : : CG : CD      cos : R : : R : sec.

Therefore $R^2 = \cos \times \sec$.

By comparing the triangles CLG and CHF,

4. CH : CL : : HF : LG, that is, R : sin : : cot : cos.
5. CG : CF : : LG : HF      R : cosec : : cos : cot.
6. CL : CH : : CG : CF      sin : R : : R : cosec.

Therefore $R^2 = \sin \times \text{cosec}$.

By comparing the triangles CAD and CHF,

7. CH : AD : : CF : CD, that is, R : tan : : cosec : sec.
8. CA : HF : : CD : CF      R : cot : : sec : cosec.
9. AD : AC : : CH : HF      tan : R : : R : cot.

Therefore $R^2 = \tan \times \cot$.

It will not be necessary for the learner to commit these proportions to memory. But he ought to make himself so familiar with the manner of stating them from the figure, as to be able to explain them, whenever they are referred to.

94. Other relations of the sine, tangent, &c. may be derived from the proposition, that the square of the hypothenuse is equal to the sum of the squares of the perpendicular sides. (Euc. 47. 1.)

In the right angled triangles CBG, CAD, and CHF, (Fig. 3.)

1. $\overline{CG}^2=\overline{CB}^2+\overline{BG}^2$, that is, $R^2=\cos^2+\sin^2$,*
2. $\overline{CD}^2=\overline{CA}^2+\overline{AD}^2$ $\qquad \sec^2=R^2+\tan^2$,
3. $\overline{CF}^2=\overline{CH}^2+\overline{HF}^2$ $\qquad \text{cosec}^2=R^2+\cot^2$.

And, extracting the root of both sides, (Alg. 296.)

$$R=\sqrt{\cos^2+\sin^2}=\sqrt{\sec^2-\tan^2}=\sqrt{\text{cosec}^2-\cot^2}$$

Hence, if R=1, (Alg. 510.)

$$\text{Sin}=\sqrt{1-\cos^2} \qquad \text{Sec}=\sqrt{1+\tan^2}$$
$$\text{Cos}=\sqrt{1-\sin^2} \qquad \text{Cosec}=\sqrt{1+\cot^2}$$

95. *The sine of* 90°, *The chord of* 60°, *And the tangent of* 45° } *are*, in any circle, *each equal to the radius*, and therefore equal to each other.

*Demonstration.*

1. In the quadrant ACH, (Fig. 5.) the arc AH is 90°. The sine of this, according to the definition, (Art. 82.) is CH, the radius of the circle.

2. Let AS be an arc of 60°. Then the angle ACS, being measured by this arc, will also contain 60°; (Art. 75.) and the triangle ACS will be equilateral. For the sum of the three angles is equal to 180°. (Art. 76.) From this, taking the angle ACS, which is 60°, the sum of the remaining two is 120°. But these two are *equal*, because they are subtended by the equal sides, CA and CS, both radii of the circle. Each, therefore, is equal to *half* 120°, that is to 60°.

---

* Sine² is here put for the square of the sine, cos² for the square of the cosine, &c.

All the angles being equal, the sides are equal, and therefore AS, the chord of 60°, is equal to CS the radius.

3. Let AR be an arc of 45°. AD will be its tangent, and the angle ACD subtended by the arc, will contain 45°. The angle CAD is a right angle, because the tangent is, by definition, perpendicular to the radius AC. (Art. 84.) Subtracting ACD, which is 45°, from 90°, (Art. 77.) the other acute angle ADC will be 45° also. Therefore the two legs of the triangle ACD are equal, because they are subtended by equal angles; (Euc. 6. 1.) that is, AD the tangent of 45°, is equal to AC the radius.

Cor. The *cotangent* of 45° is also equal to radius. For the complement of 45° is itself 45°. Thus HD, the cotangent of ACD, (Fig. 5.) is equal to AC the radius.

96. The sine of 30° is equal to *half radius.* For the sine of 30° is equal to half the chord of 60°. (Art. 82. cor.) But by the preceding article, the chord of 60° is equal to radius. Its half, therefore, which is the sine of 30°, is equal to half radius.

Cor. 1. The *cosine* of 60° is equal to half radius. For the cosine of 60° is the sine of 30°. (Art. 89.)

Cor. 2. The cosine of $30° = \frac{1}{2}\sqrt{3}$. For

$$\mathrm{Cos}^2 30° = R^2 - \sin^2 30° = 1 - \tfrac{1}{4} = \tfrac{3}{4}.$$

Therefore,

$$\mathrm{Cos}\ 30° = \sqrt{\tfrac{3}{4}} = \tfrac{1}{2}\sqrt{3}.$$

96. *b.* The sine of $45° = \frac{1}{\sqrt{2}}$. For

$$R^2 = 1 = \sin^2 45° + \cos^2 45° = 2 \sin^2 45°$$

Therefore, $\mathrm{Sin}\ 45° = \sqrt{\tfrac{1}{2}} = \frac{1}{\sqrt{2}}.$

97. The *chord* of any arc is a *mean proportional,* between the *diameter* of the circle, and the *versed sine* of the arc.

Let ADB (Fig. 6.) be an arc, of which AB is the chord, BF the sine, and AF the versed sine. The angle ABH is a right angle, (Euc. 31. 3.) and the triangles ABH and ABF are similar. (Euc. 8. 6.) Therefore,

$$AH : AB :: AB : AF.$$

Turn forward to page 57, Mensuration, [illegible] ume, where will be found the eight pages [illegible] mistake the binder has transposed them.



### PROBLEM XII.

*To find the* SOLIDITY *of a spherical* ZONE *or frustum.*

76. FROM THE SOLIDITY OF THE WHOLE SPHERE, SUBTRACT THE TWO SEGMENTS ON THE SIDES OF THE ZONE.

Or,

*Add together the squares of the radii of the two ends, and $\frac{1}{3}$ the square of their distance; and multiply the sum by three times this distance, and the product by* .5236.

If from the whole sphere, (Fig. 15.) there be taken the two segments ABP and GHO, there will remain the zone or frustum ABGH.

Or, the zone ABGH is equal to the difference between the segments GHP and ABP.

Let $\left.\begin{array}{l}NP=H\\ DP=h,\end{array}\right\}$ the *heights* of the two segments.

$\left.\begin{array}{l}GN=R\\ AD=r\end{array}\right\}$ the *radii* of their bases.

$DN=d=H-h$ the *distance* of the two bases, or the height of the zone.

$$\left.\begin{array}{l}\text{Then the larger segment}=\frac{1}{2}\pi HR^2+\frac{1}{6}\pi H^3\\ \text{And the smaller segment}=\frac{1}{2}\pi hr^2+\frac{1}{6}\pi h^3\end{array}\right\}\text{(Art. 75.)}$$

Therefore the zone $ABGH=\frac{1}{6}\pi(3HR^2+H^3-3hr^2-h^3)$

By the properties of the circle, (Euc. 35, 3.)

$ON\times H=R^2$. Therefore $(ON+H)\times H=R^2+H^2$.

$$\text{Or } OP=\frac{R^2+H^2}{H}$$

In the same manner, $OP=\dfrac{r^2+h^2}{h}$

Therefore $3H\times(r^2+h^2)=3h\times(R^2+H^2)$.

Or $3Hr^2+3Hh^2-3hR^2-3hH^2=0$. (Alg. 178.)

To reduce the expression for the solidity of the zone to the required form, without altering its value, let these terms be added to it: and it will become

$$\tfrac{1}{6}\pi(3HR^2+3Hr^2-3hR^2-3hr^2+H^3-3H^2h+3Hh^2-h^3)$$

Which is equal to

$$\tfrac{1}{6}\pi\times3(H-h)\times(R^2+r^2+\tfrac{1}{3}(H-h)^2)$$

Or, as $\frac{1}{6}\pi$ equals .5236 (Art. 71.) and $H-h$ equals $d$,

$$\text{The zone}=.5236\times3d\times(R^2+r^2+\tfrac{1}{3}d^2)$$

Ex. 1. If the diameter of one end of a spherical zone is 24 feet, the diameter of the other end 20 feet, and the distance of the two ends, or the height of the zone 4 feet; what is the solidity? Ans. 1566.6 feet.

2. If the earth be a sphere 7930 miles in diameter, and the obliquity of the ecliptic 23° 28′; what is the solidity of one of the temperate zones?

Ans. 55,390,500,000 miles.

3. What is the solidity of the torrid zone?

Ans. 147,720,000,000 miles.

| | |
|---|---|
| The solidity of the two temperate zones is | 110,781,000,000 |
| of the two frigid zones | 2,606,000,000 |
| of the torrid zone | 147,720,000,000 |
| of the whole globe | 261,107,000,000 |

4. What is the convex surface of a spherical zone, whose breadth is 4 feet, on a sphere of 25 feet diameter?

5. What is the solidity of a spherical segment, whose height is 18 feet, and the diameter of its base 40 feet?

## PROMISCUOUS EXAMPLES OF SOLIDS.

Ex. 1. How much water can be put into a cubical vessel three feet deep, which has been previously filled with cannon balls of the same size, 2, 4, 6, or 9 inches in diameter, regularly arranged in tiers, one directly above another?

Ans. $96\frac{1}{4}$ wine gallons.

2. If a cone or pyramid, whose height is three feet, be divided into three equal portions, by sections parallel to the base; what will be the heights of the several parts?

Ans. 24.961, 6.488, and 4.551 inches.

3. What is the solidity of the greatest square prism which can be cut from a cylindrical stick of timber, 2 feet 6 inches in diameter and 56 feet long?*

Ans. 175 cubic feet.

4. How many such globes as the earth are equal in bulk to the sun; if the former is 7930 miles in diameter, and the latter 890,000? Ans. 1,413,678.

5. How many cubic feet of wall are there in a conical tower 66 feet high, if the diameter of the base be 20 feet from outside to outside, and the diameter of the top 8 feet; the thickness of the wall being 4 feet at the bottom, and decreasing regularly, so as to be only 2 feet at the top?

Ans. 7188.

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* The common rule for measuring *round timber* is to multiply the square of the *quarter-girt* by the length. The quarter-girt is one fourth of the circumference. This method does not give the whole solidity. It makes an allowance of about one-fifth, for waste in hewing, bark, &c. The solidity of a cylinder is equal to the product of the length into the area of the base.

If C=the circumference, and $\pi$=3.14159, then (Art. 31.)

$$\text{The area of the base} = \frac{C^2}{4\pi} = \left(\frac{C}{\sqrt{4\pi}}\right)^2 = \left(\frac{C}{3.545}\right)^2.$$

If then the circumference were divided by 3.545, instead of 4, and the quotient squared, the area of the base would be correctly found. See note G.

6. If a metallic globe is filled with wine, which cost as much at 5 dollars a gallon, as the globe itself at 20 cents for every square inch of its surface; what is the diameter of the globe? Ans. 55.44 inches.

7. If the circumference of the earth be 25,000 miles, what must be the diameter of a metallic globe, which, when drawn into a wire $\frac{1}{20}$ of an inch in diameter, would reach round the earth? Ans. 15 feet and 1 inch.

8. If a conical cistern be 3 feet deep, $7\frac{1}{2}$ feet in diameter at the bottom, and 5 feet at the top; what will be the depth of a fluid occupying half its capacity?

Ans. 14.535 inches.

9. If a globe 20 inches in diameter be perforated by a cylinder 16 inches in diameter, the axis of the latter passing through the center of the former; what part of the solidity, and the surface of the globe will be cut away by the cylinder?

Ans. 3284 inches of the solidity, and 502,655 of the surface.

10. What is the solidity of the greatest cube which can be cut from a sphere three feet in diameter?

Ans. $5\frac{1}{5}$ feet.

11. What is the solidity of a conic frustum, the altitude of which is 36 feet, the greater diameter 16, and the lesser diameter 8?

12. What is the solidity of a spherical segment 4 feet high, cut from a sphere 16 feet in diameter?

## SECTION V.

### ISOPERIMETRY.*

Art. 77. It is often necessary to compare a number of different figures or solids, for the purpose of ascertaining which has the *greatest area*, within a given perimeter, or the *greatest capacity* under a given surface. We may have occasion to determine, for instance, what must be the form of a fort, to contain a given number of troops, with the least extent of wall; or what the shape of a metallic pipe to convey a given portion of water, or of a cistern to hold a given quantity of liquor, with the least expense of materials.

78. Figures which have equal perimeters are called *Isoperimeters*. When a quantity is *greater* than any other of the same class, it is called a *maximum*. A multitude of straight lines, of different lengths, may be drawn within a circle. But among them all, the *diameter* is a *maximum*. Of all *sines* of angles, which can be drawn in a circle, the sine of 90° is a *maximum*.

When a quantity is *less* than any other of the same class, it is called a *minimum*. Thus, of all straight lines drawn from a given point to a given straight line, that which is *perpendicular* to the given line is a *minimum*. Of all straight lines drawn from a given point in a circle, to the circumference, the *maximum* and *minimum* are the two parts of the diameter which pass through that point. (Euc. 7, 3.)

In isoperimetry, the object is to determine, on the one hand, in what cases the area is a *maximum*, within a given perimeter; or the capacity a *maximum*, within a given surface: and on the other hand, in what cases the perimeter is a *minimum* for a given area, or the surface a *minimum*, for a given capacity.

* Emerson's, Simpson's, and Legendre's Geometry, Lhuillier, Fontenelle, Hutton's Mathematics, and Lond. Phil. Trans. Vol. 75.

PROPOSITION I.

79. *An* Isosceles Triangle *has a greater area than any scalene triangle, of equal base and perimeter.*

If ABC (Fig. 26.) be an isosceles triangle whose equal sides are AC and BC; and if ABC′ be a scalene triangle on the same base AB, and having AC′+BC′=AC+BC; then the area of ABC is greater than that of ABC′.

Let perpendiculars be raised from each end of the base, extend AC to D, make C′D′ equal to AC′, join BD, and draw CH and C′H′ parallel to AB.

As the angle CAB=ABC, (Euc. 5, 1.) and ABD is a right angle, ABC+CBD=CAB+CDB=ABC+CDB. Therefore CBD=CDB, so that CD=CB; and by construction, C′D′=AC′. The perpendiculars of the equal right angled triangles CHD and CHB are equal; therefore, BH=½BD. In the same manner, AH′=½AD′. The line AD=AC+BC =AC′+BC′=D′C′+BC′. But D′C′+BC′>BD′. (Euc. 20, 1.) Therefore, AD>BD′; BD>AD′, (Euc. 47, 1.) and ½BD>½AD′. But ½BD, or BH, is the height of the isosceles triangle; (Art. 1.) and ½AD′ or AH′, the height of the scalene triangle; and the areas of two triangles which have the same base are as their heights. (Art. 8.) Therefore the area of ABC is greater than that of ABC′. Among all triangles, then, of a given perimeter, and upon a given base, the isosceles triangle is a *maximum.*

Cor. The isosceles triangle has a *less perimeter* than any scalene triangle of the same base and area. The triangle ABC′ being less than ABC, it is evident the perimeter of the former must be enlarged, to make its area equal to the area of the latter.

PROPOSITION II.

80. *A triangle in which two given sides make a* right angle, *has a greater area than any triangle in which the same sides make an oblique angle.*

If BC, BC′, and BC″ (Fig. 27.) be equal, and if BC be perpendicular to AB; then the right angled triangle ABC, has a greater area than the acute angled triangle ABC′, or the oblique angled triangle ABC″.

Let P′C′ and PC″ be perpendicular to AP. Then, as the three triangles have the same base AB, their areas are as their heights; that is, as the perpendiculars BC, P′C′, and PC″. But BC is equal to BC′, and therefore greater than P′C′. (Euc. 47, 1.) BC is also equal to BC″, and therefore greater than PC″.

PROPOSITION III.

81. *If all the sides* EXCEPT ONE *of a polygon be given, the area will be the greatest, when the given sides are so disposed, that the figure may be* INSCRIBED IN A SEMICIRCLE, *of which the undetermined side is the diameter.*

If the sides AB, BC, CD, DE, (Fig. 28.) be given, and if their position be such that the area, included between these and another side whose length is not determined, is a *maximum;* the figure may be inscribed in a semicircle, of which the undetermined side AE is the diameter.

Draw the lines AD, AC, EB, EC. By varying the angle at D, the triangle ADE may be enlarged or diminished, without affecting the area of the other parts of the figure. The whole area, therefore, cannot be a *maximum*, unless this triangle be a *maximum*, while the sides AD and ED are given. But if the triangle ADE be a *maximum*, under these conditions, the angle ADE is a right angle; (Art. 80.) and therefore the point D is in the circumference of a circle, of which AE is the diameter. (Euc. 31, 3.) In the same manner it may be proved, that the angles ACE and ABE are right angles, and therefore that the points C and B are in the circumference of the same circle.

The term *polygon* is used in this section to include *triangles*, and *four-sided* figures, as well as other right-lined figures.

82. The area of a polygon, inscribed in a semicircle, in the manner stated above, will not be altered by varying the *order* of the given sides.

The sides AB, BC, CD, DE, (Fig. 28.) are the *chords* of so many arcs. The sum of these arcs, in whatever order they are arranged, will evidently be equal to the semicircumference. And the *segments* between the given sides and the arcs will be the same, in whatever part of the circle they are situated. But the area of the polygon is equal to the area of the semicircle, diminished by the sum of these segments.

83. If a polygon, of which all the sides except one are given, be inscribed in a semicircle whose diameter is the undetermined side; a polygon having the same given sides, cannot be inscribed in any *other* semicircle which is either greater or less than this, and whose diameter is the undetermined side.

The given sides AB, BC, CD, DE, (Fig. 28.) are the chords of arcs whose sum is 180 degrees. But in a larger circle, each would be the chord of a less number of degrees, and therefore the sum of the arcs would be less than 180°: and in a smaller circle, each would be the chord of a greater number of degrees, and the sum of the arcs would be greater than 180°.

PROPOSITION IV.

84. *A polygon* INSCRIBED IN A CIRCLE *has a greater area, than any polygon of equal perimeter, and the same number of sides, which cannot be inscribed in a circle.*

If in the circle ACHF, (Fig. 30.) there be inscribed a polygon ABCDEFG; and if another polygon *abcdefg* (Fig. 31.) be formed of sides which are the same in number and length, but which are so disposed, that the figure cannot be inscri-ɔed in a circle; the area of the former polygon is greater han that of the latter.

Draw the diameter AH, and the chords DH and EH. Jpon *de* make the triangle *deh* equal and similar to DEH, nd join *ah*. The line *ah* divides the figure *abcdhefg* into two arts, of which *one at least* cannot, by supposition, be inscri-ɜd in a semicircle of which the diameter is AH, nor in any her semicircle of which the diameter is the undetermined le. (Art. 83.) It is therefore less than the corresponding rt of the figure ABCDHEFG. (Art. 81.) And the other rt of *abcdhefg* is not greater than the corresponding part of ƷCDHEFG. Therefore the whole figure ABCDHEFG is ꞉ater than the whole figure *abcdhefg*. If from these there taken the equal triangles DEH and *deh*, there will remain polygon ABCDEFG greater than the polygon *abcdefg*.

complement of the sine and cosine, in the following simple manner:

113. For the arithmetical complement of the *sine*, subtract 10 from the index of the cosecant; and for the arithmetical complement of the *cosine*, subtract 10 from the index of the secant.

By this, we may save the trouble of taking each of the figures from 9.

## SECTION III.

### SOLUTIONS OF RIGHT ANGLED TRIANGLES.

Art. 114. In a triangle, there are *six parts*, three sides, and three angles. In every trigonometrical calculation, it is necessary that some of these should be known, to enable us to find the others. *The number of parts which must be given, is* THREE, *one of which must be a* SIDE.

If only two parts be given, they will be either two sides, a side and an angle, or two angles; neither of which will limit the triangle to a particular form and size.

If *two sides* only be given, they may make any angle with each other; and may, therefore, be the sides of a thousand different triangles. Thus the two lines *a* and *b* (Fig. 7.) may belong either to the triangle ABC, or ABC′, or ABC″. So that it will be impossible, from knowing two of the sides of a triangle, to determine the other parts.

Or, if a *side and an angle* only be given, the triangle will be indeterminate. Thus, if the side AB (Fig. 8.) and the angle at A be given; they may be parts either of the triangle ABC, or ABC′, or ABC″.

Lastly, if two *angles*, or even if *all* the angles be given, they will not determine the length of the sides. For the triangles ABC, A′B′C′, A″B″C″, (Fig. 9.) and a hundred others which might be drawn, with sides parallel to these, will all have the same angles. So that one of the parts given must always be a side. If this and any other two parts, either sides or angles, be known, the other three may be found, as will be shown, in this and the following section.

115. Triangles are either *right angled* or *oblique angled.* The calculations of the former are the most simple, and those which we have the most frequent occasion to make. A great portion of the problems in the mensuration of heights and distances, in surveying, navigation and astronomy, are solved by rectangular trigonometry. Any triangle whatever may be divided into two right angled triangles, by drawing a perpendicular from one of the angles to the opposite side.

116. One of the six parts in a right angled triangle, is always given, viz. the right angle. This is a *constant* quantity; while the other angles and the sides are variable. It is also to be observed, that, if one of the *acute* angles is given, the other is known of course. For one is the complement of the other. (Art. 76, 77.) So that, *in a right angled triangle, subtracting one of the acute angles from* 90° *gives the other*: There remain, then, only *four* parts, one of the acute angles, and the three sides to be sought by calculation. If any *two* of these be given, with the right angle, the others may be found.

117. To illustrate the method of calculation, let a case be supposed in which a right angled triangle CAD (Fig. 10.) has one of its sides equal to the radius to which the trigonometrical tables are adapted.

In the first place, let the *base* of the triangle be equal to the tabular radius. Then, if a circle be described, with this radius, about the angle C as a center, DA will be the *tangent*, and DC the *secant* of that angle. (Art. 84, 85.) So that the radius, the tangent, and the secant of the angle at C, constitute the three sides of the triangle. The *tangent*, taken from the tables of natural sines, tangents, &c. will be the length of the *perpendicular;* and the *secant* will be the length of the *hypothenuse.* If the tables used be logarithmic, they will give the *logarithms* of the lengths of the two sides.

In the same manner, *any* right angled triangle whatever, whose base is equal to the radius of the tables, will have its other two sides found among the tangents and secants. Thus, if the quadrant AH (Fig. 11.) be divided into portions of 15° each; then, in the triangle

CAD, AD will be the tan, and CD the sec of 15°,
In CAD′, AD′ will be the tan, and CD′ the sec of 30°,
In CAD″, AD″ will be the tan, and CD″ the sec of 45°, &c.

118. In the next place, let the *hypothenuse* of a right angled triangle CBF (Fig. 12.) be equal to the radius of the tables. Then, if a circle be described, with the given radius, and about the angle C as a center; BF will be the *sine*, and BC the *cosine* of that angle. (Art. 82. 89.) Therefore the sine of the angle at C, taken from the tables, will be the length

of the *perpendicular*, and the cosine will be the length of the *base*.

And any right angled triangle whatever, whose hypothenuse is equal to the tabular radius, will have its other two sides found among the sines and cosines. Thus if the quadrant AH (Fig. 13.) be divided into portions of 15° each, in the points F, F′, F″, &c.; then, in the triangle,

CBF, FB will be the sin, and CB the cos, of 15°,
In CB′F′, F′B′ will be the sin, and CB′ the cos, of 30°,
In CB″F″, F″B″ will be the sin, and CB″ the cos, of 45°, &c.

119. By merely *turning to the tables*, then, we may find the parts of any right angled triangle which has one of its sides equal to the radius of the tables. But for determining the parts of triangles which have *not* any of their sides equal to the tabular radius, the following proportion is used:

*As the radius of one circle,*
*To the radius of any other;*
*So is a sine, tangent, or secant, in one,*
*To the sine, tangent, or secant, of the same number of degrees, in the other.*

In the two concentric circles AHM, *ahm*, (Fig. 4.) the arcs AG and *ag* contain the same number of degrees. (Art. 74.) The sines of these arcs are BG and *bg*, the tangents AD and *ad*, and the secants CD and C*d*. The four triangles, CAD, CBG, C*ad*, and C*bg*, are similar. For each of them, from the nature of sines and tangents, contains one right angle; the angle at C is common to them all; and the other acute angle in each is the complement of that at C. (Art. 77.) We have, then, the following proportions. (Euc. 4. 6.)

1. $CG : Cg :: BG : bg$.

That is, one radius is to the other, as one *sine* to the other.

2. $CA : Ca :: DA : da$.

That is, one radius is to the other, as one *tangent* to the other.

3. $CA : Ca :: CD : Cd$.

That is, one radius is to the other, as one *secant* to the other.

Cor. $BG : bg :: DA : da :: CD : Cd$.

That is, as the sine in one circle, to the sine in the other; so is the tangent in one, to the tangent in the other; and so is the secant in one, to the secant in the other.

This is a general principle, which may be applied to most trigonometrical calculations. If one of the sides of the proposed triangle be made radius, each of the other sides will be the sine, tangent, or secant, of an arc described by this radius. Proportions are then stated, between these lines, and the *tabular* radius, sine, tangent, &c.

120. A line is said to be *made radius*, when a circle is described, or supposed to be described, whose semi-diameter is equal to the line, and whose center is at one end of it.

121. In any right angled triangle, *if the* HYPOTHENUSE *be made radius, one of the legs will be a* SINE *of its opposite angle, and the other leg a* COSINE *of the same angle.*

Thus, if to the triangle ABC (Fig. 14.) a circle be applied, whose radius is AC, and whose center is A, then BC will be the *sine*, and BA the *cosine*, of the angle at A. (Art. 82, 89.)

If, while the same line is radius, the other end C be made the center, then BA will be the *sine*, and BC the *cosine*, of the angle at C.

122. *If either of the* LEGS *be made radius, the other leg will be a* TANGENT *of its opposite angle, and the hypothenuse will be a* SECANT *of the same angle;* that is, of the angle between the secant and the radius.

Thus, if the *base* AB (Fig. 15.) be made radius, the center being at A, BC will be the *tangent*, and AC the *secant*, of the angle at A. (Art. 84, 85.)

But, if the *perpendicular* BC (Fig. 16.) be made radius, with the center at C, then AB will be the *tangent*, and AC the *secant*, of the angle at C.

123. As the side which is the sine, tangent, or secant of one of the acute angles, is the cosine, cotangent, or cosecant of the other; (Art. 89.) the *perpendicular* BC (Fig. 14.) is the *sine* of the angle A, and the *cosine* of the angle C; while the *base* AB is the *sine* of the angle C, and the *cosine* of the angle A.

If the base is made radius, as in Fig. 15, the *perpendicular* BC is the *tangent* of the angle A, and the *cotangent* of the angle C; while the *hypothenuse* is the *secant* of the angle A, and the *cosecant* of the angle C.

If the perpendicular is made radius, as in Fig. 16, the *base* AB is the *tangent* of the angle C, and the *cotangent* of the angle A; while the *hypothenuse* is the *secant* of the angle C, and the *cosecant* of the angle A.

124. Whenever a right angled triangle is proposed, whose sides or angles are required; a *similar* triangle may be formed, from the sines, tangents, &c. of the *tables*. (Art. 117, 118.) The parts required are then found, by stating proportions between the similar sides of the two triangles. If the triangle proposed be ABC, (Fig. 17.) another, *abc*, may be formed, having the same angles with the first, but differing from it in the length of its sides, so as to correspond with the numbers in the tables. If similar sides be made radius in both, the remaining similar sides will be lines of *the same name;* that is, if the perpendicular in one of the triangles be a *sine*, the perpendicular in the other will be a sine; if the base in one be a *cosine*, the base in the other will be a cosine, &c.

If the *hypothenuse* in each triangle be made radius, as in Fig. 14, the perpendicular *bc* will be the *tabular sine* of the angle at *a;* and the perpendicular BC will be a sine of the equal angle A, in a circle of which AC is radius.

If the *base* in each triangle be made radius, as in Fig. 15, then the perpendicular *bc* will be the *tabular tangent* of the angle at *a;* and BC will be a tangent of the equal angle A, in a circle of which AB is radius, &c.

125. From the relations of the similar sides of these triangles, are derived the two following *theorems*, which are sufficient for calculating the parts of any right angled triangle whatever, when the requisite data are furnished. One is used, when a *side* is to be found; the other, when an *angle* is to be found.

### Theorem I.

126. When a *side* is required;

AS THE TABULAR SINE, TANGENT, &C. OF THE SAME NAME WITH THE GIVEN SIDE,
TO THE GIVEN SIDE;
SO IS THE TABULAR SINE, TANGENT, &C. OF THE SAME NAME WITH THE REQUIRED SIDE,
TO THE REQUIRED SIDE.

It will be readily seen, that this is nothing more than a statement, in general terms, of the proportions between the similar sides of two triangles, one proposed for solution, and the other formed from the numbers in the tables.

Thus if the hypothenuse be *given*, and the base or perpendicular be *required*; then, in Fig. 14, where *ac* is the tabular radius, *bc* the tabular sine of *a*, or its equal A, and *ab* the tabular sine of C; (Art. 124.)

*ac* : AC*b*::*c* : BC, that is, R : AC::sin A : BC.
*ac* : AC::*ab* : AB, R : AC::sin C : AB.

In Fig. 15, where *ab* is the tabular radius, *ac* the tabular secant of A, and *bc* the tabular tangent of A;

*ac* : AC::*bc* : BC, that is, sec A : AC::tan A : BC.
*ac* : AC::*ab* : AB, sec A : AC::R : AB.

In Fig. 16, where *bc* is the tabular radius, *ac* the tabular secant of C, and *ab* the tabular tangent of C;

*ac* : AC::*bc* : BC, that is, sec C : AC::R : BC.
*ac* : AC::*ab* : AB, sec C : AC::tan C : AB.

### Theorem II.

127. When an *angle* is required;

As the given side made radius,
To the tabular radius;
So is another given side,
To the tabular sine, tangent, &c. of the same name.

Thus if the side made radius, and one other side be given, then, in Fig. 14,

AC : *ac*::BC : *bc*, that is, AC : R::BC : sin A.
AC : *ac*::AB : *ab* AC : R::AB : sin C.

In Fig. 15,

AB : *ab*::BC : *bc*, that is, AB : R::BC : tan A.
AB : *ab*::AC : *ac* AB : R::AC : sec A.

In Fig. 16,

BC : *bc*::AB : *ab*, that is, BC : R::AB : tan C.
BC : *bc*::AC : *ac* BC : R::AC : sec C.

It will be observed, that in these theorems, *angles* are not introduced, though they are among the quantities which are either given or required, in the calculation of triangles. But the tabular sines, tangents, &c. may be considered the *representatives* of angles, as one may be found from the other, by merely turning to the tables.

128. In the theorem for finding a *side*, the first term of the proportion is a *tabular number*. But, in the theorem for finding an *angle*, the first term is a side. Hence, in applying the proportions to particular cases, this rule is to be observed;

*To find a* SIDE, *begin with a tabular number,*
*To find an* ANGLE, *begin with a side.*

*Radius* is to be reckoned among the tabular numbers.

129. In the theorem for finding an *angle*, the first term is a *side made radius*. As in every proportion, the three first terms must be given, to enable us to find the fourth, it is evident, that where this theorem is applied, the side made radius must be a *given* one. But, in the theorem for finding a *side*, it is not necessary that either of the terms should be radius. Hence,

130. *To find a* SIDE, ANY *side may be made radius.*
*To find an* ANGLE, *a* GIVEN *side must be made radius.*

It will generally be expedient, in both cases, to make radius one of the terms in the proportion; because, in the tables of natural sines, tangents, &c. radius is 1, and in the logarithmic tables it is 10. (Art. 103.)

131. The proportions in Trigonometry are of the same nature as other simple proportions. The fourth term is found, therefore, as in the Rule of Three in Arithmetic, by *multiplying together the second and third terms, and dividing their product by the first term.* This is the mode of calculation, when the tables of *natural* sines, tangents, &c. are used. But the operation by logarithms is so much more expeditious, that it has almost entirely superseded the other method. In logarithmic calculations, addition takes the place of multiplication; and subtraction the place of division.

The logarithms expressing the lengths of the *sides* of a triangle, are to be taken from the tables of common logarithms. The logarithms of the *sines*, *tangents*, &c. are found in the tables of artificial sines, &c. The calculation is then made *by* *adding the second and third terms, and subtracting the first.* (Art. 52.)

132. The logarithmic radius 10, or, as it is written in the tables, 10.00000, is so easily added and subtracted, that the three terms of which it is one, may be considered as, in effect, reduced to two. Thus, if the tabular radius is in the *first* term, we have only to add the other two terms, and then take 10 from the index; for this is subtracting the first term. If radius occurs in the *second* term, the first is to be subtracted from the third, after its index is increased by 10. In the same manner, if radius is in the *third* term, the first is to be subtracted from the second.

133. Every species of right angled triangles may be solved upon the principle, that the sides of similar triangles are proportional, according to the two theorems mentioned above. There will be some advantages, however, in giving the examples in distinct classes.

There must be given, in a right angled triangle, *two* of the parts, besides the right angle. (Art. 116.) These may be;

1. The hypothenuse and an angle; or
2. The hypothenuse and a leg; or
3. A leg and an angle; or
4. The two legs.

## Case I.

134. Given { The hypothenuse, And an angle, } to find { The base and Perpendicular.

Ex. 1. If the hypothenuse AC (Fig. 17.*) be 45 miles, and the angle at A 32° 20′, what is the length of the base AB, and the perpendicular BC?

In this case, as *sides* only are required, *any* side may be made radius. (Art. 130.)

If the hypothenuse be made radius, as in Fig. 14, BC will be the sine of A, and AB the sine of C, or the cosine of A. (Art. 121.) And if *abc* be a similar triangle, whose hypothenuse is equal to the *tabular* radius, *bc* will be the tabular sine of A, and *ab* the tabular sine of C. (Art. 124.)

* The parts which are *given* are distinguished by a mark across the line, or at the opening of the angle, and the parts *required*, by a cipher.

To find the *perpendicular*, then, by Theorem I, we have this proportion;

$$ac : AC :: bc : BC.$$
$$\text{Or } R : AC :: \text{Sin } A : BC.$$

Whenever the terms Radius, Sine, Tangent, &c. occur in a proportion like this, the *tabular* Radius, &c. is to be understood, as in Arts. 126, 127.

The numerical calculation, to find the length of BC, may be made, either by *natural* sines, or by *logarithms*. See Art. 131.

*By natural Sines.*

$$1 : 45 :: 0.53484 : 24.068 = BC.$$

*Computation by Logarithms.*

| | | |
|---|---|---|
| As Radius | | 10.00000 |
| To the hypothenuse | 45 | 1.65321 |
| So is the Sine of A | 32° 20′ | 9.72823 |
| To the perpendicular | 24.068 | 1.38144 |

Here the logarithms of the second and third terms are added, and from the sum, the first term 10 is subtracted. (Art. 132.) The remainder is the logarithm of 24.068=BC.

Subtracting the angle at A from 90°, we have the angle at C=57° 40′. (Art. 116.) Then to find the *base* AB;

$$ac : AC :: ab : AB$$
$$\text{Or } R : AC :: \text{Sin } C : AB = 38.023.$$

Both the sides required are now found, by making the hypothenuse radius. The results here obtained may be verified, by making either of the other sides radius.

If the *base* be made radius, as in Fig. 15, the perpendicular will be the *tangent*, and the hypothenuse the *secant* of the angle at A. (Art. 122.) Then,

$$\text{Sec } A : AC :: R : AB$$
$$R : AB :: \text{Tan } A : BC$$

By making the arithmetical calculations, in these two proportions, the values of AB and BC will be found the same as before.

If the *perpendicular* be made radius, as in Fig. 16, AB will be the *tangent*, and AC the *secant* of the angle at C. Then,

Sec C : AC::R : BC
R : BC::Tan C : AB

Ex. 2. If the hypothenuse of a right angled triangle be 250 rods, and the angle at the base 46° 30′; what is the length of the base and perpendicular?

Ans. The base is 172.1 rods, and the perpendic. 181.35.

### Case II.

135. Given { The hypothenuse, And one leg } to find { The angles and The other leg.

Ex. 1. If the hypothenuse (Fig. 18.) be 35 leagues, and the base 26; what is the length of the perpendicular, and the quantity of each of the acute angles?

To find the angles it is necessary that one of the *given* sides be made radius. (Art. 130.)

If the *hypothenuse* be radius, the base and perpendicular will be sines of their opposite angles. Then,

AC : R::AB : Sin C=47° 58′$\frac{1}{2}$

And to find the *perpendicular* by Theorem I;

R : AC::Sin A : BC=23.43

If the *base* be radius, the perpendicular will be *tangent*, and the hypothenuse *secant* of the angle at A. Then,

AB : R::AC : Sec A
R : AB::Tan A : BC

In this example, where the hypothenuse and base are given, the angles can not be found by making the *perpendicular* radius. For to find an angle, a *given* side must be made radius. (Art. 130.)

136. Ex. 2. If the hypothenuse (Fig. 19.) be 54 miles, and the perpendicular 48 miles, what are the angles, and the base?

Making the *hypothenuse* radius.

AC : R : : BC : Sin A
R : AC : : Sin C : AB

The numerical calculation will give A=62° 44′ and AB =24.74.

Making the *perpendicular* radius.

BC : R : : AC : Sec C
R : BC : : Tan C : AB

The angles cannot be found by making the *base* radius, when its length is not given.

CASE III.

137. Given { The angles, And one leg } to find { The hypothenuse, And the other leg.

Ex. 1. If the base (Fig. 20.) be 60, and the angle at the base 47° 12′, what is the length of the hypothenuse and the perpendicular?

In this case, as *sides* only are required, *any* side may be radius.

Making the *hypothenuse* radius.

Sin C : AB : : R : AC=88.31
R : AC : : Sin A : BC=64.8

Making the *base* radius.

R : AB : : Sec A : AC
R : AB : : Tan A : BC

Making the *perpendicular* radius.

Tan C : AB : : R : BC
R : BC : : Sec C : AC

138. Ex. 2. If the perpendicular (Fig. 21.) be 74, and the angle C 61° 27′, what is the length of the base and the hypothenuse?

Making the *hypothenuse* radius.

Sin A : BC : : R : AC
R : AC : : sin C : AB

Making the *base* radius.

Tan A : BC : : R : AB
R : AB : : sec A : AC

Making the *perpendicular* radius.

R : BC : : sec C : AC
R : BC : : tan C : AB

The hypothenuse is 154.83 and the base 136.

CASE IV.

139. Given { The base, and Perpendicular } to find { The hypothenuse, And the angles.

Ex. 1. If the base (Fig. 22.) be 284, and the perpendicular 192, what are the angles, and the hypothenuse?

In this case, one of the legs must be made radius, to find an angle; because the hypothenuse is not given.

Making the *base* radius.

AB : R : : BC : tan A=34° 4′
R : AB : : sec A : AC=342.84

Making the *perpendicular* radius.

BC : R : : AB : tan C
R : BC : : sec C : AC

Ex. 2. If the base be 640, and the perpendicular 480, what are the angles and hypothenuse?

Ans. The hypothenuse is 800, and the angle at the base 36° 52′ 12″.

*Examples for practice.*

1. Given the hypothenuse 68, and the angle at the base 39° 17′; to find the base and perpendicular.
2. Given the hypothenuse 850, and the base 594, to find the angles, and the perpendicular.
3. Given the hypothenuse 78, and perpendicular 57, to find the base, and the angles.
4. Given the base 723, and the angle at the base 64° 18′, to find the hypothenuse and perpendicular.
5. Given the perpendicular 632, and the angle at the base 81° 36′, to find the hypothenuse and the base.
6. Given the base 32, and the perpendicular 24, to find the hypothenuse, and the angles.

140. The preceding solutions are all effected, by means of the tabular sines, tangents, and secants. But, when any *two sides* of a right angled triangle are given, the third side may be found, without the aid of the trigonometrical tables, by the proposition, that *the square of the hypothenuse is equal to the sum of the squares of the two perpendicular sides.* (Euc. 47. 1.)

If the legs be given, extracting the square root of the *sum* of their squares, will give the hypothenuse. Or, if the hypothenuse and one leg be given, extracting the square root of the *difference* of the squares, will give the other leg.

Let $h$=the hypothenuse, $p$=the perpendicular, $b$=the base } of a right angled triangle.

Then $h^2=b^2+p^2$, or (Alg. 296.) $h=\sqrt{b^2+p^2}$

By trans. $b^2=h^2-p^2$, or $b=\sqrt{h^2-p^2}$

And $p^2=h^2-b^2$, or $p=\sqrt{h^2-b^2}$

Ex. 1. If the base is 32, and the perpendicular 24, what is the hypothenuse? Ans. 40.

2. If the hypothenuse is 100, and the base 80, what is the perpendicular? Ans. 60.

3. If the hypothenuse is 300, and the perpendicular 220, what is the base?

Ans. $\overline{300}^2-\overline{220}^2=41600$, the root of which is 204 nearly.

141. It is generally most convenient to find the difference of the squares by *logarithms.* But this is not to be done by *subtraction.* For subtraction, in logarithms, performs the office of *division.* (Art. 41.) If we subtract the logarithm of $b^2$ from the logarithm of $h^2$, we shall have the logarithm, not of the *difference* of the squares, but of their *quotient.* There is, however, an indirect, though very simple method, by which the difference of the squares may be obtained by logarithms. It depends on the principle, that the *difference of the squares of two quantities is equal to the product of the sum and difference of the quantities.* (Alg. 235.) Thus,

$$h^2 - b^2 = (h+b) \times (h-b)$$

as will be seen at once, by performing the multiplication. The two factors may be multiplied by *adding* their logarithms. Hence,

142. *To obtain the difference of the squares of two quantities, add the logarithm of the sum of the quantities, to the logarithm of their difference.* After the logarithm of the difference of the squares is found; the *square root* of this difference is obtained, by dividing the logarithm by 2. (Art. 47.)

Ex. 1. If the hypothenuse be 75 inches, and the base 45, what is the length of the perpendicular?

| | | |
|---|---|---|
| Sum of the given sides | 120 | log. 2.07918 |
| Difference of do. | 30 | 1.47712 |
| | Dividing by | 2)3.55630 |
| Side required | 60 | 1.77815 |

2. If the hypothenuse is 135, and the perpendicular 108, what is the length of the base? Ans. 81.

## SECTION IV.

### SOLUTIONS OF OBLIQUE ANGLED TRIANGLES.

ART. 143. THE sides and angles of oblique angled triangles may be calculated by the following theorems.

### THEOREM I.

In any plane triangle, THE SINES OF THE ANGLES ARE AS THEIR OPPOSITE SIDES.

Let the angles be denoted by the letters A, B, C, and their opposite sides by $a, b, c$, as in Fig. 23 and 24. From one of the angles, let the line $p$ be drawn perpendicular to the opposite side. This will fall either within or without the triangle.

1. Let it fall *within* as in Fig. 23. Then, in the right angled triangles ACD and BCD, according to art. 126,

$$R : b :: \sin A : p$$
$$R : a :: \sin B : p$$

Here, the two *extremes* are the same in both proportions. The other four terms are, therefore, *reciprocally* proportional: (Alg. 387.*) that is,

$$a : b :: \sin A : \sin B.$$

2. Let the perpendicular $p$ fall *without* the triangle, as in Fig. 24. Then, in the right angled triangles ACD and BCD;

$$R : b :: \sin A : p$$
$$R : a :: \sin B : p$$

Therefore as before,

$$a : b :: \sin A : \sin B.$$

* Euclid 23. 5.

Sin A is here put both for the sine of DAC, and for that of BAC. For, as one of these angles is the *supplement* of the other, they have the same sine. (Art. 90.)

The sines which are mentioned here, and which are used in calculation, are *tabular* sines. But the proportion will be the same, if the sines be adapted to any other radius. (Art. 119.)

### THEOREM II.

144. In a plane triangle,

AS THE SUM OF ANY TWO OF THE SIDES,
TO THEIR DIFFERENCE;
SO IS THE TANGENT OF HALF THE SUM OF THE OPPOSITE ANGLES;
TO THE TANGENT OF HALF THEIR DIFFERENCE.

Thus the sum of AB and AC (Fig. 25.) is to their difference; as the tangent of half the sum of the angles ACB and ABC, to the tangent of half their difference.

*Demonstration.*

Extend CA to G, making AG equal to AB; then CG is *the sum of the two sides* AB and AC. On AB set off AD equal to AC; then BD *is the difference of the sides* AB and AC.

The sum of the two angles ACB and ABC, is equal to the sum of ACD and ADC; because each of these sums is the supplement of CAD. (Art. 79.) But as AC=AD by construction, the angle ADC=ACD. (Euc. 5. 1.) Therefore ACD is *half the sum* of ACB and ABC. As AB=AG, the angle AGB=ABG or DBE. Also GCE or ACD=ADC=BDE. (Euc. 15. 1.) Therefore, in the triangles GCE and DBE, the two remaining angles DEB and CEG are equal; (Art. 79.) So that CE is perpendicular to BG. (Euc. Def. 10. 1.) If then CE is made radius, GE is the tangent of GCE. (Art. 84.) that is, *the tangent of half the sum of the angles opposite to* AB and AC.

If from the greater of the two angles ACB and ABC, there be taken ACD their half sum; the remaining angle ECB will be their half difference. (Alg. 341.) The tangent of this angle, CE being radius, is EB, that is, *the tangent of half the difference of the angles opposite to* AB and AC. We have then,

CG=the sum of the sides AB and AC;
DB=their difference;
GE=the tangent of half the sum of the opposite angles;
EB=the tangent of half their difference.

But by similar triangles,

CG : DB : GE : EB. Q. E. D.

### Theorem III.

145. If upon the longest side of a triangle, a perpendicular be drawn from the opposite angle;

As the longest side,
To the sum of the two others;
So is the difference of the latter,
To the difference of the segments made by the perpendicular.

In the triangle ABC, (Fig. 26.) if a perpendicular be drawn from C upon AB;

$$AB : CB+CA :: CB-CA : BP-PA.*$$

*Demonstration.*

Describe a circle on the center C, and with the radius BC. Through A and C, draw the diameter LD, and extend BA to H. Then by Euc. 35, 3,

$$AB\times AH=AL\times AD$$

Therefore,

$$AB : AD :: AL : AH$$

But $AD=CD+CA=CB+CA$
And $AL=CL-CA=CB-CA$
And $AH=HP-PA=BP-PA$ (Euc. 3. 3.)

If then, for the three last terms in the proportion, we substitute their equals, we have,

$$AB : CB+CA :: CB-CA : PB-PA.$$

146. It is to be observed, that the greater segment is next the greater side. If BC is greater than AC, (Fig. 26.) PB is

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* See note F.

greater than AP. With the radius AC, describe the arc AN. The segment NP=AP. (Euc. 3. 3.) But BP is greater than NP.

147. The two segments are to each other, as the tangents of the opposite angles, or the cotangents of the adjacent angles. For, in the right angled triangles ACP and BCP, (Fig. 26.) if CP be made radius, (Art. 126.)

R : PC : : Tan ACP : AP
R : PC : Tan BCP : BP

Therefore, by equality of ratios, (Alg. 384.*)
Tan ACP : AP : : Tan BCP : BP

That is, the segments are as the tangents of the opposite angles. And the tangents of these are the *cotangents* of the adjacent angles A and B. (Art. 89.)

Cor. The greater segment is opposite to the greater angle. And of the angles at the base, the less is next the greater side. If BP is greater than AP, the angle BCP is greater than ACP ; and B is less than A. (Art. 77.)

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148. To enable us to find the sides and angles of an oblique angled triangle, *three* of them must be *given*. (Art. 114.)

These may be, either

1. Two angles and a side, or
2. Two sides and an angle *opposite* one of them, or
3. Two sides and the *included* angle, or
4. The three sides.

The two first of these cases are solved by theorem I, (Art. 143.) the third by theorem II, (Art. 144.) and the fourth by theorem III, (Art. 145.)

149. In making the calculations, it must be kept in mind, that the greater side is always opposite to the greater angle, (Euc. 18, 19. 1.) that there can be only one *obtuse* angle in a triangle, (Art. 76.) and therefore, that the angles opposite to the two least sides must be *acute*.

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* Euc. 11. 5.

## Case I.

150. Given,

| | | |
|---|---|---|
| Two angles, and<br>A side, | to find | The remaining angle, and<br>The other two sides. |

The third angle is found by merely subtracting the sum of the two which are given from 180°. (Art. 79.)

The sides are found, by stating, according to theorem I, the following proportion;

As the sine of the angle opposite the *given* side,
To the length of the given side;
So is the sine of the angle opposite the *required* side,
To the length of the required side.

As a *side* is to be found, it is necessary to begin with a *tabular number*.

Ex. 1. In the triangle ABC (Fig. 27.) the side $b$ is given 32 rods, the angle A 56° 20′, and the angle C 49° 10′, to find the angle B, and the sides $a$ and $c$.

The sum of the two given angles 56° 20′+49° 10′=105° 30′; which subtracted from 180°, leaves 74° 30′ the angle B. Then,

$$\text{Sin B} : b :: \begin{cases} \text{Sin A} : a \\ \text{Sin C} : c \end{cases}$$

Calculation by logarithms.

| | | | |
|---|---|---|---|
| As the sine of B | 74° 30′ | *a. c.* | 0.01609 |
| To the side $b$ | 32 | | 1.50515 |
| So is the sine of A | 56°20′ | | 9.92027 |
| To the side $a$ | 27.64 | | 1.44151 |

| | | | |
|---|---|---|---|
| As the sine of B | 74° 30′ | *a. c.* | 0.01609 |
| To the side $b$ | 32 | | 1.50515 |
| So is the sine of C | 49° 10′ | | 9.87887 |
| To the side $c$ | 25.13 | | 1.40011 |

The *arithmetical complement* used in the first term here, may be found, in the usual way, or by taking out the *cosecant* of the given angle, and rejecting 10 from the index. (Art. 113.)

Ex. 2. Given the side $b$ 71, the angle A 107° 6′ and the angle C 27° 40′; to find the angle B, and the sides $a$ and $c$. The angle B is 45° 14′. Then

$$\text{Sin B} : b :: \begin{cases} \text{Sin A} : a = 95.58 \\ \text{Sin C} : c = 46.43 \end{cases}$$

When one of the given angles is *obtuse*, as in this example, the sine of its *supplement* is to be taken from the tables. (Art. 99.)

## Case II.

151. Given

| Given | | |
|---|---|---|
| Two sides, and An opposite angle, | to find | The remaining side, and The other two angles. |

One of the required angles is found, by beginning with a side, and, according to theorem I, stating the proportion,

> As the side opposite the given angle,
> To the sine of that angle;
> So is the side opposite the required angle,
> To the sine of that angle.

The third angle is found, by subtracting the sum of the other two from 180°; and the remaining side is found, by the proportion in the preceding article.

152. In this second case, if the side opposite to the given angle be shorter than the other given side, the solution will be *ambiguous*. Two different triangles may be formed, each of which will satisfy the conditions of the problem.

Let the side $b$, (Fig. 28.) the angle A, and the length of the side opposite this angle be given. With the latter for radius, (if it be shorter than $b$,) describe an arc, cutting the line AH in the points B and B′. The lines BC and B′C will be equal. So that, with the same data, there may be formed two different triangles, ABC and AB′C.

There will be the same ambiguity in the numerical calculation. The answer found by the proportion will be the *sine* of an angle. But this may be the sine, either of the *acute* angle AB′C, or of the *obtuse* angle ABC. For, BC being equal to B′C, the angle CB′B is equal to CBB′. Therefore ABC, which is the supplement of CBB′ is also the supplement of CB′B. But the sine of an angle is the same, as the sine of its supplement. (Art. 90.) The result of the calculation will, therefore be ambiguous. In practice however, there will generally be some circumstances which will determine whether the angle required is acute or obtuse.

If the side opposite the given angle be *longer* than the other given side, the angle which is subtended by the latter, will necessarily be acute. For there can be but one obtuse angle in a triangle, and this is always subtended by the longest side. (Art. 149.)

If the *given* angle be obtuse, the other two will, of course, be acute. There can, therefore, be no ambiguity in the solution.

Ex. 1. Given the angle A, (Fig. 28.) 35° 20′, the opposite side $a$ 50, and the side $b$ 70; to find the remaining side, and the other two angles.

To find the angle opposite to $b$, (Art. 151.)

$$a : \sin A :: b : \sin B$$

The calculation here gives the acute angle AB′C 54° 3′ 50″, and the obtuse angle ABC 125° 56′ 10″. If the latter be added to the angle at A 35° 20′, the sum will be 161° 16′ 10″, the supplement of which 18° 43′ 50″ is the angle ACB. Then in the triangle ABC, to find the side $c$=AB,

$$\text{Sin } A : a :: \sin C : c = 27.76$$

If the *acute* angle AB′C 54° 3′ 50″ be added to the angle at A 35° 20′, the sum will be 89° 23′ 50″, the supplement of which 90° 36′ 10″ is the angle ACB′. Then, in the triangle AB′C,

$$\text{Sin } A : CB' :: \sin C : AB' = 86.45$$

Ex. 2. Given the angle at A 63° 35′ (Fig. 29.) the side $b$ 64, and the side $a$ 72; to find the side $c$, and the angles B and C.

$a : \sin A :: b : \sin B = 52° 45' 25''$
$\sin A : a :: \sin C : c = 72.05$

The sum of the angles A and B is 116° 20′ 25″, the supplement of which 63° 39′ 35″ is the angle C.

In this example the solution is *not ambiguous*, because the side opposite the given angle is longer than the other given side.

Ex. 3. In a triangle of which the angles are A, B, and C, and the opposite sides *a*, *b*, and *c*, as before; if the angle A be 121° 40′, the opposite side *a* 68 rods, and the side *b* 47 rods; what are the angles B and C, and what is the length of the side *c*? Ans. B is 36° 2′ 4″, C 22° 17′ 56″, and *c* 30.3.

In this example also, the solution is not ambiguous, because the *given* angle is obtuse.

## Case III.

153. Given

| | | |
|---|---|---|
| Two sides, and<br>The included angle, | to find | The remaining side, and<br>The other two angles. |

In this case, the angles are found by theorem II. (Art. 144.) The required side may be found by theorem I.

In making the solutions, it will be necessary to observe, that by subtracting the given angle from 180°, the *sum* of the other two angles is found; (Art. 79.) and, that *adding half the difference of two quantities to their half sum gives the greater quantity, and subtracting the half difference from the half sum gives the less.* (Alg. 341.) The latter proposition may be geometrically demonstrated thus;

Let AE (Fig. 32.) be the greater of two magnitudes, and BE the less. Bisect AB in D, and make AC equal to BE. Then

AB is the *sum* of the two magnitudes;
CE their *difference;*
DA or DB *half* their *sum;*
DE or DC *half* their *difference;*
But DA+DE=AE the *greater* magnitude;
And DB−DE=BE the *less.*

Ex. 1. In the triangle ABC (Fig. 30.) the angle A is given 26° 14′ the side $b$ 39, and the side $c$ 53; to find the angles B and C, and the side $a$.

The *sum* of the sides $b$ and $c$ is 53+39=92,
And their *difference* 53−39=14.
The sum of the angles B and C=180°−26° 14′=153° 46′
And *half the sum* of B and C is 76° 53′

Then, by theorem II,

$$(b+c):(b-c)::\tan\tfrac{1}{2}(B+C):\tan\tfrac{1}{2}(B-C)$$

| | | | |
|---|---|---|---|
| To and from the half sum | 76° | 53′ | |
| Adding and subtracting the half difference | 33 | 8 | 50 |
| We have the greater angle | 110 | 1 | 50 |
| And the less angle | 43 | 44 | 10 |

As the greater of the two given sides is $c$, the greater angle is C, and the less angle B. (Art. 149.)

To find the side $a$, by theorem I,
Sin B : $b$::sin A : $a$=24.94.

Ex. 2. Given the angle A 101° 30′, the side $b$ 76, and the side $c$ 109; to find the angles B and C, and the side $a$.
B is 30° 57½′, C 47° 32½′, and $a$ 144.8.

## Case IV.

154. Given the three sides, to find the angles.

In this case, the solutions may be made, by drawing a perpendicular to the longest side, from the opposite angle. This will divide the given triangle into two *right angled* triangles. The two segments may be found by theorem III. (Art. 145.) There will then be given, in each of the right angled triangles, the hypothenuse and one of the legs, from which the angles may be determined, by rectangular trigonometry. (Art. 135.)

Ex. 1. In the triangle ABC (Fig. 31.) the side AB is 39, AC 35, and BC 27. What are the angles?

Let a perpendicular be drawn from C, dividing the longest side AB into the two segments AP and BP. Then by theorem III,

$$AB : AC+BC :: AC-BC : AP-BP$$

| | | | |
|---|---|---|---|
| As the longest side | 39 | a. c. | 8.40894 |
| To the sum of the two others | 62 | | 1.79239 |
| So is the difference of the latter | 8 | | 0.90309 |
| To the difference of the segments | 12.72 | | 1.10442 |

The greater of the two segments is AP, because it is next the side AC, which is greater than BC. (Art. 146.)

| | |
|---|---|
| To and from half the sum of the segments | 19.5 |
| Adding and subtracting half their difference, (Art. 153.) | 6.36 |
| We have the greater segment AP | 25.86 |
| And the less BP | 13.14 |

Then, in each of the right angled triangles APC and BPC, we have given the hypothenuse and base; and by art. 135,

$$AC : R :: AP : \cos A = 42° \ 21' \ 57''$$
$$BC : R :: BP : \cos B = 60° \ 52' \ 42''$$

And subtracting the sum of the angles A and B from 180°, we have the remaining angle ACB=76° 45′ 21″.

Ex. 2. If the three sides of a triangle are 78, 96, and 104; what are the angles?

Ans. 45° 41′ 48″, 61° 43′ 27″, and 72° 34′ 45″.

### *Examples for Practice.*

1. Given the angle A 54° 30′, the angle B 63° 10′, and the side *a* 164 rods; to find the angle C, and the sides *b* and *c*.
2. Given the angle A 45° 6′, the opposite side *a* 93, and the side *b* 108; to find the angles B and C, and the side *c*.
3. Given the angle A 67° 24′, the opposite side *a* 62, and the side *b* 46; to find the angles B and C, and the side *c*.
4. Given the angle A 127° 42′, the opposite side *a* 381, and the side *b* 184; to find the angles B and C, and the side *c*.

5. Given the side $b$ 58, the side $c$ 67, and the included angle $A=36°$; to find the angles B and C, and the side $a$.
6. Given the three sides, 631, 268, and 546; to find the angles.

155. The three theorems demonstrated in this section, have been here applied to *oblique angled* triangles only. But they are equally applicable to *right angled* triangles.

Thus, in the triangle ABC, (Fig. 17.) according to theorem I, (Art. 143.)

Sin B : AC : : sin A : BC

This is the same proportion as one stated in art. 134, except that, in the first term here, the *sine of* B is substituted for *radius*. But, as B is a right angle, its sine is *equal* to *radius*. (Art. 95.)

Again, in the triangle ABC, (Fig. 21.) by the same theorem;

Sin A : BC : : sin C : AB

This is also one of the proportions in rectangular trigonometry, when the hypothenuse is made radius.

The other two theorems might be applied to the solution of right angled triangles. But, when one of the angles is *known* to be a right angle, the methods explained in the preceding section, are much more simple in practice.*

* For the application of Trigonometry to the Mensuration of Heights and Distances, see Navigation and Surveying.

# SECTION V.

## GEOMETRICAL CONSTRUCTION OF TRIANGLES, BY THE PLANE SCALE.

Art. 156. To facilitate the construction of geometrical figures, a number of graduated lines are put upon the common two feet scale; one side of which is called the *Plane Scale*, and the other side, *Gunter's Scale*. The most important of these are the scales of *equal parts*, and the line of *chords*. In forming a given triangle, or any other right lined figure, the parts which must be made to agree with the conditions proposed, are the *lines*, and the *angles*. For the former, a scale of equal parts is used; for the latter, a line of chords.

157. The line on the upper side of the plane scale, is divided into *inches* and *tenths* of an inch. Beneath this, on the left hand, are two *diagonal* scales of equal parts,* divided into inches and half inches, by perpendicular lines. On the larger scale, one of the inches is divided into tenths, by lines which pass *obliquely* across, so as to intersect the parallel lines which run from right to left. The use of the oblique lines is to measure *hundredths* of an inch, by inclining more and more to the right, as they cross each of the parallels.

To take off, for instance, an extent of 3 inches, 4 tenths, and 6 hundredths;

Place one foot of the compasses at the intersection of the perpendicular line marked 3 with the parallel line marked 6, and the other foot at the intersection of the latter with the oblique line marked 4.

The other diagonal scale is of the same nature. The divisions are smaller, and are numbered from left to right.

158. In geometrical constructions, what is often required, is to make a figure, not *equal* to a given one, but only *similar*. Now figures are similar which have equal angles, and the

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* These lines are not represented in the plate, as the learner is supposed to have the scale before him.

sides about the equal angles *proportional.* (Euc. Def. 1. 6.) Thus a land surveyor, in plotting a field, makes the several lines in his plan to have the same proportion to each other, as the sides of the field. For this purpose, a scale of equal parts may be used, of any dimensions whatever. If the sides of the field are 2, 5, 7, and 10 *rods*, and the lines in the plan are 2, 5, 7, and 10 *inches*, and if the angles are the same in each, the figures are similar. One is a copy of the other, upon a smaller scale.

So any two right lined figures are similar, if the angles are the same in both, and if the number of smaller parts in each side of one, is equal to the number of larger parts in the corresponding sides of the other. The several divisions on the scale of equal parts may, therefore, be considered as representing any measures of length, as feet, rods, miles, &c. All that is necessary is, that the scale be not changed, in the construction of the same figure ; and that the several divisions and subdivisions be properly proportioned to each other. If the larger divisions, on the diagonal scale, are units, the smaller ones are tenths and hundredths. If the larger are tens, the smaller are units and tenths.

159. In laying down an *angle*, of a given number of degrees, it is necessary to *measure* it. Now the proper measure of an angle is an arc of a circle. (Art. 74.) And the measure of an arc, where the radius is given, is its *chord.* For the chord is the distance, in a straight line, from one end of the arc to the other. Thus the chord AB (Fig. 33.) is a measure of the arc ADB, and of the angle ACB.

To form the *line of chords*, a circle is described, and the lengths of its chords determined for every degree of the quadrant. These measures are put on the plane scale, on the line marked CHO.

160. The chord of 60° is equal to *radius.* (Art. 95.) In laying down or measuring an angle, therefore, an arc must be drawn, with a radius which is equal to the extent from 0 to 60 on the line of chords. There are generally on the scale, two lines of chords. Either of these may be used ; but the angle must be measured by the same line from which the radius is taken.

161. To *make an angle*, then, of a given number of degrees ; From one end of a straight line as a center, and with a radius equal to the chord of 60° on the line of chords, describe an arc of a circle cutting the straight line. From the

point of intersection, extend the chord of the given number of degrees, applying the other extremity to the arc; and through the place of meeting, draw the other line from the angular point.

If the given angle is *obtuse*, take from the scale the chord of *half* the number of degrees, and apply it *twice* to the arc. Or make use of the chords of any two arcs whose *sum* is equal to the given number of degrees.

A *right angle* may be constructed, by drawing a perpendicular without using the line of chords.

Ex. 1. To make an angle of 32 degrees. (Fig. 33.) With the point C, in the line CH, for a center, and with the chord of 60° for radius, describe the arc ADF. Extend the chord of 32° from A to B; and through B, draw the line BC. Then is ACB an angle of 32 degrees.

2. To make an angle of 140 degrees. (Fig. 34.) On the line CH, with the chord of 60°, describe the arc ADF; and extend the chord of 70° from A to D, and from D to B. The arc ADB$=70°\times2=140.°$

On the other hand;

162. To *measure an angle;* On the angular point as a center, and with the chord of 60° for radius, describe an arc to cut the two lines which include the angle. The distance between the points of intersection, applied to the line of chords, will give the measure of the angle in degrees. If the angle be *obtuse*, divide the arc into two parts.

Ex. 1. To measure the angle ACB. (Fig. 33.) Describe the arc ADF cutting the lines CH and CB. The distance AB will extend 32° on the line of chords.

2. To measure the angle ACB. (Fig. 34.) Divide the arc ADB into two parts, either equal or unequal, and measure each part, by applying its chord to the scale. The sum of the two will be 140°.

163. Besides the lines of chords, and of equal parts, on the plane scale; there are also lines of natural *sines*, *tangents*, and *secants*, marked Sin. Tan. and Sec.; of *semitangents*, marked S. T.; of *longitude*, marked Lon. or M. L.; of *rhumbs*, marked Rhu. or Rum. &c. These are not necessary in trigonometrical constructions. Some of them are used in Navigation; and some of them, in the projections of the Sphere.

164. In Navigation, the quadrant, instead of being graduated in the usual manner, is divided into *eight* portions, called

*Rhumbs.* The *Rhumb line*, on the scale, is a line of chords, divided into rhumbs and quarter-rhumbs, instead of degrees.

165. The line of *Longitude* is intended to show the number of geographical miles in a degree of longitude, at different distances from the equator. It is placed over the line of chords, with the numbers in an inverted order: so that the figure above shows the length of a degree of longitude, in any latitude denoted by the figure below.* Thus at the equator, where the latitude is 0, a degree of longitude is 60 geographical miles. In latitude 40, it is 46 miles; in latitude 60, 30 miles, &c.

166. The graduation on the line of *secants* begins where the line of sines ends. For the greatest sine is only equal to radius; but the secant of the least arc is greater than radius.

167. The *semitangents* are the tangents of *half* the given arcs. Thus the semitangent of 20° is the tangent of 10°. The line of semitangents is used in one of the projections of the sphere.

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168. In the construction of *triangles*, the sides and angles which are *given*, are laid down according to the directions in Arts. 158, 161. The parts *required* are then measured, according to Arts. 158, 162. The following problems correspond with the four cases of oblique angled triangles; (Art. 148.) but are equally adapted to right angled triangles.

169. Prob. I. *The angles and one side* of a triangle being given; to find, by construction, the other two sides.

Draw the given side. From the ends of it, lay off two of the given angles. Extend the other sides till they intersect; and then measure their lengths on a scale of equal parts.

Ex. 1. Given the side *b* 32 rods, (Fig. 27.) the angle A 56° 20′, and the angle C 49° 10′; to construct the triangle, and find the lengths of the sides *a* and *c*.

Their lengths will be 25 and 27½.

2. In a right angled triangle, (Fig. 17.) given the hypothenuse 90, and the angle A 32° 20′, to find the base and perpendicular.

The length of AB will be 76, and of BC 48.

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* Sometimes the line of longitude is placed *under* the line of chords.

3. Given the side AC 68, the angle A 124°, and the angle C 37°: to construct the triangle.

170. PROB. II. *Two sides and an opposite angle* being given, to find the remaining side, and the other two angles.

Draw one of the given sides; from one end of it, lay off the given angle; and extend a line indefinitely for the required side. From the other end of the first side, with the remaining given side for radius, describe an arc cutting the indefinite line. The point of intersection will be the end of the required side.

If the side opposite the given angle be less than the other given side, the case will be *ambiguous.* (Art. 152.)

Ex. 1. Given the angle A 63° 35′ (Fig. 29.) the side $b$ 32, and the side $a$ 36.

The side AB will be 36 nearly, the angle B 52° 45½′, and C 63° 39½′.

2. Given the angle A (Fig. 28.) 35° 20′, the opposite side $a$ 25, and the side $b$ 35.

Draw the side $b$ 35, make the angle A 35° 20′, and extend AH indefinitely. From C with radius 25, describe an arc cutting AH in B and B′. Draw CB and CB′, and two triangles will be formed, ABC, and AB′C, each corresponding with the conditions of the problem.

3. Given the angle A 116°, the opposite side $a$ 38, and the side $b$ 26; to construct the triangle.

171. PROB. III. *Two sides and the included angle* being given; to find the other side and angles.

Draw one of the given sides. From one end of it lay off the given angle, and draw the other given side. Then connect the extremities of this and the first line.

Ex. 1. Given the angle A (Fig. 30.) 26° 14′, the side $b$ 78, and the side $c$ 106; to find B, C, and $a$.

The side $a$ will be 50, the angle B 43° 44′, and C 110° 2′.

2. Given A 86°, $b$ 65, and $c$ 83; to find B, C, and $a$.

172. PROB. IV. The *three sides* being given; to find the angles.

Draw one of the sides, and from one end of it, with an extent equal to the second side, describe an arc. From the other end, with an extent equal to the third side, describe a second arc cutting the first; and from the point of intersection draw the two sides. (Euc. 22. 1.)

Ex. 1. Given AB (Fig. 31.) 78, AC 70, and BC, 54; to find the angles.

The angles will be A 42° 22′, B 60° 52$\frac{2}{3}$′, and C 76° 45$\frac{1}{3}$′.

2. Given the three sides 58, 39, and 46; to find the angles.

173. Any right lined figure whatever, whose sides and angles are given, may be constructed, by laying down the sides from a scale of equal parts, and the angles from a line of chords.

Ex. Given the sides AB (Fig. 35.) =20, BC=22, CD=30, DE=12; and the angles B=102°, C=130°, D=108°, to construct the figure.

Draw the side AB=20, make the angle B=102°, draw BC=22, make C=130°, draw CD=30, make D=108°, draw DE=12, and connect E and A.

The last line EA may be measured on the scale of equal parts; and the angles E and A, by a line of chords.

## SECTION VI.

### DESCRIPTION AND USE OF GUNTER'S SCALE.

ART. 174. AN expeditious method of solving the problems in trigonometry, and making other logarithmic calculations, in a mechanical way, has been contrived by Mr. Edmund Gunter. The logarithms of numbers, of sines, tangents, &c. are represented by *lines*. By means of these, multiplication, division, the rule of three, involution, evolution, &c. may be performed much more rapidly, than in the usual method by figures.

The logarithmic lines are generally placed on one side only of the scale in common use. They are,

| | | |
|---|---|---|
| A line of artificial *Sines* divided into *Rhumbs*, and marked | | S. R. |
| A line of artificial *Tangents*, | do. | T. R. |
| A line of the logarithms of *numbers*, | | Num. |
| A line of artificial *Sines*, to every *degree*, | | SIN. |
| A line of artificial *Tangents*, | do. | TAN. |
| A line of *Versed Sines*, | | V. S. |

To these are added a line of *equal parts*, and a line of *Meridional Parts*, which are not logarithmic. The latter is used in Navigation.

#### *The Line of Numbers.*

175. Portions of the line of *Numbers*, are intended to represent the *logarithms* of the natural series of numbers 2, 3, 4, 5, &c.

The logarithms of 10, 100, 1000, &c. are 1, 2, 3, &c. (Art. 3.)

If then, the log. of 10 be represented by a line of 1 foot;
the log. of 100 will be repres'd by one of 2 feet;
the log. of 1000 by one of 3 feet;
the lengths of the several lines being proportional to the corresponding logarithms in the tables. *Portions* of a foot will represent the logarithms of numbers between 1 and 10;

and portions of a line 2 feet long, the logarithms of numbers between 1 and 100.

On Gunter's scale, the line of the logarithms of numbers begins at a brass pin on the left, and the divisions are numbered 1, 2, 3, &c. to another pin near the middle. From this the numbers are repeated, 2, 3, 4, &c. which may be read 20, 30, 40, &c. The logarithms of numbers between 1 and 10 are represented by portions of the first half of the line; and the logarithms of numbers between 10 and 100, by portions greater than half the line, and less than the whole.

176. The logarithm of 1, which is 0, is denoted, not by any extent of line, but by a *point* under 1, at the commencement of the scale. The distances from this point to different parts of the line, represent other logarithms, of which the *figures* placed over the several divisions are the *natural numbers*. For the intervening logarithms, the intervals between the figures, are divided into tenths, and sometimes into smaller portions. On the right hand half of the scale, as the divisions which are numbered are *tens*, the subdivisions are units.

Ex. 1. To take from the scale the logarithm of 3.6; set one foot of the compasses under 1 at the beginning of the scale, and extend the other to the 6th division after the first figure 3.

2. For the logarithm of 47; extend from 1 at the beginning, to the 7th subdivision after the second figure 4.*

177. It will be observed, that the divisions and subdivisions *decrease*, from left to right; as in the tables of *logarithms*, the differences decrease. The difference between the logarithms of 10 and 100 is no greater, than the difference between the logarithms of 1 and 10.

178. The line of numbers, as it has been here explained, furnishes the logarithms of all numbers between 1 and 100.

And if the indices of the logarithms be neglected, the same scale may answer for all numbers whatever. For the *decimal* part of the logarithm of any number is the same, as that of the number multiplied or divided by 10, 100, &c. (Art. 14.) In logarithmic calculations, the use of the indices is to determine the distance of the several figures of the natural numbers from the place of units. (Art. 11.) But in those cases in which the logarithmic line is commonly used, it will

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* If the compasses will not reach the distance required; first open them so as to take off *half*, or any part of the distance, and then the remaining part.

not generally be difficult to determine the local value of the figures in the result.

179. We may, therefore, consider the *point* under 1 at the left hand, as representing the logarithm of 1, or 10, or 100; or $\frac{1}{10}$, or $\frac{1}{100}$, &c. for the decimal part of the logarithm of each of these is 0. But if the first 1 is reckoned 10, all the succeeding numbers must also be increased in a tenfold ratio; so as to read, on the first half of the line, 20, 30, 40, &c. and on the other half, 200, 300, &c.

The whole extent of the logarithmic line,

| | | | | | |
|---|---|---|---|---|---|
| is from 1 | to 100, | | or from 0.1 | to 10, | |
| or from 10 | to 1000, | | or from 0.01 | to 1, | |
| or from 100 | to 10000, | &c. | or from 0.001 | to 0.1, | &c. |

Different values may, on different occasions, be assigned to the several numbers and subdivisions marked on this line. But for any one calculation, the value must remain the same.

Ex. Take from the scale 365.

As this number is between 10 and 1000, let the 1 at the beginning of the scale, be reckoned 10. Then, from this point to the second 3 is 300; to the 6th dividing stroke is 60; and half way from this to the next stroke is 5.

180. Multiplication, division, &c. are performed by the line of numbers, on the same principle, as by common logarithms. Thus,

To *multiply* by this line, *add* the logarithms of the two factors; (Art. 37.) that is, take off, with the compasses, that length of line which represents the logarithm of *one* of the factors, and apply this so as to extend forward from the end of that which represents the logarithm of the *other* factor. The sum of the two will reach to the end of the line representing the logarithm of the product.

Ex. Multiply 9 into 8. The extent from 1 to 8, added to that from 1 to 9, will be equal to the extent from 1 to 72 the product.

181. To *divide* by the logarithmic line, *subtract* the logarithm of the divisor from that of the dividend; (Art. 41.) that is, take off the logarithm of the divisor, and this extent set back from the end of the logarithm of the dividend, will reach to the logarithm of the quotient.

Ex. Divide 42 by 7. The extent from 1 to 7, set back from 42, will reach to 6 the quotient.

182. *Involution* is performed in logarithms, by multiplying the logarithm of the quantity into the index of the power

(Art. 45.) that is, by *repeating* the logarithms as many times as there are units in the index. To involve a quantity on the scale, then, take in the compasses the linear logarithm, and *double it*, *treble it*, &c. according to the index of the proposed power.

Ex. 1. Required the square of 9. Extend the compasses from 1 to 9. *Twice* this extent will reach to 81 the square.

2. Required the cube of 4. The extent from 1 to 4, repeated *three times*, will reach to 64 the cube of 4.

183. On the other hand, to perform *evolution* on the scale; take *half*, *one third*, &c. of the logarithm of the quantity, according to the index of the proposed root.

Ex. 1. Required the square root of 49. *Half* the extent from 1 to 49, will reach from 1 to 7 the root.

2. Required the cube root of 27. *One third* the distance from 1 to 27, will extend from 1 to 3 the root.

184. The *Rule of Three* may be performed on the scale, in the same manner as in logarithms, by adding the two middle terms, and from the sum, subtracting the first term. (Art. 52.) But it is more convenient in practice to *begin* by subtracting the first term from one of the others. If four quantities are proportional, the quotient of the first divided by the second, is equal to the quotient of the third divided by the fourth. (Alg. 364.)

Thus if $a : b :: c : d$, then $\frac{a}{b}=\frac{c}{d}$, and $\frac{a}{c}=\frac{b}{d}$. (Alg. 380.)

But in logarithms, *subtraction* takes the place of division; so that,

$\log. a - \log. b = \log. c - \log. d$. Or $\log. a - \log. c = \log. b - \log. d$.

Hence,

185. On the scale, *the difference between the first and second terms of a proportion, is equal to the difference between the third and fourth.* Or, the difference between the first and third terms, is equal to the difference between the second and fourth.

The difference between the two terms is taken, by extending the compasses from one to the other. If the second term be greater than the first; the fourth must be greater than the third; if less, less. (Alg. 395.*) Therefore if the compasses extend *forward* from *left to right*, that is, from a

* Euc. 14. 5.

less number to a greater, from the first term to the second; they must also extend forward from the third to the fourth. But if they extend *backward*, from the first term to the second; they must extend the same way, from the third to the fourth.

Ex. 1. In the proportion 3 : 8 :: 12 : 32, the extent from 3 to 8, will reach from 12 to 32; Or, the extent from 3 to 12, will reach from 8 to 32.

2. If 54 yards of cloth cost 48 dollars, what will 18 yards cost? 54 : 48 :: 18 : 16

The extent from 54 to 48, will reach *backwards* from 18 to 16.

3. If 63 gallons of wine cost 81 dollars, what will 35 gallons cost? 63 : 81 :: 35 : 45

The extent from 63 to 81, will reach from 35 to 45.

## *The Line of Sines.*

186. The line on Gunter's scale marked SIN. is a line of logarithmic sines, made to correspond with the line of numbers. The whole extent of the line of numbers, (Art. 179.)

| | | | |
|---|---|---|---|
| is from 1 | to 100, | whose logs. are 0.00000 and | 2.00000, |
| or from 10 | to 1000, | whose logs. are 1.00000 and | 3.00000, |
| or from 100 | to 10000, | whose logs. are 2.00000 | 4.00000, |

the *difference of the indices* of the two extreme logarithms being in each case 2.

Now the logarithmic sine of 0° 34′ 22″ 41‴ is 8.00000

And the sine of 90° (Art. 95.) is 10.00000

Here also the difference of the indices is 2. If then the point directly beneath one extremity of the line of numbers, be marked for the sine of 0° 34′ 22″ 41‴; and the point beneath the other extremity, for the sine of 90°; the interval may furnish the intermediate sines; the divisions on it being made to correspond with the decimal part of the logarithmic sines in the tables.*

* To represent the sines *less* than 34′ 22″ 41‴, the scale must be extended on the left indefinitely. For, as the sine of an arc approaches to 0, its logarithm, which is negative, increases without limit. (Art. 15.)

The first dividing stroke in the line of Sines is generally at 0° 40′, a little farther to the right than the beginning of the line of numbers. The next division is at 0° 50′; then begins the numbering of the degrees, 1, 2, 3, 4, &c. from left to right.

### *The Line of Tangents.*

187. The first 45 degrees on this line are numbered from left to right, nearly in the same manner as on the line of Sines.

| | |
|---|---|
| The logarithmic tangent of 0° 34′ 22″ 35‴ is | 8.00000 |
| And the tangent of 45°, (Art. 95.) | 10.00000 |

The difference of the indices being 2, 45 degrees will reach to the end of the line. For those above 45° the scale ought to be continued much farther to the right. But as this would be inconvenient, the numbering of the degrees, after reaching 45, is *carried back* from right to left. The same dividing stroke answers for an arc and its *complement*, one above and the other below 45°. For, (Art. 93. Propor. 9.)

$$\tan : R :: R : \cot.$$

In logarithms, therefore, (Art. 184.)

$$\tan - R = R - \cot.$$

That is, the *difference* between the tangent and radius, is equal to the difference between radius and the cotangent: in other words, one is as much *greater* than the tangent of 45°, as the other is *less*. In taking, then, the tangent of an arc greater than 45°, we are to suppose the distance between 45 and the division marked with a given number of degrees, to be added to the whole line, in the same manner as if the line were continued out. In working proportions, extending the compasses *back*, from a less number to a greater, must be considered the same as carrying them *forward* in other cases. See art. 185.

### *Trigonometrical Proportions on the Scale.*

188. In working proportions in trigonometry by the scale; *the extent from the first term to the middle term of the same*

*name, will reach from the other middle term to the fourth term.* (Art. 185.)

In a trigonometrical proportion, two of the terms are the lengths of sides of the given triangle; and the other two are tabular sines, tangents, &c. The former are to be taken from the line of numbers; the latter, from the lines of logarithmic sines and tangents. If one of the terms is a *secant*, the calculation cannot be made on the scale, which has commonly no line of secants. It must be kept in mind that *radius* is equal to the sine of 90°, or to the tangent of 45°. (Art. 95.) Therefore, whenever radius is a term in the proportion, one foot of the compasses must be set on the end of the line of sines or of tangents.

189. The following examples are taken from the proportions which have already been solved by numerical calculation.

Ex. 1. In Case I, of right angled triangles, (Art. 134. ex. 1.)

R : 45 : : sin 32° 20′ : 24

Here the third term is a *sine*; the first term radius is, therefore, to be considered as the sine of 90°. Then the extent from 90° to 32° 20′ on the line of sines, will reach from 45 to 24 on the line of numbers. As the compasses are set *back* from 90° to 32° 20′; they must also be set back from 45. (Art. 185.)

2. In the same case, if the base be made radius, (page 60.)

R : 38 : : tan 32° 20′ : 24

Here, as the third term is a *tangent*, the first term radius is to be considered the tangent of 45°. Then the extent from 45° to 32° 20′ on the line of tangents, will reach from 38 to 24 on the line of numbers.

3. If the perpendicular be made radius, (page 60.)

R : 24 : : tan 57° 40′ : 38

The extent from 45° to 57° 40′ on the line of tangents, will reach from 24 to 38 on the line of numbers. For the tangent of 57° 40′ on the scale, look for its *complement* 32° 20′. (Art. 187.) In this example, although the compasses extend *back* from 45° to 57° 40′; yet, as this is from a *less* number to a *greater*, they must extend *forward* on the line of numbers. (Arts. 185, 187.)

4. In art. 135, 35 : R : : 26 : sin 48°

The extent from 35 to 26 will reach from 90° to 48°.

5. In art. 136, R : 48 : : tan $27\frac{1}{4}°$ : $24\frac{3}{4}$

The extent from 45° to $27\frac{1}{4}°$, will reach from 48 to $24\frac{3}{4}$.

6. In art. 150, ex. 1. Sin 74° 30′ : 32 : : sin 56° 20′ : $27\frac{1}{2}$.

For other examples, see the several cases in Sections III. and IV.

190. Though the solutions in trigonometry may be effected by the logarithmic scale, or by geometrical construction, as well as by arithmetical computation; yet the latter method is by far the most accurate. The first is valuable principally for the *expedition* with which the calculations are made by it. The second is of use, in presenting the *form* of the triangle to the eye. But the accuracy which attends arithmetical operations, is not to be expected, in taking lines from a scale with a pair of compasses.*

* See note G.

## SECTION VII.*

### THE FIRST PRINCIPLES OF TRIGONOMETRICAL ANALYSIS.

Art. 191. In the preceding sections, sines, tangents, and secants have been employed in calculating the sides and angles of triangles. But the use of these lines is not confined to this object. Important assistance is derived from them, in conducting many of the investigations in the higher branches of analysis, particularly in physical astronomy. It does not belong to an elementary treatise of trigonometry, to prosecute these inquiries to any considerable extent. But this is the proper place for *preparing the formulæ*, the applications of which are to be made elsewhere.

*Positive and negative* SIGNS *in trigonometry.*

192. Before entering on a particular consideration of the algebraic expressions which are produced by combinations of the several trigonometrical lines, it will be necessary to attend to the positive and negative *signs* in the different quarters of the circle. The sines, tangents, &c. in the tables, are calculated for a single quadrant only. But these are made to answer for the whole circle. For they are of the same length in each of the four quadrants. (Art. 90.) Some of them, however are *positive;* while others are *negative.* In algebraic processes, this distinction must not be neglected.

193. For the purpose of tracing the changes of the signs, in different parts of the circle, let it be supposed that a straight line CT (Fig. 36.) is fixed at one end C, while the other end is carried round, like a rod moving on a pivot; so that the point S shall describe the circle ABDH. If the two diameters AD and BH be perpendicular to each other, they will divide the circle into quadrants.

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* Euler's Analysis of Infinites, Hutton's Mathematics, Lacroix's Differential Calculus, Mansfield's Essays, Legendre's, Lacroix's, Playfair's, Cagnoli's, and Woodhouse's Trigonometry.

194. In the *first quadrant* AB, the sine, cosine, tangent, &c. are considered *all positive*. In the *second quadrant* BD, the sine P'S' continues *positive;* because it is still on the *upper* side of the diameter AD, from which it is measured. But the *cosine*, which is measured from BH, becomes *negative*, as soon as it changes from the *right* to the *left* of this line. (Alg. 507.) In the *third quadrant*, the *sine* becomes negative, by changing from the upper side to the under side of DA. The *cosine* continues negative, being still on the left of BH. In the *fourth quadrant*, the *sine* continues negative. But the *cosine* becomes positive, by passing to the right of BH.

195. The signs of the *tangents* and *secants* may be derived from those of the sines and cosines. The relations of these several lines to each other must be such, that a uniform method of calculation may extend through the different quadrants.

In the first quadrant, (Art. 93. Propor. 1.)

$$R : \cos :: \tan : \sin, \text{ that is, } \mathrm{Tan}=\frac{R\times\sin}{\cos}.$$

The sign of the quotient is determined from the signs of the divisor and dividend. (Alg. 123.) The radius is considered as always positive. If then the sine and cosine be both positive or both negative, the tangent will be positive. But if one of these be positive, while the other is negative, the tangent will be negative.

Now by the preceding article,

In the 2d quadrant, the sine is positive, and the cosine negative.

The tangent must therefore be *negative.*

In the 3d quadrant, the sine and cosine are both negative.

The tangent must therefore be *positive.*

In the 4th quadrant, the sine is negative, and the cosine positive.

The tangent must therefore be *negative.*

196. By the 9th, 3d, and 6th proportions in Art. 93.

$$1.\ \mathrm{Tan} : R :: R : \cot, \text{ that is } \mathrm{Cot}=\frac{R^2}{\tan}.$$

Therefore, as radius is uniformly positive, the *cotangent* must have the same sign as the tangent.

2. Cos : R : : R : sec, that is, Sec$=\frac{R^2}{\cos}$.

The *secant*, therefore, must have the same sign as the cosine.

3. Sin : R : : R : cosec, that is, Cosec$=\frac{R^2}{\sin}$.

The *cosecant*, therefore, must have the same sign as the sine.

The *versed sine*, as it is measured from A, in one direction only, is invariably positive.

197. The *tangent* AT (Fig. 36.) increases, as the arc extends from A towards B. See also Fig. 11. Near B the increase is very rapid; and when the difference between the arc and 90°, is *less* than any assignable quantity, the tangent is *greater* than any assignable quantity, and is said to be *infinite.* (Alg. 447.) If the arc is *exactly* 90 degrees, it has, strictly speaking, *no* tangent. For a tangent is a line drawn perpendicular to the diameter which passes through one end of the arc, and extended till it *meets* a line proceeding from the center through the other end. (Art. 34.) But if the arc is 90 degrees, as AB, (Fig. 36.) the angle ACB is a right angle, and therefore AT is *parallel* to CB; so that, if these lines be extended ever so far, they never can meet. Still, as an arc infinitely near to 90° has a tangent infinitely great, it is frequently said, in concise terms, that the tangent of 90° is infinite.

In the second quadrant, the tangent, is, at first, infinitely great, and gradually diminishes, till at D it is reduced to nothing. In the third quadrant it increases again, becomes infinite near H, and is reduced to nothing at A.

The *cotangent* is inversely as the tangent. It is therefore nothing at B and H, (Fig. 36.) and infinite near A and D.

198. The *secant* increases with the tangent, through the first quadrant, and becomes infinite near B; it then diminishes, in the second quadrant, till at D it is equal to the radius CD. In the third quadrant it increases again, becomes infinite near H, after which it diminishes, till it becomes equal to radius.

The *cosecant* decreases, as the secant increases, and *v. v.* It is therefore equal to radius at B and H, and infinite near A and D.

199. The *sine* increases through the first quadrant, till at B (Fig. 36.) it is equal to radius. See also Fig. 13. It then diminishes, and is reduced to nothing at D. In the third quadrant, it increases again, becomes equal to radius at H, and is reduced to nothing at A.

The *cosine* decreases through the first quadrant, and is reduced to nothing at B. In the second quadrant, it increases, till it becomes equal to radius at D. It then diminishes again, is reduced to nothing at H, and afterwards increases till it becomes equal to radius at A.

In all these cases, the arc is supposed to *begin* at A, and to extend round in the direction of BDH.

200. The *sine* and *cosine* vary from nothing to radius, which they never exceed. The *secant* and *cosecant* are never less than radius, but may be greater than any given length. The *tangent* and *cotangent* have every value from nothing to infinity. Each of these lines, after reaching its *greatest* limit begins to *decrease;* and as soon as it arrives at its *least* limit, begins to *increase.* Thus the sine begins to decrease, after becoming equal to radius, which is its greatest limit. But the secant begins to increase after becoming equal to radius, which is its least limit.

201. The substance of several of the preceding articles is comprised in the following tables. The first shows the *signs* of the trigonometrical lines, in each of the quadrants of the circle. The other gives the *values* of these lines, at the extremity of each quadrant.

| Quadrant | 1st | 2d | 3d | 4th |
|---|---|---|---|---|
| Sine and cosecant | + | + | − | − |
| Cosine and secant | + | − | − | + |
| Tangent and cotangent | + | − | + | − |

| | 0° | 90° | 180° | 270° | 360° |
|---|---|---|---|---|---|
| Sine | 0 | $r$ | 0 | $r$ | 0 |
| Cosine | $r$ | 0 | $r$ | 0 | $r$ |
| Tangent | 0 | $\infty$ | 0 | $\infty$ | 0 |
| Cotangent | $\infty$ | 0 | $\infty$ | 0 | $\infty$ |
| Secant | $r$ | $\infty$ | $r$ | $\infty$ | $r$ |
| Cosecant | $\infty$ | $r$ | $\infty$ | $r$ | $\infty$ |

Here $r$ is put for radius, and $\infty$ for infinite.

202. By comparing these two tables, it will be seen, that each of the trigonometrical lines changes from positive to negative, or from negative to positive, in that part of the circle

in which the line is either *nothing* or *infinite.* Thus the tangent changes from positive to negative, in passing from the first quadrant to the second, through the place where it is infinite. It becomes positive again, in passing from the second quadrant to the third, through the point in which it is nothing.

203. There can be no more than 360 degrees in any circle. But a body may have a number of successive revolutions in the same circle; as the earth moves round the sun, nearly in the same orbit, year after year. In astronomical calculations, it is frequently necessary to add together parts of different revolutions. The sum may be more than 360°. But a body which has made more than a complete revolution in a circle, is only brought back to a point which it had passed over before. So the sine, tangent, &c. of an arc greater than 360°, is the same as the sine, tangent, &c. of some arc less than 360°. If an entire circumference, or a number of circumferences be added to any arc, it will terminate in the same point as before. So that, if C be put for a whole circumference, or 360°, and $x$ be any arc whatever;

$$\sin x = \sin (C+x) = \sin (2C+x) = \sin (3C+x), \text{ \&c.}$$
$$\tan x = \tan (C+x) = \tan (2C+x) = \tan (3C+x), \text{ \&c.}$$

204. It is evident also, that, in a number of successive revolutions, in the same circle;

| | |
|---|---|
| The first quadrant must coincide with the | 5th, 9th, 13th, 17th, |
| The second, with the | 6th, 10th, 14th, 18th, &c. |
| The third, with the | 7th, 11th, 15th, 19th, &c. |
| The fourth, with the | 8th, 12th, 16th, 20th, &c. |

205. If an arc extending in a certain direction from a given point, be considered *positive;* an arc extending from the same point, in an *opposite* direction, is to be considered *negative.* (Alg. 507.) Thus, if the arc extending from A to S (Fig. 36.) be positive; an arc extending from A to S‴ will be negative. The latter will not terminate in the *same quadrant* as the other; and the signs of the tabular lines must be accommodated to this circumstance. Thus the sine of AS will be positive; while that of AS‴ will be negative. (Art. 194.) When a greater arc is subtracted from a less, if the latter be positive, the *remainder* must be negative. (Alg. 58, 9.)

### TRIGONOMETRICAL FORMULÆ.

206. From the view which has here been taken of the changes in the trigonometrical lines, it will be easy to see, in

what parts of the circle each of them increases or decreases. But this does not determine their exact values, except at the extremities of the several quadrants. In the analytical investigations which are carried on by means of these lines, it is necessary to calculate the changes produced in them, by a given increase or diminution of the arcs to which they belong. In this there would be no difficulty, if the sines, tangents, &c. were *proportioned* to their arcs. But this is far from being the case. If an arc is doubled, its sine is *not* exactly doubled. Neither is its tangent or secant. We have to inquire, then, in what manner, the sine, tangent, &c. of one arc may be obtained, from those of other arcs already known.

The problem on which almost the whole of this branch of analysis depends, consists in deriving, from the sines and cosines of two given arcs, expressions for the sine and cosine of their *sum* and *difference*. For, by addition and subtraction, a few arcs may be so combined and varied, as to produce others of almost every dimension. And the expressions for the tangents and secants may be deduced from those of the sines and cosines.

### *Expressions for the* SINE *and* COSINE *of the* SUM *and* DIFFERENCE *of arcs.*

207. Let $a$=AH, the greater of the given arcs,
And $b$=HL=HD, the less. (Fig. 37.)

Then $a+b$=AH+HL=AL, the *sum* of the two arcs,
And $a-b$=AH−HD=AD, their *difference*.

Draw the chord DL, and the radius CH, which may be represented by R. As DH is, by construction, equal to HL; DQ is equal to QL, and therefore DL is perpendicular to CH. (Euc. 3. 3.) Draw DO, HN, QP, and LM, each perpendicular to AC; and DS and QB parallel to AC.

From the definitions of the sine and cosine, (Arts. 82, 9.) it is evident, that

The sine { of AH, that is, $\sin a$=HN,
of HL, $\sin b$=QL,
of AL, $\sin(a+b)$=LM,
of AD, $\sin(a-b)$=DO.

The cosine $\begin{cases} \text{of AH, that is, } \cos a = CN, \\ \text{of HL, } \cos b = CQ, \\ \text{of AL, } \cos(a+b) = CM, \\ \text{of AD, } \cos(a-b) = CO. \end{cases}$

The triangle CHN is obviously similar to CQP; and it is also similar to BLQ, because the sides of the one are perpendicular to those of the other, each to each. We have, then,

1. CH : CQ :: HN : QP, that is, R : cos $b$ :: sin $a$ : QP,
2. CH : QL :: CN : BL, R : sin $b$ :: cos $a$ : BL,
3. CH : CQ :: CN : CP, R : cos $b$ :: cos $a$ : CP,
4. CH : QL :: HN : QB, R : sin $b$ :: sin $a$ : QB.

Converting each of these proportions into an equation;

1. $QP = \frac{\sin a \cos b}{R}$* 3. $CP = \frac{\cos a \cos b}{R}$

2. $BL = \frac{\sin b \cos a}{R}$ 4. $QB = \frac{\sin a \sin b}{R}$

Then adding the first and second,

$$QP + BL = \frac{\sin a \cos b + \sin b \cos a}{R}$$

Subtracting the second from the first,

$$QP - BL = \frac{\sin a \cos b - \sin b \cos a}{R}$$

Subtracting the fourth from the third,

$$CP - QB = \frac{\cos a \cos b - \sin a \sin b}{R}$$

Adding the third and fourth,

$$CP + QB = \frac{\cos a \cos b + \sin a \sin b}{R}$$

* In these formulæ, the sign of multiplication is omitted; sin $a$ cos $b$ being put for sin $a \times \cos b$, that is, the product of the sine of $a$ into the cosine of $b$.

But it will be seen, from the figure, that

$$QP+BL=BM+BL=LM=\sin(a+b)$$
$$QP-BL=QP-QS=DO=\sin(a-b)$$
$$CP-QB=CP-PM=CM=\cos(a+b)$$
$$CP+QB=CP+SD=CO=\cos(a-b)$$

208. If then, for the first member of each of the four equations above, we substitute its value, we shall have,

$$\text{I. } \sin(a+b)=\frac{\sin a\cos b+\sin b\cos a}{R}$$

$$\text{II. } \sin(a-b)=\frac{\sin a\cos b-\sin b\cos a}{R}$$

$$\text{III. } \cos(a+b)=\frac{\cos a\cos b-\sin a\sin b}{R}$$

$$\text{IV. } \cos(a-b)=\frac{\cos a\cos b+\sin a\sin b}{R}$$

Or, multiplying both sides by R,

$$R\sin(a+b)=\sin a\cos b+\sin b\cos a$$
$$R\sin(a-b)=\sin a\cos b-\sin b\cos a$$
$$R\cos(a+b)=\cos a\cos b-\sin a\sin b$$
$$R\cos(a-b)=\cos a\cos b+\sin a\sin b$$

That is, the product of radius and the *sine* of the *sum* of two arcs, is equal to the product of the sine of the first arc into the cosine of the second + the product of the sine of the second into the cosine of the first.

The product of radius and the *sine* of the *difference* of two arcs, is equal to the product of the sine of the first arc into the cosine of the second — the product of the sine of the second into the cosine of the first.

The product of radius and the *cosine* of the *sum* of two arcs, is equal to the product of the cosines of the arcs — the product of their sines.

The product of radius and the *cosine* of the *difference* of two arcs, is equal to the product of the cosines of the arcs + the product of their sines.

These four equations may be considered as fundamental propositions, in what is called the *Arithmetic of Sines and Cosines, or Trigonometrical Analysis.*

*Expressions for the sine and cosine of a* DOUBLE *arc.*

209. When the sine and cosine of any arc are given, it is easy to derive from the equations in the preceding article, expressions for the sine and cosine of *double* that arc. As the two arcs $a$ and $b$ may be of any dimensions, they may be supposed to be *equal.* Substituting, then, $a$ for its equal $b$, the first and the third of the four preceding equations will become,

$$\text{R} \sin(a+a)=\sin a \cos a+\sin a \cos a$$
$$\text{R} \cos(a+a)=\cos a \cos a-\sin a \sin a$$

That is, writing $\sin^2 a$ for the square of the sine of $a$, and $\cos^2 a$ for the square of the cosine of $a$,

$$\text{I.}\quad \text{R} \sin 2a=2\sin a \cos a$$
$$\text{II.}\quad \text{R} \cos 2a=\cos^2 a-\sin^2 a.$$

*Expressions for the sine and cosine of* HALF *a given arc.*

210. The arc in the preceding equations, not being necessarily limited to any particular value, may be *half a*, as well as $a$. Substituting then $\frac{1}{2}a$ for $a$, we have,

$$\text{R} \sin a=2\sin \tfrac{1}{2}a \cos \tfrac{1}{2}a$$
$$\text{R} \cos a=\cos^2 \tfrac{1}{2}a-\sin^2 \tfrac{1}{2}a$$

Putting the sum of the squares of the sine and cosine equal to the square of radius, (Art. 94.) and inverting the members of the last equation,

$$\cos^2 \tfrac{1}{2}a+\sin^2 \tfrac{1}{2}a=\text{R}^2$$
$$\cos^2 \tfrac{1}{2}a-\sin^2 \tfrac{1}{2}a=\text{R} \cos a$$

If we *subtract* one of these from the other, the terms containing $\cos^2 \frac{1}{2}a$ will disappear; and if we *add* them, the terms containing $\sin^2 \frac{1}{2}a$ will disappear: therefore,

$$2\sin^2 \tfrac{1}{2}a=\text{R}^2-\text{R} \cos a$$
$$2\cos^2 \tfrac{1}{2}a=\text{R}^2+\text{R} \cos a$$

Dividing by 2, and extracting the root of both sides,

$$\text{I. } \sin\tfrac{1}{2}a=\sqrt{\tfrac{1}{2}R^2-\tfrac{1}{2}R\times\cos a}$$

$$\text{II. } \cos\tfrac{1}{2}a=\sqrt{\tfrac{1}{2}R^2+\tfrac{1}{2}R\times\cos a}$$

*Expressions for the sines and cosines of* MULTIPLE *arcs.*

211. In the same manner, as expressions for the sine and cosine of a *double* arc, are derived from the equations in art. 208; expressions for the sines and cosines of other multiple arcs may be obtained, by substituting successively $2a$, $3a$, &c. for $b$, or for $b$ and $a$ both. Thus,

$$\text{I. } \begin{cases} R\sin 3a=R\sin(a+2a)=\sin a\cos 2a+\sin 2a\cos a \\ R\sin 4a=R\sin(a+3a)=\sin a\cos 3a+\sin 3a\cos a \\ R\sin 5a=R\sin(a+4a)=\sin a\cos 4a+\sin 4a\cos a \end{cases}$$

&c.

$$\text{II. } \begin{cases} R\cos 3a=R\cos(a+2a)=\cos a\cos 2a-\sin a\sin 2a \\ R\cos 4a=R\cos(a+3a)=\cos a\cos 3a-\sin a\sin 3a \\ R\cos 5a=R\cos(a+4a)=\cos a\cos 4a-\sin a\sin 4a \end{cases}$$

&c,

*Expressions for the* PRODUCTS *of sines and cosines.*

212. Expressions for the products of sines and cosines may be obtained, by adding and subtracting the four equations in art. 208, viz.

$$R\sin(a+b)=\sin a\cos b+\sin b\cos a$$
$$R\sin(a-b)=\sin a\cos b-\sin b\cos a$$
$$R\cos(a+b)=\cos a\cos b-\sin a\sin b$$
$$R\cos(a-b)=\cos a\cos b+\sin a\sin b$$

Adding the first and second,

$$R\sin(a+b)+R\sin(a-b)=2\sin a\cos b$$

Subtracting the second from the first,

$$R\sin(a+b)-R\sin(a-b)=2\sin b\cos a$$

Adding the third and fourth,

$$R\cos(a-b)+R\cos(a+b)=2\cos a\cos b$$

Subtracting the third from the fourth,

$$R\cos(a-b)-R\cos(a+b)=2\sin a\sin b$$

Inverting the members of each of these equations, and dividing by 2, we have,

I. $\sin a \cos b = \frac{1}{2}R \sin(a+b) + \frac{1}{2}R \sin(a-b)$
II. $\sin b \cos a = \frac{1}{2}R \sin(a+b) - \frac{1}{2}R \sin(a-b)$
III. $\cos a \cos b = \frac{1}{2}R \cos(a-b) + \frac{1}{2}R \cos(a+b)$
IV. $\sin a \sin b = \frac{1}{2}R \cos(a-b) - \frac{1}{2}R \cos(a+b)$

213. If $b$ be taken equal to $a$, then $a+b=2a$, and $a-b=0$, the sine of which is 0, (Art. 201.); and the term in which this is a *factor*, is reduced to 0. (Alg. 112.) But the *cosine* of 0 is equal to radius, so that $R \times \cos 0 = R^2$. Reducing, then, the preceding equations,

| | |
|---|---|
| The first becomes | $\sin a \cos a = \frac{1}{2}R \sin 2a$ |
| The third, | $\cos^2 a = \frac{1}{2}R^2 + \frac{1}{2}R \cos 2a$ |
| The fourth, | $\sin^2 a = \frac{1}{2}R^2 - \frac{1}{2}R \cos 2a$ |

214. If $s$ be the *sum*, and $d$ the *difference* of two arcs, $\frac{1}{2}(s+d)$ will be equal to the greater, and $\frac{1}{2}(s-d)$ to the less. (Art. 153.) Substituting then, in the four equations in art. 212,

| | |
|---|---|
| $s$ for $a+b$, | $\frac{1}{2}(s+d)$ for $a$ |
| $d$ for $a-b$, | $\frac{1}{2}(s-d)$ for $b$, we have, |

I. $\sin \frac{1}{2}(s+d) \cos \frac{1}{2}(s-d) = \frac{1}{2}R (\sin s + \sin d)$
II. $\sin \frac{1}{2}(s-d) \cos \frac{1}{2}(s+d) = \frac{1}{2}R (\sin s - \sin d)$
III. $\cos \frac{1}{2}(s+d) \cos \frac{1}{2}(s-d) = \frac{1}{2}R (\cos d + \cos s)$
IV. $\sin \frac{1}{2}(s+d) \sin \frac{1}{2}(s-d) = \frac{1}{2}R (\cos d - \cos s)$

Or, making $R=1$,

I. $\sin(a+b) + \sin(a-b) = 2 \sin a \cos b$
II. $\sin(a+b) - \sin(a-b) = 2 \sin b \cos a$
III. $\cos(a-b) + \cos(a+b) = 2 \cos a \cos b$
IV. $\cos(a-b) - \cos(a+b) = 2 \sin a \sin b$

215. If radius be taken equal to 1, the two first equations in art. 208, are,

$$\sin(a+b) = \sin a \cos b + \sin b \cos a$$
$$\sin(a-b) = \sin a \cos b - \sin b \cos a$$

Multiplying these into each other,

$$\sin(a+b) \times \sin(a-b) = \sin^2 a \cos^2 b - \sin^2 b \cos^2 a$$

But by art. 94, if radius is 1,

$$\cos^2 b = 1 - \sin^2 b, \text{ and } \cos^2 a = 1 - \sin^2 a$$

Substituting, then, for $\cos^2 b$ and $\cos^2 a$, their values, multiplying the factors, and reducing the terms, we have,

$$\sin(a+b) \times \sin(a-b) = \sin^2 a - \sin^2 b$$

Or, because the difference of the squares of two quantities is equal to the product of their sum and difference, (Alg. 235.)

$$\sin(a+b) \times \sin(a-b) = (\sin a + \sin b) \times (\sin a - \sin b)$$

That is, the product of the sine of the sum of two arcs, into the sine of their difference; is equal to the product of the sum of their sines, into the difference of their sines.

*Expressions for the* TANGENTS *of arcs.*

216. Expressions for the *tangents* of arcs may be derived from those already obtained for the sines and cosines. By art. 93, proportion 1st,

$$R : \tan :: \cos : \sin$$

That is, $\frac{R}{\tan} = \frac{\cos}{\sin}$, and $\frac{\tan}{R} = \frac{\sin}{\cos}$, and $\tan = \frac{R \times \sin}{\cos}$,

$$\text{Thus } \tan(a+b) = \frac{R \sin(a+b)}{\cos(a+b)}.$$

If, for $\sin(a+b)$ and $\cos(a+b)$ we substitute their values, as given in art. 208, we shall have,

$$\tan(a \mp b) = \frac{R(\sin a \cos b \mp \sin b \cos a)}{\cos a \cos b \pm \sin a \sin b}.$$

217. Here, the value of the tangent of the sum of two arcs is expressed, in terms of the *sines* and *cosines* of the arcs. To exchange these for terms of the *tangents*, let the numerator and denominator of the second member of the equation be both divided by $\cos a \cos b$. This will not alter the value of the fraction. (Alg. 140.)

The *numerator*, divided by $\cos a \cos b$, is

$$\frac{R(\sin a \cos b \mp \sin b \cos a)}{\cos a \cos b} = R\left(\frac{\sin a}{\cos a} \mp \frac{\sin b}{\cos b}\right) = \tan a \mp \tan b$$

And the *denominator*, divided by cos $a$ cos $b$, is

$$\frac{\cos a \cos b \pm \sin a \sin b}{\cos a \cos b} = 1 \pm \frac{\sin a}{\cos a} \times \frac{\sin b}{\cos b} = 1 \pm \frac{\tan a}{R} \times \frac{\tan b}{R}$$

Therefore $\tan(a \mp b) = \dfrac{\tan a \mp \tan b}{1 \pm \dfrac{\tan a \tan b}{R^2}}$

The denominator of the fraction may be cleared of the divisor $R^2$, by multiplying both the numerator and denominator into $R^2$. And if we proceed in a similar manner to find the tangent of $a-b$, we shall have,

218. I. $\tan(a+b) = \dfrac{R^2(\tan a + \tan b)}{R^2 - \tan a \tan b}$

II. $\tan(a-b) = \dfrac{R^2(\tan a - \tan b)}{R^2 + \tan a \tan b}$.

If the arcs $a$ and $b$ are *equal*, then substituting $\frac{1}{2}a$, $a$, $2a$, $3a$, &c. as in Art. 210, 211.

$$\tan a = \tan(\tfrac{1}{2}a + \tfrac{1}{2}a) = \frac{R^2(2\tan\frac{1}{2}a)}{R^2 - \tan^2\frac{1}{2}a}$$

$$\tan 2a = \tan(a+a) = \frac{R^2(2\tan a)}{R^2 - \tan^2 a}$$

$$\tan 3a = \tan(a+2a) = \frac{R^2(\tan a + \tan 2a)}{R^2 - \tan a \tan 2a}, \text{ \&c.}$$

219. If we divide the first of the equations in Art. 214, by the second; we shall have, after rejecting $\frac{1}{2}R$ from the numerator and denominator, (Alg. 140.)

$$\frac{\sin\frac{1}{2}(s+d)\cos\frac{1}{2}(s-d)}{\sin\frac{1}{2}(s-d)\cos\frac{1}{2}(s+d)} = \frac{\sin s + \sin d}{\sin s - \sin d}$$

But the first member of this equation, (Alg. 155,) is equal to

$$\frac{\sin\frac{1}{2}(s+d)}{\cos\frac{1}{2}(s+d)} \times \frac{\cos\frac{1}{2}(s-d)}{\sin\frac{1}{2}(s-d)} = \frac{\tan\frac{1}{2}(s+d)}{R} \times \frac{R}{\tan\frac{1}{2}(s-d)}. \text{(Art. 216.)}$$

Therefore,

$$\frac{\sin s + \sin d}{\sin s - \sin d} = \frac{\tan\frac{1}{2}(s+d)}{\tan\frac{1}{2}(s-d)}$$

220. According to the notation in Art. 214, $s$ stands for the *sum* of two arcs, and $d$ for their *difference*. But it is evident that arcs may be taken, whose sum shall be equal to *any* arc $a$, and whose difference shall be equal to any arc $b$, provided that $a$ be *greater* than $b$. Substituting then, in the preceding equation $a$ for $s$, and $b$ for $d$,

$$\frac{\sin a+\sin b}{\sin a-\sin b}=\frac{\tan \frac{1}{2}(a+b)}{\tan \frac{1}{2}(a-b)}. \quad \text{Or,}$$

$$\sin a+\sin b : \sin a-\sin b :: \tan \tfrac{1}{2}(a+b) : \tan \tfrac{1}{2}(a-b.)$$

That is, *The sum of the sines of two arcs or angles, is to the difference of those sines; as the tangent of half the sum of the arcs or angles, to the tangent of half their difference.*

By Art. 143, the *sides of triangles* are as the sines of their opposite angles. It follows, therefore, from the preceding proposition, (Alg. 389.) that the sum of any two sides of a triangle, is to their difference; as the tangent of half the sum of the opposite angles, to the tangent of half their difference.

This is the second theorem applied to the solution of oblique angled triangles, which was *geometrically* demonstrated in Art. 144.

Expressions for the *cotangents* may be obtained by putting

$$\cot=\frac{R^2}{\tan} \text{ (Art. 93.)}$$

$$\text{Thus } \cot(a+b)=\frac{R^2}{\tan(a+b)}=\frac{R^2 \mp \tan a \tan b}{\tan a \pm \tan b} \text{ (Art. 218.)}$$

Substituting $\frac{R^2}{\cot a}$ for $\tan a$, and $\frac{R^2}{\cot b}$ for $\tan b$,

$$\cot(a+b)=\frac{R^2 \mp \frac{R^2}{\cot a}\times\frac{R^2}{\cot b}}{\frac{R^2}{\cot a}+\frac{R^2}{\cot b}}$$

Multiplying both the numerator and denominator by $\cot a \cot b$, dividing by $R^2$, and proceeding in the same manner, for $\cot(a-b)$ we have,

$$\text{I. } \cot(a+b)=\frac{\cot a \cot b-R^2}{\cot b+\cot a}$$

II. $\cot(a-b)=\frac{\cot a\cot b+R^2}{\cot b-\cot a}$

220. *b.* By comparing the expressions for the sines, and cosines, with those for the tangents and cotangents, a great variety of formulæ may be obtained. Thus the tangent of the sum or the difference of two arcs, may be expressed in terms of the cotangent.

Putting radius $=1$, we have (Arts. 93, 220.)

I. $\tan(a+b)=\frac{1}{\cot(a+b)}=\frac{\cot b+\cot a}{\cot a\cot b-1}$

II. $\tan(a-b)=\frac{1}{\cot(a-b)}=\frac{\cot b-\cot a}{\cot a\cot b+1}$

By art. 208,

$$\frac{\sin(a+b)}{\sin(a-b)}=\frac{\sin a\cos b+\sin b\cos a}{\sin a\cos b-\sin b\cos a}$$

Dividing the last member of the equation, in the first place by $\cos a\cos b$, as in art. 217, and then by $\sin a\sin b$, we have

$$\frac{\sin(a+b)}{\sin(a-b)}=\frac{\tan a+\tan b}{\tan a-\tan b}=\frac{\cot b+\cot a}{\cot b-\cot a}$$

In a similar manner, dividing the expressions for the cosines, in the first place by $\sin b\cos a$, and then by $\sin a\cos b$, we obtain

$$\frac{\cos(a+b)}{\cos(a-b)}=\frac{\cot b-\tan a}{\cot b+\tan a}=\frac{\cot a-\tan b}{\cot a+\tan b}$$

Dividing the numerator and denominator of the expression for the tangent of $a$, (Art. 218.) by $\tan\frac{1}{2}a$, we have

$$\tan a=\frac{2}{\cot\frac{1}{2}a-\tan\frac{1}{2}a}$$

These formulæ may be multiplied almost indefinitely, by combining the expressions for the sines, tangents, &c. The following are put down without demonstrations, for the exercise of the student.

$\tan \frac{1}{2}a = \cot\frac{1}{2}a - 2\cot a.$ $\tan \frac{1}{2}a = \frac{1-\cos a}{\sin a}$

$\tan \frac{1}{2}a = \frac{\sin a}{1+\cos a}$ $\tan^2 \frac{1}{2}a = \frac{1-\cos a}{1+\cos a}$

$\sin a = \frac{2\tan\frac{1}{2}a}{1+\tan^2\frac{1}{2}a}$ $\cos a = \frac{1-\tan^2\frac{1}{2}a}{1+\tan^2\frac{1}{2}a}$

$\cos a = \frac{\cot\frac{1}{2}a - \tan\frac{1}{2}a}{\cot\frac{1}{2}a + \tan\frac{1}{2}a}$ $\sin a = \frac{2}{\cot\frac{1}{2}a + \tan\frac{1}{2}a}$

$\sin a = \frac{1}{\cot\frac{1}{2}a - \cot a}$ $\sin a = \frac{1}{\cot a + \tan\frac{1}{2}a}$

---

Expression for the *area* of a triangle, in terms of the sides.

221. Let the sides of the triangle ABC (Fig. 23.) be expressed by $a$, $b$, and $c$, the perpendicular CD by $p$, the segment AD by $d$, and the area by S.

Then $a^2 = b^2 + c^2 - 2cd$, (Euc. 13. 2.)

Transposing and dividing by $2c$;

$d = \frac{b^2+c^2-a^2}{2c}$. Therefore $d^2 = \frac{(b^2+c^2-a^2)^2}{4c^2}$. (Alg. 223.)

By Euc. 47. 1, $p^2 = b^2 - d^2 = b^2 - \frac{(b^2+c^2-a^2)^2}{4c^2}$

Reducing the fraction, (Alg. 150.) and extracting the root of both sides,

$$p=\frac{\sqrt{4b^2c^2-(b^2+c^2-a^2)^2}}{2c}\text{*}$$

This gives the length of the *perpendicular*, in terms of the sides of the triangle. But the *area* is equal to the product of the base into half the perpendicular height. (Alg. 518.) that is,

$$S=\tfrac{1}{2}cp=\tfrac{1}{4}\sqrt{4b^2c^2-(b^2+c^2-a^2)^2}$$

Here we have an expression for the area, in terms of the sides. But this may be reduced to a form much better adapted to arithmetical computation. It will be seen, that the quantities $4b^2c^2$, and $(b^2+c^2-a^2)^2$ are both *squares;* and that the whole expression under the radical sign is the *difference* of these squares. But the difference of two squares is equal to the product of the sum and difference of their roots. (Alg. 235.) Therefore $4b^2c^2-(b^2+c^2-a^2)^2$ may be resolved into the two factors,

$$\begin{cases} 2bc+(b^2+c^2-a^2) \text{ which is equal to } (b+c)^2-a^2 \\ 2bc-(b^2+c^2-a^2) \text{ which is equal to } a^2-(b-c)^2 \end{cases}$$

Each of these also, as will be seen in the expressions on the right, is the difference of two squares; and may, on the same principle, be resolved into factors, so that,

$$\begin{cases} (b+c)^2-a^2=(b+c+a)\times(b+c-a) \\ a^2-(b-c)^2=(a+b-c)\times(a-b+c) \end{cases}$$

Substituting, then, these four factors, in the place of the quantity which has been resolved into them, we have,

$$S=\tfrac{1}{4}\sqrt{(b+c+a)\times(b+c-a)\times(a+b-c)\times(a-b+c)}$$

---

* The expression for the perpendicular is the same, when one of the angles is *obtuse*, as in Fig. 24. Let $AD=d$.

Then $a^2=b^2+c^2+2cd$. (Euc. 12. 2.) And $d=\dfrac{-b^2-c^2+a^2}{2c}$.

Therefore $d^2=\dfrac{(-b^2-c^2+a^2)^2}{4c^2}=\dfrac{(b^2+c^2-a^2)^2}{4c^2}$ (Alg. 219.)

And $p=\dfrac{\sqrt{4b^2c^2-(b^2+c^2-a^2)^2}}{2c}$ as above.

Here it will be observed, that all the three sides, $a$, $b$, and $c$, are in each of these factors.

Let $h=\frac{1}{2}(a+b+c)$ *half the sum* of the sides. Then

$$S=\sqrt{h\times(h-a)\times(h-b)\times(h-c)}$$

222. For finding the area of a triangle, then, when the three sides are given, we have this general rule;

*From half the sum of the sides, subtract each side severally; multiply together the half sum and the three remainders; and extract the square root of the product.*

# SECTION VIII.

## COMPUTATION OF THE CANON.

ART. 223. THE trigonometrical canon is a set of tables containing the sines, cosines, tangents, &c. to every degree and minute of the quadrant. In the computation of these tables, it is common to find, in the first place, the sine and cosine of *one minute;* and then, by successive additions and multiplications, the sines, cosines, &c. of the larger arcs. For this purpose, it will be proper to begin with an arc, whose sine or cosine is a known portion of the radius. The cosine of 60° is equal to *half radius.* (Art. 96. Cor.) A formula has been given, (Art. 210.) by which, when the cosine of an arc is known, the cosine of *half* that arc may be obtained.

By successive bisections of 60°, we have the arcs

| | |
|---|---|
| 30° | 0° 28′ 7″ 30‴ |
| 15° | 0 14 3 45 |
| 7° 30′ | 0 7 1 52 30 |
| 3° 45′ | 0 3 30 56 15 |
| 1° 52′ 30″ | 0 1 45 28 7 30 |
| 0° 56′ 15″ | 0 0′ 52″ 44‴ 3⁗ 45′′′′′ |

By formula II, art. 210,

$$\cos \tfrac{1}{2}a=\sqrt{\tfrac{1}{2}R^2+\tfrac{1}{2}R\times\cos a}$$

If the radius be 1, and if $a=60°$, $b=30°$, $c=15°$, &c.; then

Decimally

$$\cos b=\cos \tfrac{1}{2}a=\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\times\tfrac{1}{2}}=0.8660254$$

$$\cos c=\cos \tfrac{1}{2}b=\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\cos b}=0.9659258$$

$$\cos d=\cos \tfrac{1}{2}c=\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\cos c}=0.9914449$$

$$\cos e=\cos \tfrac{1}{2}d=\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\cos d}=0.9978589$$

Proceeding in this manner, by repeated extractions of the square root, we shall find the cosine of

$0° 0' 52'' 44''' 3'''' 45'''''$ to be 0.99999996732

And the sine (Art. 94.) $=\sqrt{1-\cos^2}=0.00025566346$

This, however, does not give the sine of *one minute* exactly. The arc is a little *less* than a minute. But the ratio of very small arcs to each other, is so nearly equal to the ratio of their sines, that one may be taken for the other, without sensible error. Now the circumference of a circle is divided into 21600 parts, for the arc of 1′; and into 24576, for the arc of $0° 0' 52'' 44''' 3'''' 45'''''$

Therefore,

21600 : 24576 :: 0.00025566346 : 0.0002908882,

which is the sine of 1 minute very nearly.*

And the cosine $=\sqrt{1-\sin^2}=0.9999999577$.

224. Having computed the sine and cosine of one minute, we may proceed, in a contrary order, to find the sines and cosines of *larger* arcs.

Making radius $=1$, and adding the two first equations in art. 208, we have

$$\sin(a+b)+\sin(a-b)=2\sin a\cos b$$

Adding the third and fourth,

$$\cos(a+b)+\cos(a-b)=2\cos a\cos b$$

Transposing $\sin(a-b)$ and $\cos(a-b)$

I. $\sin(a+b)=2\sin a\cos b-\sin(a-b)$
II. $\cos(a+b)=2\cos a\cos b-\cos(a-b)$

If we put $b=1'$, and $a=1'$, $2'$, $3'$, &c. successively, we shall have expressions for the sines and cosines of a series of arcs increasing regularly by one minute. Thus,

* See note H.

$$\sin(1'+1')=2\sin 1'\times\cos 1'-\sin 0= 0.0005817764,$$
$$\sin(2'+1')=2\sin 2'\times\cos 1'-\sin 1'=0.0008726645,$$
$$\sin(3'+1')=2\sin 3'\times\cos 1-'\sin 2'=0.0011635526,$$
&c. &c.

$$\cos(1'+1')=2\cos 1'\times\cos 1'-\cos 0 =0.9999993308$$
$$\cos(2'+1')=2\cos 2'\times\cos 1'-\cos 1'=0.9999996192$$
$$\cos(3'+1')=2\cos 3'\times\cos 1'-\cos 2'=0.9999993230$$
&c. &c.

The constant multiplier here, cos 1′ is 0.9999999577, which is equal to 1 − 0.0000000423.

225. Calculating, in this manner, the sines and cosines from 1 minute up to 30 degrees, we shall have also the sines and cosines from 60° to 90°. For the sines of arcs between 0° and 30°, are the *cosines* of arcs between 60° and 90°. And the cosines of arcs between 0° and 30°, are the *sines* of arcs between 60° and 90°. (Art. 104.)

226. For the interval between 30° and 60°, the sines and cosines may be obtained by subtraction merely. As twice the sine of 30° is equal to radius (Art. 96.); by making $a=30°$, the equation marked I, in Article 224 will become

$$\sin(30°+b)=\cos b-\sin(30°-b.)$$

And putting $b=1'$, $2'$, $3'$, &c. successively,

$$\sin(30° 1')=\cos 1'-\sin(29° 59')$$
$$(30° 2')=\cos 2'-\sin(29° 58')$$
$$(30° 3')=\cos 3'-\sin(29° 57')$$
&c. &c.

If the *sines* be calculated from 30° to 60°, the *cosines* will also be obtained. For the sines of arcs between 30° and 45°, are the cosines of arcs between 45° and 60°. And the sines of arcs between 45° and 60°, are the cosines of arcs between 30° and 45°.* (Art. 96.)

227. By the methods which have here been explained, the *natural* sines and cosines are found.

The *logarithms* of these, 10 being in each instance added to the index, will be the *artificial* sines and cosines by which trigonometrical calculations are commonly made. (Arts. 102, 3.)

228. The *tangents*, *cotangents*, *secants*, and *cosecants*, are easily derived from the sines and cosines. By Art. 93,

---

* See note I.

R : cos : : tan : sin        cos : R : : R : sec
R : sin : : cot : cos        sin : R : : R : cosec

Therefore,

The tangent$=\frac{R\times\sin}{\cos}$  The secant$=\frac{R^2}{\cos}$

The cotangent$=\frac{R\times\cos}{\sin}$  The cosecant$=\frac{R^2}{\sin}$

Or if the computations are made by *logarithms*,

The tangent$=10+\sin-\cos$,  The secant $=20-\cos$,
The cotangent$=10+\cos-\sin$,  The cosecant$=20-\sin$.

## SECTION IX.

### PARTICULAR SOLUTIONS OF TRIANGLES.*

Art. 231. Any triangle whatever may be solved, by the theorems in Sections III. IV. But there are other methods, by which, in certain circumstances, the calculations are rendered more expeditious, or more accurate results are obtained.

The differences in the *sines* of angles near 90°, and in the *cosines* of angles near 0°, are so small as to leave an uncertainty of several seconds in the result. The solutions should be varied, so as to avoid finding a very small angle by its cosine, or one near 90° by its sine.

The differences in the logarithmic *tangents* and *cotangents* are least at 45°, and increase towards each extremity of the quadrant. In no part of it, however, are they very small. In the tables which are carried to 7 places of decimals, the least difference for one second is 42. Any angle may be found within one second, by its tangent, if tables are used which are calculated to seconds.

But the differences in the logarithmic sines and tangents, within a few minutes of the beginning of the quadrant, and in cosines and tangents within a few minutes of 90°, though they are very large, are too *unequal* to allow of an exact determination of their corresponding angles, by taking *proportional parts* of the differences. Very small angles may be accurately found, from their sines and tangents, by the rules given in a note at the end.†

232. The following formulæ may be applied to *right angled* triangles, to obtain accurate results, by finding the sine or tangent of *half* an arc, instead of the whole.

In the triangle ABC (Fig. 20, Pl. II.) making AC radius,

$$AC : AB :: 1 : \text{Cos } A.$$

By conversion, (Alg. 389, 5.)

$$AC : AC - AB :: 1 : 1 - \text{Cos } A.$$

* Simson's, Woodhouse's, and Cagnoli's Trigonometry. † See note K.

Therefore,

$$\frac{AC-AB}{AC}=1-\cos A=2\sin^2\tfrac{1}{2}A. \text{ (Art. 210.)}$$

Or,

$$\text{Sin } \tfrac{1}{2} A=\sqrt{\left(\frac{AC-AB}{2AC}\right)}$$

Again, from the first proportion, adding and subtracting terms, (Alg. 389, 7.)

$$AC+AB : AC-AB :: 1+\cos A : 1-\cos A.$$

Therefore,

$$\frac{AC-AB}{AC+AB}=\frac{1-\cos A}{1+\cos A}=\tan^2\tfrac{1}{2}A. \text{ (Page 120.)}$$

Or,

$$\text{Tan } \tfrac{1}{2}A=\sqrt{\left(\frac{AC-AB}{AC+AB}\right)}$$

233. Sometimes, instead of having two parts of a right angled triangle given, in addition to the right angle; we have only one of the parts, and the *sum* or *difference* of two others. In such cases, solutions may be obtained by the following proportions.

By the preceding formulæ, and Arts. 140, 141,

1. $\text{Tan}^2 \tfrac{1}{2} A=\dfrac{AC-AB}{AC+AB}$

2. $BC^2=(AC-AB)(AC+AB)$

Multiplying these together, and extracting the root, we have,

$$\text{Tan } \tfrac{1}{2}A \times BC=AC-AB$$

Therefore,

I. $\text{Tan } \tfrac{1}{2}A : 1 :: AC-AB : BC$

That is, the tangent of half of one of the acute angles, is to 1, as the difference between the hypothenuse and the side at the angle, to the other side.

If, instead of multiplying, we *divide* the first equation above by the second, we have

$$\frac{\text{Tan } \tfrac{1}{2}A}{BC}=\frac{1}{AC+AB}$$

Therefore,

II. $1 : \tan \frac{1}{2}A :: AC+AB : BC$

Again, in the triangle ABC, Fig. 20,

$AB : BC : 1 :: \tan A$

Therefore,

$AB+BC : AB-BC :: 1+\tan A : 1-\tan A$

Or,

$$AB+BC : AB-BC :: 1 : \frac{1-\tan A}{1+\tan A}$$

By art. 218, one of the arcs being A, and the other 45°; the tangent of which is equal to radius, we have,

$$\text{Tan}\,(45^\circ - A) = \frac{1-\tan A}{1+\tan A}$$

Therefore,

III. $1 : \tan(45^\circ - A) :: AB+BC : AB-BC.$

That is, unity is to the tangent of the difference between 45° and one of the acute angles; as the sum of the perpendicular sides is to their difference.

Ex. 1. In a right angled triangle, if the difference of the hypothenuse and base be 64 feet, and the angle at the base $33\frac{3}{4}$°, what is the length of the perpendicular?

Ans. 211.

2. If the sum of the hypothenuse and base be 1853, and the angle at the base 37°; what is the perpendicular?

Ans. 620.

3. Given the sum of the base and perpendicular 128.4, and the angle at the base $41\frac{1}{4}$°, to find the sides.

$1 : \tan(45^\circ - 41\frac{1}{4}^\circ) :: 128.4 : 8.4,$

the difference of the base and perpendicular. Half the difference added to, and subtracted from, the half sum, gives the base 68.4, and the perpendicular 60.

4. Given the sum of the hypothenuse and perpendicular 83, and the angle at the perpendicular 40°, to find the base.

5. Given the difference of the hypothenuse and perpendicular 16.5, and the angle at the perpendicular $37\frac{1}{4}$°, to find the base.

6. Given the difference of the base and perpendicular 35, and the angle at the perpendicular $27\frac{1}{4}$°, to find the sides.

234. The following solutions may be applied to the *third* and *fourth* cases of *oblique* angled triangles; in one of which, two sides and the included angle are given, and in the other, the three sides. See pages 87 and 88.

## Case III.

In astronomical calculations, it is frequently the case, that two sides of a triangle are given by their *logarithms.* By the following proposition, the necessity of finding the corresponding natural numbers is avoided.

Theorem A. *In any plane triangle, of the two sides which include a given angle, the less is to the greater; as radius to the tangent of an angle greater than* 45°:

*And radius is to the tangent of the excess of this angle above* 45°; *as the tangent of half the sum of the opposite angles, to the tangent of half their difference.*

In the triangle ABC, (Fig. 39.) let the sides AC and AB, and the angle A be given. Through A draw DH perpendicular to AC. Make AD and AF each equal to AC, and AH equal to AB. And let HG be perpendicular to a line drawn from C through F.

$$\text{Then } AC : AB :: R : \tan ACH$$

$$\text{And } R : \tan(ACH-45^\circ) :: \tan\tfrac{1}{2}(ACB+B) : \tan\tfrac{1}{2}(ACB-B)$$

### *Demonstration.*

In the right angled triangle ACD, as the acute angles are subtended by the equal sides AC and AD, each is 45°. For the same reason, the acute angles in the triangle CAF are each 45°. Therefore, the angle DCF is a right angle, the angles GFH and GHF are each 45°, and the line GH is equal to GF and parallel to DC.

In the triangle ACH, if AC be radius, AH which is equal to AB will be the tangent of ACH. Therefore,

$$AC : AB :: R : \tan ACH.$$

In the triangle CGH, if CG be radius, GH which is equal to FG will be the tangent of HCG. Therefore,

$$R : \tan(ACH-45^\circ) :: CG : FG.$$

And, as GH and DC are parallel, (Euc. 2. 6.)

$$CG : FG :: DH : FH.$$

But DH is, by construction, equal to the *sum*, and FH to the *difference* of AC and AB. And by theorem II, (Art. 144.) the sum of the sides is to their difference; as the tangent of half the sum of the opposite angles, to the tangent of half their difference. Therefore,

$$R : \tan(ACH-45°) :: \tan\tfrac{1}{2}(ACB+B) : \tan\tfrac{1}{2}(ACB \backsim B)$$

Ex. In the triangle ABC, (Fig. 30.) given the angle A= 26° 14′, the side AC=39, and the side AB=53.

| | | | | |
|---|---|---|---|---|
| AC | 39 | 1.5910646 | R | 10. |
| AB | 53 | 1.7242759 | Tan 8° 39′ 9″ | 9.1823381 |
| R | | 10. | Tan ½(B+C)76° 53′ | 10.6326181 |
| Tan 53° 39′ 9″ | | 10.1332113 | Tan ½(B∽C)33° 8′ 50″ | 9.8149562 |

The same result is obtained here, as by theorem II, p. 75.

To find the required *side* in this third case, by the theorems in section IV, it is necessary to find, in the first place, an *angle* opposite one of the given sides. But the required side may be obtained, in a different way, by the following proposition.

THEOREM B. *In a plane triangle, twice the product of any two sides, is to the difference between the sum of the squares of those sides, and the square of the third side, as radius to the cosine of the angle included between the two sides.*

In the triangle ABC, (Fig. 23.) whose sides are $a$, $b$, and $c$.

$$2bc : b^2+c^2-a^2 :: R : \cos A$$

For in the right angled triangle ACD, $b : d :: R : \cos A$
Multiplying by $2c$, $2bc : 2dc :: R : \cos A$
But, by Euclid 13. 2, $2dc = b^2+c^2-a^2$
Therefore, $2bc : b^2+c^2-a^2 :: R : \cos A.$

The demonstration is the same, when the angle A is *obtuse*, as in the triangle ABC, (Fig. 24.) except that $a^2$ is *greater* than $b^2+c^2$; (Euc. 12. 2.) so that the cosine of A is *negative*. See art. 194.

From this theorem are derived expressions, both for the *sides* of a triangle, and for the cosines of the *angles*. Converting the last proportion into an equation, and proceeding in the same manner with the other sides and angles, we have the following expressions;

For the angles. For the sides.

$$\begin{cases} \text{Cos A}=\text{R}\times\dfrac{b^2+c^2-a^2}{2bc} \\ \text{Cos B}=\text{R}\times\dfrac{a^2+c^2-b^2}{2ac} \\ \text{Cos C}=\text{R}\times\dfrac{a^2+b^2-c^2}{2ab} \end{cases} \qquad \begin{cases} a=\sqrt{\left(b^2+c^2-\dfrac{2bc\cos A}{\text{R}}\right)} \\ b=\sqrt{\left(a^2+c^2-\dfrac{2ac\cos B}{\text{R}}\right)} \\ c=\sqrt{\left(a^2+b^2-\dfrac{2ab\cos C}{\text{R}}\right)} \end{cases}$$

These formulæ are useful, in many trigonometrical investigations; but are not well adapted to logarithmic computation.

## Case IV.

When the *three sides* of a triangle are given, the angles may be found, by either of the following theorems; in which $a$, $b$, and $c$ are the sides, A, B, and C, the opposite angles, and $h$=half the sum of the sides.

$$\text{Theorem C.} \begin{cases} \text{Sin A}=\dfrac{2\text{R}}{bc}\sqrt{h(h-a)(h-b)(h-c)} \\ \text{Sin B}=\dfrac{2\text{R}}{ac}\sqrt{h(h-a)(h-b)(h-c)} \\ \text{Sin C}=\dfrac{2\text{R}}{ab}\sqrt{h(h-a)(h-b)(h-c)} \end{cases}$$

The quantities under the radical sign are the same in all the equations.

In the triangle ACD, (Fig. 23.)

R : $b$ : : sin A : $p$. Therefore, sin A$\times b$=R$\times p$.

But $p=\dfrac{\sqrt{4b^2c^2-(b^2+c^2-a^2)^2}}{2c}$. (Art. 221, p. 121.)

This, by the reductions in page 122, becomes

$$p=\frac{\sqrt{2h\times2(h-a)\times2(h-b)\times2(h-c)}}{2c}$$

Substituting this value of $p$, and reducing,

$$\text{Sin A}=\frac{2R}{bc}\sqrt{h(h-a)(h-b)(h-c)}$$

The arithmetical calculations may be made, by adding the logarithms of the factors under the radical sign, dividing the sum by 2, and to the quotient, adding the logarithms of radius and 2, and the arithmetical complements of the logarithms of $b$ and $c$. (Arts. 39, 47, 59.)

Ex. Given $a=134$, $b=108$, and $c=80$, to find A, B, and C.

For the angle A.

| | | | |
|---|---|---|---|
| $h$ | 161 | log. | 2.2068259 |
| $h-a$ | 27 | log. | 1.4313638 |
| $h-b$ | 53 | log. | 1.7242759 |
| $h-c$ | 81 | log. | 1.9084850 |
| | | | 2)7.2709506 |
| | | | *3.6354753 |
| R×2 | | log. | 10.3010300 |
| | | | 13.9365053 |
| $b$ | 108 | a. c. | 7.9665762 |
| $c$ | 80 | a. c. | 8.0969100 |
| Sin A. | | | 9.9999915 |
| A=89° 38′ 31″ | | | |

For the angle B.

| | | | |
|---|---|---|---|
| | | | 13.9365053 |
| $a$ | 134 | a. c. | 7.8728952 |
| $c$ | 80 | a. c. | 8.0969100 |
| Sin B. | | | 9.9063105 |
| B=53° 42′ 9″ | | | |

For the angle C.

| | | | |
|---|---|---|---|
| | | | 13.9365053 |
| $a$ | 134 | a. c. | 7.8728952 |
| $b$ | 108 | a. c. | 7.9665762 |
| Sin C. | | | 9.7759767 |
| C=36° 39′ 20″ | | | |

THEOREM D.
$$\begin{cases}\text{Sin }\tfrac{1}{2}A=R\sqrt{\dfrac{(h-b)(h-c)}{bc}}\\ \text{Sin }\tfrac{1}{2}B=R\sqrt{\dfrac{(h-a)(h-c)}{ac}}\\ \text{Sin }\tfrac{1}{2}C=R\sqrt{\dfrac{(h-a)(h-b)}{ab}}\end{cases}$$

By Art. 210, $2\,\text{Sin}^2\tfrac{1}{2}A=R^2-R\times\cos A$.

Substituting for cos A, its value, as given in page 132,

$$2\text{Sin}^2\tfrac{1}{2}A=R^2-R^2\times\frac{b^2+c^2-a^2}{2bc}$$

* This is the logarithm of the *area* of the triangle. (Art. 222.)

But $R^2 = R^2 \times \frac{2bc}{2bc}$. And $-R^2 \times \frac{b^2+c^2-a^2}{2bc} = R^2 \times \frac{a^2-b^2-c^2}{2bc}$.

$$\text{Therefore } 2\text{Sin}^2 \tfrac{1}{2}A = R^2 \times \frac{2bc+a^2-b^2-c^2}{2bc}$$

But $2bc+a^2-b^2-c^2 = a^2-(b-c)^2 = (a+b-c)(a-b+c)$ (Alg. 235.)

Putting then $h=\frac{1}{2}(a+b+c)$, reducing, and extracting;

$$\text{Sin } \tfrac{1}{2}A = R\sqrt{\frac{(h-b)\,(h-c)}{bc}}$$

Ex. Given $a$, $b$, and $c$, as before, to find A and B.

For the angle A.

| | | | |
|---|---|---|---|
| $h-b$ | 53 | | 1.7242759 |
| $h-c$ | 81 | | 1.9084850 |
| $b$ | 108 | a. c. | 7.9665762 |
| $c$ | 80 | a. c. | 8.0969100 |
| | | | 2)19.6962471 |
| Sin $\frac{1}{2}$A | | | 9.8481235 |

A=89° 38′ 31″

For the angle B.

| | | | |
|---|---|---|---|
| $h-a$ | 27 | | 1.4313638 |
| $h-c$ | 81 | | 1.9084850 |
| $a$ | 134 | a. c. | 7.8728952 |
| $c$ | 80 | a. c. | 8.0969100 |
| | | | 2)19.3096540 |
| Sin $\frac{1}{2}$B | | | 9.6548270 |

B=53°42′ 9″

THEOREM E.
$$\begin{cases} \text{Cos } \tfrac{1}{2}A = R\sqrt{\frac{h\,(h-a)}{bc}} \\ \text{Cos } \tfrac{1}{2}B = R\sqrt{\frac{h\,(h-b)}{ac}} \\ \text{Cos } \tfrac{1}{2}C = R\sqrt{\frac{h\,(h-c)}{ab}} \end{cases}$$

By Art. 210, $2\text{Cos}^2 \tfrac{1}{2}A = R^2 + R \times \cos. A$.

Substituting and reducing, as in the demonstration of the last theorem,

$$2\text{Cos}^2 \tfrac{1}{2}A = R^2 \times \frac{2bc+b^2+c^2-a^2}{2bc} = R^2 \times \frac{(b+c+a)(b+c-a)}{2bc}$$

Putting $h=\frac{1}{2}(a+b+c)$, reducing and extracting,

$$\text{Cos } \tfrac{1}{2}A = R\sqrt{\frac{h\,(h-a)}{bc}}$$

Ex. Given the sides 134, 108, 80; to find B and C.

For the angle B.

| | | | |
|---|---|---|---|
| $h$ | 161 | | 2.2068259 |
| $h-b$ | 53 | | 1.7242759 |
| $a$ | 134 | a. c. | 7.8728952 |
| $c$ | 80 | a. c. | 8.0969100 |
| | | | 2)19.9009070 |
| Cos $\frac{1}{2}$B | | | 9.9504535 |

B=53° 42′ 9″

For the angle C.

| | | | |
|---|---|---|---|
| $h$ | 161 | | 2.2068259 |
| $h-c$ | 81 | | 1.9084850 |
| $a$ | 134 | a. c. | 7.8728952 |
| $b$ | 108 | a. c. | 7.9665762 |
| | | | 2)19.9547823 |
| Cos $\frac{1}{2}$C | | | 9.9773911 |

C=36° 39′ 20″

THEOREM F.
$$\begin{cases} \text{Tan } \tfrac{1}{2}A = R\sqrt{\dfrac{(h-b)(h-c)}{h(h-a)}} \\ \text{Tan } \tfrac{1}{2}B = R\sqrt{\dfrac{(h-a)(h-c)}{h(h-b)}} \\ \text{Tan } \tfrac{1}{2}C = R\sqrt{\dfrac{(h-a)(h-b)}{h(h-c)}} \end{cases}$$

The tangent is equal to the product of radius and the sine, divided by the cosine. (Art. 216.) By the last two theorems, then,

$$\text{Tan } \tfrac{1}{2}A = \frac{R \sin \frac{1}{2}A}{\cos \frac{1}{2}a} = R^2\sqrt{\frac{(h-b)(h-c)}{bc}} \div R\sqrt{\frac{h(h-a)}{bc}}$$

$$\text{That is, tan } \tfrac{1}{2}A = R\sqrt{\frac{(h-b)(h-c)}{h(h-a)}}$$

Ex. Given the sides as before, to find A and C.

For the angle A.

| | | | |
|---|---|---|---|
| $h-b$ | 53 | | 1.7242759 |
| $h-c$ | 81 | | 1.9084850 |
| $h-a$ | 27 | a. c. | 8.5686362 |
| $h$ | 161 | a. c. | 7.7931741 |
| | | | 2)19.9945712 |
| Tan $\frac{1}{2}$A | | | 9.9972856 |

A=89° 38′ 31″

For the angle C.

| | | | |
|---|---|---|---|
| $h-a$ | 27 | | 1.4313638 |
| $h-b$ | 53 | | 1.7242759 |
| $h-c$ | 81 | a. c. | 8.0915150 |
| $h$ | 161 | a. c. | 7.7931741 |
| | | | 2)19.0403288 |
| Tan $\frac{1}{2}$C | | | 9.5201644 |

C=36° 39′ 20″

The three last theorems give the angle required, *without ambiguity*. For the *half* of any angle must be less than 90°.

Of these different methods of solution, each has its advantages in particular cases. It is expedient to find an angle, sometimes by its sine, sometimes by its cosine, and sometimes by its tangent.

By the first of the four preceding theorems marked C, D, E, and F, the calculation is made for the *sine* of the *whole* angle; by the others, for the *sine*, *cosine*, or *tangent*, of *half* the angle. For finding an angle near 90°, each of the three last theorems is preferable to the first. In the example above, A would have been uncertain to several seconds, by theorem C, if the other two angles had not been determined also.

But for a very *small* angle, the first method has an advantage over the others. The third, by which the calculation is made for the *cosine* of half the required angle, is in this case the most defective of the four. The second will not answer well for an angle which is almost 180°. For the *half* of this is almost 90°; and near 90°, the differences of the sines are very small.

# NOTES.

## Note A. Page 1.

The *name* Logarithm is from λόγος, *ratio*, and ἀριθμὸς, *number*. Considering the ratio of $a$ to 1 as a *simple* ratio, that of $a^2$ to 1 is a *duplicate* ratio, of $a^3$ to 1 a *triplicate* ratio, &c. (Alg. 354.) Here the *exponents* or *logarithms* 2, 3, 4, &c. show how many times the simple ratio is *repeated as a factor*, to form the compound ratio. Thus the ratio of 100 to 1, is the *square* of the ratio of 10 to 1; the ratio of 1000 to 1, is the *cube* of the ratio of 10 to 1, &c. On this account, logarithms are called the *measures* of ratios; that is, of the ratios which different numbers bear to unity. See the Introduction to Hutton's Tables, and Mercator's Logarithmo-Technia, in Maseres' Scriptores Logarithmici.

## Note B. p. 4.

If 1 be added to $-.09691$, it becomes $1-.09691$, which is equal to $+.90309$. The decimal is here rendered positive, by *subtracting* the figures from 1. But it is made 1 too great. This is compensated, by adding $-1$ to the *integral* part of the logarithm. So that $-2-.09691=-3+.90309$.

In the same manner, the decimal part of any logarithm which is wholly negative, may be rendered positive, by subtracting it from 1, and adding $-1$ to the index. The subtraction is most easily performed, by taking the right hand significant figure from 10, and each of the other figures from 9. (Art. 55.)

On the other hand, if the index of a logarithm be negative, while the decimal part is positive; the whole may be rendered negative, by subtracting the decimal part from 1, and taking $-1$ from the index.

## Note C. p. 7.

It is common to *define* logarithms to be a series of numbers in arithmetical progression, corresponding with another series in geometrical progression. This is calculated to perplex the learner, when, upon opening the tables, he finds that the natural numbers, as they stand there, instead of being in *geometrical*, are in *arithmetical* progression; and that the logarithms are *not* in arithmetical progression.

It is true, that a geometrical series may be obtained, by taking out, here and there, a few of the natural numbers; and that the logarithms of these will form an arithmetical series. But the definition is not applicable to the whole of the numbers and logarithms, as they stand in the tables.

The supposition that positive and negative numbers have the same series of logarithms, (p. 7.) is attended with some theoretical difficulties. But these do not affect the practical rules for calculating by logarithms.

## Note D. p. 43.

To revert a series, of the form

$$x=an+bn^2+cn^3+dn^4+en^5+, \&c.$$

that is, to find the value of $n$, in terms of $x$, *assume* a series, with indeterminate co-efficients, (Alg. 490. *b.*)

$$\text{Let } n=Ax+Bx^2+Cx^3+Dx^4+Ex^5+, \&c.$$

Finding the powers of this value of $n$, by multiplying the series into itself, and arranging the several terms according to the powers of $x$; we have

$$n^2=A^2x^2+2ABx^3+\left.\begin{matrix}+2AC\\+B^2\end{matrix}\right\}x^4\left.\begin{matrix}+2BC\\+2AD\end{matrix}\right\}x^5+, \&c.$$

$$n^3=\quad A^3x^3+3A^2Bx^4\left.\begin{matrix}+3A^2C\\+3AB^2\end{matrix}\right\}x^5+, \&c.$$

$$n^4=\quad A^4x^4+4A^3Bx^5+, \&c.$$

$$n^5=\quad A^5x^5+, \&c.$$

Substituting these values, for $n$ and its powers, in the first series above, we have

$$x=\left\{\begin{array}{l} aAx + \left.\begin{array}{l} aB \\ +bA^2 \end{array}\right\} x^2 + \left.\begin{array}{l} aC \\ +2bAB \\ +cA^3 \end{array}\right\} x^3 + \left.\begin{array}{l} aD \\ +2bAC \\ +bB^2 \\ +3cA^2B \\ +dA^4 \end{array}\right\} x^4 + \left.\begin{array}{l} aE \\ +2bBC \\ +2bAD \\ +3cA^2C \\ +3cAB^2 \\ +4dA^3B \\ +eA^5 \end{array}\right\} x^5 \end{array}\right.$$

Transposing $x$, and making the co-efficients of the several powers of $x$ each equal to 0, we have

$aA-1=0,$
$aB+bA^2=0,$
$aC+2bAB+cA^3=0,$
$aD+2bAC+bB^2+3cA^2B+dA^4=0,$
$aE+2bBC+2bAD+3cA^2C+3cAB^2+4dA^3B+eA^5=0.$

And reducing the equations,

$$A=\frac{1}{a}$$

$$B=-\frac{b}{a^3}$$

$$C=\frac{2b^2-ac}{a^5}$$

$$D=-\frac{5b^3-5abc+a^2d}{a^7}$$

$$E=b\frac{14b^4-21ab^2c+3a^2c^2+6a^2bd-a^3e}{a^9}$$

These are the values of the co-efficients A, B, C, &c. in the assumed series

$$n=Ax+Bx^2+Cx^3+Dx^4+Ex^5+, \&c.$$

Applying these results to the logarithmic series; (Art. 66. p. 43.)

$$x=n-\tfrac{1}{2}n^2+\tfrac{1}{3}n^3-\tfrac{1}{4}n^4+\tfrac{1}{5}n^5-\&c.$$

in which

$a=1, b=-\frac{1}{2}, c=\frac{1}{3}, d=-\frac{1}{4}, e=\frac{1}{5}$,

we have, in the inverted series

$n=Ax+Bx^2+Cx^3+Dx^4+Ex^5+$, &c.

$A=\frac{1}{a}=1$ $\quad D=\frac{1}{2.3.4}$

$B=-b=\frac{1}{2}$

$C=2b^2-ac=\frac{1}{2.3}$ $\quad E=\frac{1}{2.3.4.5}$

Therefore,

$$n=x+\frac{x^2}{2}+\frac{x^3}{2.3}+\frac{x^4}{2.3.4}+\frac{x^5}{2.3.4.5}+,\&c.$$

## Note E. p. 50.

According to the scheme lately introduced into France, of dividing the denominations of weights, measures, &c. into tenths, hundredths, &c. the fourth part of a circle is divided into 100 degrees, a degree into 100 minutes, a minute into 100 seconds, &c. The whole circle contains 400 of these degrees; a plane triangle 200. If a right angle be taken for the measuring *unit;* degrees, minutes and seconds, may be written as decimal fractions. Thus 36° 5′ 49″ is 0.360549.

According to the French division $\left\{\begin{array}{l}10^\circ=9^\circ\\ 100'=54'\\ 1000''=324''\end{array}\right\}$ English.

## Note F. p. 82.

If the perpendicular be drawn from the angle opposite the longest side, it will always fall *within* the triangle; because the other two angles must, of course, be acute. But if one of the angles at the base be *obtuse,* the perpendicular will fall *without* the triangle, as CP, (Fig. 38.)

In this case, the side on which the perpendicular falls, is to the sum of the other two; as the difference of the latter, to the *sum* of the segments made by the perpendicular.

The demonstration is the same, as in the other case, except that AH=BP+PA, instead of BP−PA.

Thus in the circle BDHL (Fig. 38.) of which C is the center,

$$AB \times AH = AL \times AD; \text{ therefore } AB : AD :: AL :: AH.$$

$$\begin{aligned}&\text{But } AD = CD + CA = CB + CA\\&\text{And } AL = CL - CA = CB - CA\\&\text{And } AH = HP + PA = BP + PA.\end{aligned}$$

Therefore

$$AB : CB + CA :: CB - CA : BP + PA$$

When the three sides are given, it may be known whether one of the angles is obtuse. For any angle of a triangle is obtuse or acute, according as the square of the side subtending the angle is *greater*, or *less*, than the sum of the squares of the sides containing the angle. (Euc. 12, 13. 2.)

### Note G. p. 104.

Gunter's *Sliding Rule*, is constructed upon the same principle as his scale, with the addition of a slider, which is so contrived as to answer the purpose of a pair of compasses, in working proportions, multiplying, dividing, &c. The lines on the *fixed part* are the same as on the scale. The *slider* contains two lines of numbers, a line of logarithmic sines, and a line of logarithmic tangents.

To *multiply* by this, bring 1 on the slider, against one of the factors on the fixed part; and against the other factor on the slider, will be the product on the fixed part. To divide, bring the divisor on the slider, against the dividend on the fixed part; and against 1 on the slider, will be the quotient on the fixed part. To work a *proportion*, bring the first term on the slider, against one of the middle terms on the fixed part; and against the other middle term on the slider, will be the fourth term on the fixed part. Or the first term may be taken on the fixed part; and then the fourth term will be found on the slider.

Another instrument frequently used in trigonometrical constructions, is

## THE SECTOR.

This consists of two equal scales movable about a point as a center. The lines which are drawn on it are of two kinds; some being parallel to the sides of the instrument, and others diverging from the central point, like the radii of a circle. The latter are called the *double* lines, as each is repeated upon the two scales. The *single* lines are of the same nature, and have the same use, as those which are put upon the common scale; as the lines of equal parts, of chords, of latitude, &c. on one face; and the logarithmic lines of numbers, of sines, and of tangents, on the other.

The *double* lines are

| | |
|---|---|
| A line of *Lines*, or equal parts, marked | Lin. or L. |
| A line of *Chords*, | Cho. or C. |
| A line of natural *Sines*, | Sin. or S. |
| A line of natural *Tangents* to 45°, | Tan. or T. |
| A line of tangents *above* 45°, | Tan. or T. |
| A line of natural *Secants*, | Sec. or S. |
| A line of *Polygons*, | Pol. or P. |

The double lines of *chords*, of *sines*, and of *tangents* to 45°, are all of the same radius; beginning at the central point, and terminating near the other extremity of each scale; the chords at 60°, the sines at 90°, and the tangents at 45°. (See Art. 95.) The line of *lines* is also of the same length, containing ten equal parts which are numbered, and which are again subdivided. The radius of the lines of secants, and of tangents above 45°, is about one fourth of the length of the other lines. From the end of the radius, which for the secants is at 0, and for the tangents at 45°, these lines extend to between 70° and 80°. The line of polygons is numbered 4, 5, 6, &c. from the extremity of each scale, towards the center.

The simple principle on which the utility of these several pairs of lines depends is this, that *the sides of similar triangles are proportional.* (Euc. 4. 6.) So that sines, tangents, &c. are furnished *to any radius*, within the extent of the opening of the two scales. Let AC and AC′ (Fig. 40.) be any pair of lines on the sector, and AB and AB′ equal portions of these lines. As AC and AC′ are equal, the triangle ACC′ is isosceles, and similar to ABB′. Therefore,

$$AB : AC :: BB' : CC'.$$

Distances measured from the center on either scale, as AB and AC, are called *lateral distances*. And the distances between corresponding points of the two scales, as BB′ and CC′, are called *transverse distances*.

Let AC and CC′ be radii of two circles. Then if AB be the chord, sine, tangent, or secant, of any number of degrees in one; BB′ will be the chord, sine, tangent, or secant, of the same number of degrees in the other. (Art. 119.) Thus, to find the *chord* of 30°, to a radius of four inches, open the sector so as to make the transverse distance from 60 to 60, on the lines of chords, four inches; and the distance from 30 to 30, on the same lines, will be the chord required. To find the *sine* of 28°, make the distance from 90 to 90, on the lines of sines, equal to radius; and the distance from 28 to 28 will be the sine. To find the *tangent* of 37°, make the distance from 45 to 45, on the lines of tangents, equal to radius; and the distance from 37 to 37 will be the tangent. In finding *secants*, the distance from 0 to 0 must be made radius. (Art. 201.)

To lay down an *angle* of 34°, describe a circle, of any convenient radius, open the sector, so that the distance from 60 to 60 on the lines of chords shall be equal to this radius, and to the circle apply a chord equal to the distance from 34 to 34. (Art. 161.) For an angle above 60°, the chord of *half* the number of degrees may be taken, and applied *twice* on the arc, as in art. 161.

The line of *polygons* contains the chords of arcs of a circle which is divided into equal portions. Thus the distances from the center of the sector to 4, 5, 6, and 7, are the chords of $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, and $\frac{1}{7}$ of a circle. The distance 6 is the radius. (Art. 95.) This line is used to make a regular polygon, or to inscribe one in a given circle. Thus, to make a *pentagon* with the transverse distance from 6 to 6 for radius, describe a circle, and the distance from 5 to 5 will be the length of one of the sides of a pentagon inscribed in that circle.

The line of *lines* is used to divide a line into equal or proportional parts, to find fourth proportionals, &c. Thus, to divide a line into 7 equal parts, make the length of the given line the transverse distance from 7 to 7, and the distance from 1 to 1 will be one of the parts. To find $\frac{3}{5}$ of a line, make the transverse distance from 5 to 5 equal to the given line; and the distance from 3 to 3 will be $\frac{3}{5}$ of it.

In working the *proportions in trigonometry* on the sector, the lengths of the sides of triangles are taken from the line of lines, and the degrees and minutes from the lines of sines, tangents, or secants. Thus in art. 135, ex. 1,

$$35 : R :: 26 : \sin 48°.$$

To find the fourth term of this proportion by the sector, make the lateral distance 35 on the line of lines, a transverse distance from 90 to 90 on the lines of sines; then the lateral distance 26 on the line of lines, will be the transverse distance from 48 to 48 on the lines of sines.

For a more particular account of the construction and uses of the Sector, see Stone's edition of Bion on Mathematical Instruments, Hutton's Dictionary, and Robertson's Treatise on Mathematical Instruments.

## Note H. p. 124.

The error in supposing that arcs less than 1 minute are proportional to their sines, cannot affect the first ten places of decimals. Let AB and AB′ (Fig. 41.) each equal 1 minute. The tangents of these arcs BT and B′T are equal, as are also the sines BS and B′S. The arc BAB′ is greater than BS+B′S, but less than BT+B′T. Therefore BA is greater than BS, but less than BT: that is, *the difference between the sine and the arc is less than the difference between the sine and the tangent.*

| | |
|---|---|
| Now the sine of 1 minute is | 0.000290888216 |
| And the tangent of 1 minute is | 0.000290888204 |
| The difference is | 0.0000000000012 |

The difference between the sine and the arc of 1 minute is less than this; and the error in supposing that the sines of 1′, and of 0′ 52″ 44‴ 3⁗ 45′′′′′ are proportional to their arcs, as in art. 223, is still less.

## Note I. p. 125.

There are various ways in which sines and cosines may be more *expeditiously* calculated, than by the method which is given here. But as we are already supplied with accurate trigonometrical tables, the computation of the canon is, to the great body of our students, a subject of speculation, rather than of practical utility. Those who wish to enter into a minute examination of it, will of course consult the treatises in which it is particularly considered.

There are also numerous formulæ of *verification*, which are used to detect the errors with which any part of the calculation is liable to be affected. For these, see Legendre's and Woodhouse's Trigonometry, Lacroix's Differential Calculus, and particularly Euler's Analysis of Infinites.

## Note K. p. 127.

The following rules for finding the sine or tangent of a very small arc, and, on the other hand, for finding the arc from its sine or tangent, are taken from Dr. Maskelyne's Introduction to Taylor's Logarithms.

### *To find the logarithmic* SINE *of a very small arc.*

From the sum of the constant quantity 4.6855749, and the logarithm of the given arc reduced to seconds and decimals, subtract one third of the arithmetical complement of the logarithmic cosine.

### *To find the logarithmic* TANGENT *of a very small arc.*

To the sum of the constant quantity 4.6855749, and the logarithm of the given arc reduced to seconds and decimals, add two thirds of the arithmetical complement of the logarithmic cosine.

### *To find a small arc from its logarithmic* SINE.

To the sum of the constant quantity 5.3144251, and the given logarithmic sine, add one third of the arithmetical complement of the logarithmic cosine. The remainder diminished by 10, will be the logarithm of the number of seconds in the arc.

### *To find a small arc from its logarithmic* TANGENT.

From the sum of the constant quantity 5.3144251, and the given logarithmic tangent, subtract two thirds of the arithmetical complement of the logarithmic cosine. The remainder diminished by 10, will be the logarithm of the number of seconds in the arc.

For the demonstration of these rules, see Woodhouse's Trigonometry, p. 189.

A TABLE OF

# NATURAL SINES AND TANGENTS;

TO EVERY TEN MINUTES OF A DEGREE.

---

IF the given angle is less than 45°, look for the title of the column, at the *top* of the page; and for the degrees and minutes, on the *left*. But if the angle is between 45° and 90°, look for the title of the column, at the *bottom*; and for the degrees and minutes, on the *right*.

| D. M. | Sine. | Tangent. | Cotangent. | Cosine. | D. M. |
|---|---|---|---|---|---|
| 0° 0′ | 0.0000000 | 0.0000000 | Infinite. | 1.0000000 | 90° 0′ |
| 10 | 0029089 | 0029089 | 343.77371 | 0.9999958 | 50 |
| 20 | 0058177 | 0058178 | 171.88540 | 9999831 | 40 |
| 30 | 0087265 | 0087269 | 114.58865 | 9999619 | 30 |
| 40 | 0116353 | 0116361 | 85.939791 | 9999323 | 20 |
| 0° 50′ | 0145439 | 0145454 | 68.750087 | 9998942 | 89° 10′ |
| 1° 0′ | 0.0174524 | 0.0174551 | 57.289962 | 0.9998477 | 89° 0′ |
| 10 | 0203608 | 0203650 | 49.103881 | 9997927 | 50 |
| 20 | 0232690 | 0232753 | 42.964077 | 9997292 | 40 |
| 30 | 0261769 | 0261859 | 38.188459 | 9996573 | 30 |
| 40 | 0290847 | 0290970 | 34.367771 | 9995770 | 20 |
| 1° 50′ | 0319922 | 0320086 | 31.241577 | 9994881 | 88° 10′ |
| 2° 0′ | 0.0348995 | 0.0349208 | 28.636253 | 0.9993908 | 88° 0′ |
| 10 | 0378065 | 0378335 | 26.431600 | 9992851 | 50 |
| 20 | 0407131 | 0407469 | 24.541758 | 9991709 | 40 |
| 30 | 0436194 | 0436609 | 22.903766 | 9990482 | 30 |
| 40 | 0465253 | 0465757 | 21.470401 | 9989171 | 20 |
| 2° 50′ | 0494308 | 0494913 | 20.205553 | 9987775 | 87° 10′ |
| 3° 0′ | 0.0523360 | 0.0524078 | 19.081137 | 0.9986295 | 87° 0′ |
| 10 | 0552406 | 0553251 | 18.074977 | 9984731 | 50 |
| 20 | 0581448 | 0582434 | 17.169337 | 9983082 | 40 |
| 30 | 0610485 | 0611626 | 16.349855 | 9981348 | 30 |
| 40 | 0639517 | 0640829 | 15.604784 | 9979530 | 20 |
| 3° 50′ | 0668544 | 0670043 | 14.924417 | 9977627 | 86° 10′ |
| 4° 0′ | 0.0697565 | 0.0699268 | 14.300666 | 0.9975641 | 86° 0′ |
| 10 | 0726580 | 0728505 | 13.726738 | 9973569 | 50 |
| 20 | 0755589 | 0757755 | 13.196883 | 9971413 | 40 |
| 30 | 0784591 | 0787017 | 12.706205 | 9969173 | 30 |
| 40 | 0813587 | 0816293 | 12.250505 | 9966849 | 20 |
| 4° 50′ | 0842576 | 0845583 | 11.826167 | 9964440 | 85° 10′ |
| 5° 0′ | 0.0871557 | 0.0874887 | 11.430052 | 0.9961947 | 85° 0′ |
| 10 | 0900532 | 0904206 | 11.059431 | 9959370 | 50 |
| 20 | 0929499 | 0933540 | 10.711913 | 9956708 | 40 |
| 30 | 0958458 | 0962890 | 10.385397 | 9953962 | 30 |
| 40 | 0987408 | 0992257 | 10.078031 | 9951132 | 20 |
| 5° 50′ | 1016351 | 1021641 | 9.7881732 | 9948217 | 84° 10′ |
| D. M. | Cosine. | Cotangent. | Tangent. | Sine. | D. M. |

| D. M. | Sine. | Tangent. | Cotangent. | Cosine. | D. M. |
|---|---|---|---|---|---|
| 6° 0′ | 0.1045285 | 0.1051042 | 9.5143645 | 0.9945219 | 84° 0′ |
| 10 | 1074210 | 1080462 | 9.2553035 | 9942136 | 50 |
| 20 | 1103126 | 1109899 | 9.0098261 | 9938969 | 40 |
| 30 | 1132032 | 1139356 | 8.7768874 | 9935719 | 30 |
| 40 | 1160929 | 1168832 | 8.5555468 | 9932384 | 20 |
| 6° 50′ | 1189816 | 1198329 | 8.3449558 | 9928065 | 83° 10′ |
| 7° 0′ | 0.1218693 | 0.1227846 | 8.1443464 | 0.9925462 | 83° 0′ |
| 10 | 1247560 | 1257384 | 7.9530224 | 9921874 | 50 |
| 20 | 1276416 | 1286943 | 7.7703506 | 9918204 | 40 |
| 30 | 1305262 | 1316525 | 7.5957541 | 9914449 | 30 |
| 40 | 1334096 | 1346129 | 7.4287064 | 9910610 | 20 |
| 7° 50′ | 1362919 | 1375757 | 7.2687255 | 9906687 | 82° 10′ |
| 8° 0′ | 0.1391731 | 0.1404085 | 7.1153697 | 0.9902681 | 82° 0′ |
| 10 | 1420531 | 1435084 | 6.9682335 | 9898590 | 50 |
| 20 | 1449319 | 1464784 | 6.8269437 | 9894416 | 40 |
| 30 | 1478094 | 1494510 | 6.6911562 | 9890159 | 30 |
| 40 | 1506857 | 1524262 | 6.5605538 | 9885817 | 20 |
| 8° 50′ | 1535607 | 1554040 | 6.4348428 | 9881392 | 81° 10′ |
| 9° 0′ | 0.1564345 | 0.1583844 | 6.3137515 | 0.9876883 | 81° 0′ |
| 10 | 1593069 | 1613677 | 6.1970279 | 9872291 | 50 |
| 20 | 1621779 | 1643537 | 6.0844381 | 9867615 | 40 |
| 30 | 1650476 | 1673426 | 5.9757644 | 9862856 | 30 |
| 40 | 1679159 | 1703344 | 5.8708042 | 9858013 | 20 |
| 9° 50′ | 1707828 | 1733292 | 5.7693688 | 9853087 | 80° 10′ |
| 10° 0′ | 0.1736482 | 0.1763270 | 5.6712818 | 0.9848078 | 80° 0′ |
| 10 | 1765121 | 1793279 | 5.5763786 | 9842985 | 50 |
| 20 | 1793746 | 1823319 | 5.4845052 | 9837808 | 40 |
| 30 | 1822355 | 1853390 | 5.3955172 | 9832549 | 30 |
| 40 | 1850949 | 1883495 | 5.3092793 | 9827206 | 20 |
| 10° 50′ | 1879528 | 1913632 | 5.2256647 | 9821781 | 79° 10′ |
| 11° 0′ | 0.1908090 | 0.1943803 | 5.1445540 | 0.9816272 | 79° 0′ |
| 10 | 1936636 | 1974008 | 5.0658352 | 9810680 | 50 |
| 20 | 1965166 | 2004248 | 4.9894027 | 9805005 | 40 |
| 30 | 1993679 | 2034523 | 4.9151570 | 9799247 | 30 |
| 40 | 2022176 | 2064834 | 4.8430045 | 9793406 | 20 |
| 11° 50′ | 2050655 | 2095181 | 4.7728568 | 9787483 | 78° 10′ |
| D. M. | Cosine. | Cotangent. | Tangent. | Sine. | D. M. |

| D. M. | Sine. | Tangent. | Cotangent. | Cosine. | D. M. |
|---|---|---|---|---|---|
| 12° 0′ | 0.2079117 | 0.2125566 | 4.7046301 | 0.9781476 | 78° 0′ |
| 10 | 2107561 | 2155988 | 4.6382457 | 9775387 | 50 |
| 20 | 2135988 | 2186448 | 4.5736287 | 9769215 | 40 |
| 30 | 2164396 | 2216947 | 4.5107085 | 9762960 | 30 |
| 40 | 2192786 | 2247485 | 4.4494181 | 9756623 | 20 |
| 12° 50′ | 2221158 | 2278063 | 4.3896940 | 9750203 | 77° 10′ |
| 13° 0′ | 0.2249511 | 0.2308682 | 4.3314759 | 0.9743701 | 77° 0′ |
| 10 | 2277844 | 2339342 | 4.2747066 | 9737116 | 50 |
| 20 | 2306159 | 2370044 | 4.2193318 | 9730449 | 40 |
| 30 | 2334454 | 2400788 | 4.1652998 | 9723699 | 30 |
| 40 | 2362729 | 2431575 | 4.1125614 | 9716867 | 20 |
| 13° 50′ | 2390984 | 2462405 | 4.0610700 | 9709953 | 76° 10′ |
| 14° 0′ | 0.2419219 | 0.2493280 | 4.0107809 | 0.9702957 | 76° 0′ |
| 10 | 2447433 | 2524200 | 3.9616518 | 9695879 | 50 |
| 20 | 2475627 | 2555165 | 3.9136420 | 9688719 | 40 |
| 30 | 2503800 | 2586176 | 3.8667131 | 9681476 | 30 |
| 40 | 2531952 | 2617234 | 3.8208281 | 9674152 | 20 |
| 14° 50′ | 2560082 | 2648339 | 3.7759519 | 9666746 | 75° 10′ |
| 15° 0′ | 0.2588190 | 0.2679492 | 3.7320508 | 0.9659258 | 75° 0′ |
| 10 | 2616277 | 2710694 | 3.6890927 | 9651689 | 50 |
| 20 | 2644342 | 2741945 | 3.6470467 | 9644037 | 40 |
| 30 | 2672384 | 2773245 | 3.6058835 | 9636305 | 30 |
| 40 | 2700403 | 2804597 | 3.5655749 | 9628490 | 20 |
| 15° 50′ | 2728400 | 2835999 | 3.5260938 | 9620594 | 74° 10′ |
| 16° 0′ | 0.2756374 | 0.2867454 | 3.4874144 | 0.9612617 | 74° 0′ |
| 10 | 2784324 | 2898961 | 3.4495120 | 9604558 | 50 |
| 20 | 2812251 | 2930521 | 3.4123626 | 9596418 | 40 |
| 30 | 2840153 | 2962135 | 3.3759434 | 9588197 | 30 |
| 40 | 2868032 | 2993803 | 3.3402326 | 9579895 | 20 |
| 16° 50′ | 2895887 | 3025527 | 3.3052091 | 9571512 | 73° 10′ |
| 17° 0′ | 0.2923717 | 0.3057307 | 3.2708526 | 0.9563048 | 73° 0′ |
| 10 | 2951522 | 3089143 | 3.2371438 | 9554502 | 50 |
| 20 | 2979303 | 3121036 | 3.2040638 | 9545876 | 40 |
| 30 | 3007058 | 3152988 | 3.1715948 | 9537170 | 30 |
| 40 | 3034788 | 3184998 | 3.1397194 | 9528382 | 20 |
| 17° 50 | 3062492 | 3217067 | 3.1084210 | 9519514 | 72° 10′ |
| D. M. | Cosine. | Cotangent. | Tangent. | Sine. | D. M. |

| D. M. | Sine. | Tangent. | Cotangent. | Cosine. | D. M. |
|---|---|---|---|---|---|
| 18° 0′ | 0.3090170 | 0.3249197 | 3.0776835 | 0.9510565 | 72° 0′ |
| 10 | 3117822 | 3281387 | 3.0474915 | 9501536 | 50 |
| 20 | 3145448 | 3313639 | 3.0178301 | 9492426 | 40 |
| 30 | 3173047 | 3345953 | 2.9886850 | 9483237 | 30 |
| 40 | 3200619 | 3378330 | 2.9600422 | 9473966 | 20 |
| 18° 50′ | 3228164 | 3410771 | 2.9318885 | 9464616 | 71° 10′ |
| 19° 0′ | 0.3255682 | 0.3443276 | 2.9042109 | 0.9455186 | 71° 0′ |
| 10 | 3283172 | 3475846 | 2.8769970 | 9445675 | 50 |
| 20 | 3310634 | 3508483 | 2.8502349 | 9436085 | 40 |
| 30 | 3338069 | 3541186 | 2.8239129 | 9426415 | 30 |
| 40 | 3365475 | 3573956 | 2.7980198 | 9416665 | 20 |
| 19° 50′ | 3392852 | 3606795 | 2.7725448 | 9406835 | 70° 10′ |
| 20° 0′ | 0.3420201 | 0.3639702 | 2.7474774 | 0.9396926 | 70° 0′ |
| 10 | 3447521 | 3672680 | 2.7228076 | 9386938 | 50 |
| 20 | 3474812 | 3705728 | 2.6985254 | 9376869 | 40 |
| 30 | 3502074 | 3738847 | 2.6746215 | 9366722 | 30 |
| 40 | 3529306 | 3772038 | 2.6510867 | 9356495 | 20 |
| 20° 50′ | 3556508 | 3805302 | 2.6279121 | 6346189 | 69° 10′ |
| 21° 0′ | 0.3583679 | 0.3838640 | 2.6050891 | 0.9335804 | 69° 0′ |
| 10 | 3610821 | 3872053 | 2.5826094 | 9325340 | 50 |
| 20 | 3637932 | 3905541 | 2.5604649 | 9314797 | 40 |
| 30 | 3665012 | 3939105 | 2.5386479 | 9304176 | 30 |
| 40 | 3692061 | 3972746 | 2.5171507 | 9293475 | 20 |
| 21° 50′ | 3719079 | 4006465 | 2.4959661 | 9282696 | 68° 10′ |
| 22° 0′ | 0.3746066 | 0.4040262 | 2.4750869 | 0.9271839 | 68° 0′ |
| 10 | 3773021 | 4074139 | 2.4545061 | 9260902 | 50 |
| 20 | 3799944 | 4108097 | 2.4342172 | 9249888 | 40 |
| 30 | 3826834 | 4142136 | 2.4142136 | 9238795 | 30 |
| 40 | 3853693 | 4176257 | 2.3944889 | 9227624 | 20 |
| 22° 50′ | 3880518 | 4210460 | 2.3750372 | 9216375 | 67° 10′ |
| 23° 0′ | 0.3907311 | 0.4244748 | 2.3558524 | 0.9205049 | 67° 0′ |
| 10 | 3934071 | 4279121 | 2.3369287 | 9193644 | 50 |
| 20 | 3960798 | 4313579 | 2.3182606 | 9182161 | 40 |
| 30 | 3987491 | 4348124 | 2.2998425 | 9170601 | 30 |
| 40 | 4014150 | 4382756 | 2.2816693 | 9158963 | 20 |
| 23° 50′ | 4040775 | 4417477 | 2.2637357 | 9147247 | 66° 10′ |
| D. M. | Cosine. | Cotangent. | Tangent. | Sine. | D. M. |

| D. M. | Sine. | Tangent. | Cotangent. | Cosine. | D. M. |
|---|---|---|---|---|---|
| 24° 0′ | 0.4067366 | 0.4452287 | 2.2460368 | 0.9135455 | 66° 0′ |
| 10 | 4093923 | 4487187 | 2.2285676 | 9123584 | 50 |
| 20 | 4120445 | 4522179 | 2.2113234 | 9111637 | 40 |
| 30 | 4146932 | 4557263 | 2.1942997 | 9099613 | 30 |
| 40 | 4173385 | 4592439 | 2.1774920 | 9087511 | 20 |
| 24° 50′ | 4199801 | 4627710 | 2.1608958 | 9075333 | 65° 10′ |
| 25° 0′ | 0.4226183 | 0.4663077 | 2.1445069 | 0.9063078 | 65° 0′ |
| 10 | 4252528 | 4698539 | 2.1283213 | 9050746 | 50 |
| 20 | 4278838 | 4734098 | 2.1123348 | 9038338 | 40 |
| 30 | 4305111 | 4769755 | 2.0965436 | 9025853 | 30 |
| 40 | 4331348 | 4805512 | 2.0809438 | 9013292 | 20 |
| 25° 50′ | 4357548 | 4841368 | 2.0655318 | 9000654 | 64° 10′ |
| 26° 0′ | 0.4383711 | 0.4877326 | 2.0503038 | 0.8987940 | 64° 0′ |
| 10 | 4409838 | 4913386 | 2.0352565 | 8975151 | 50 |
| 20 | 4435927 | 4949549 | 2.0203862 | 8962285 | 40 |
| 30 | 4461978 | 4985816 | 2.0056897 | 8949344 | 30 |
| 40 | 4487992 | 5022189 | 1.9911637 | 8936326 | 20 |
| 26° 50′ | 4513967 | 5058668 | 1.9768050 | 8923234 | 63° 10′ |
| 27° 0′ | 0.4539905 | 0.5095254 | 1.9626105 | 0.8910065 | 63° 0′ |
| 10 | 4565804 | 5131950 | 1.9485772 | 8896822 | 50 |
| 20 | 4591665 | 5168755 | 1.9347020 | 8883503 | 40 |
| 30 | 4617486 | 5205671 | 1.9209821 | 8870108 | 30 |
| 40 | 4643269 | 5242698 | 1.9074147 | 8856639 | 20 |
| 27° 50′ | 4669012 | 5279839 | 1.8939971 | 8843095 | 62° 10′ |
| 28° 0′ | 0.4694716 | 0.5317094 | 1.8807265 | 0.8829476 | 62° 0′ |
| 10 | 4720380 | 5354465 | 1.8676003 | 8815782 | 50 |
| 20 | 4746004 | 5391952 | 1.8546159 | 8802014 | 40 |
| 30 | 4771588 | 5429557 | 1.8417709 | 8788171 | 30 |
| 40 | 4797131 | 5467281 | 1.8290628 | 8774254 | 20 |
| 28° 50′ | 4822634 | 5505125 | 1.8164892 | 8760263 | 61° 10′ |
| 29° 0′ | 0.4848096 | 0.5543091 | 1.8040478 | 0.8746197 | 61° 0′ |
| 10 | 4873517 | 5581179 | 1.7917362 | 8732058 | 50 |
| 20 | 4898897 | 5619391 | 1.7795524 | 8717844 | 40 |
| 30 | 4924236 | 5657728 | 1.7674940 | 8703557 | 30 |
| 40 | 4949532 | 5696191 | 1.7555590 | 8689196 | 20 |
| 29° 50′ | 4974787 | 5734783 | 1.7437453 | 8674762 | 60° 10′ |
| D. M. | Cosine. | Cotangent. | Tangent. | Sine. | D. M. |

A

# PRACTICAL APPLICATION

OF

# THE PRINCIPLES OF GEOMETRY

TO THE

# MENSURATION

OF

# SUPERFICIES AND SOLIDS:

BEING

THE THIRD PART

OF

A COURSE OF MATHEMATICS,

ADAPTED TO THE METHOD OF INSTRUCTION IN THE AMERICAN COLLEGES.

BY JEREMIAH DAY, D.D. LL. D.
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THIRD EDITION,

WITH ADDITIONS AND ALTERATIONS.

NEW HAVEN:
PUBLISHED AND SOLD BY HEZEKIAH HOWE.
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1831.

THE following short Treatise contains little more than an application of the principles of Geometry, to the numerical calculation of the superficial and solid contents of such figures as are treated of in the Elements of Euclid. As the plan proposed for the work of which this number is a part, does not admit of introducing rules and propositions which are not demonstrated; the particular consideration of the areas of the Conic Sections and other curves, with the contents of solids produced by their revolution, is reserved for succeeding parts of the course. The student would be little profited by applying arithmetical calculation, in a mechanical way, to figures of which he has not yet learned even the definitions. But as this number may fall into the hands of some who will not read those which are to follow, the principal rules for conic areas and solids, and for the gauging of casks, are given, without demonstrations, in the appendix. Those who wish to take a complete view of Mensuration, in all its parts, are referred to the valuable treatise of Dr. Hutton on the subject.

# CONTENTS.



## APPENDIX.—Part I.



## Part II.

## SECTION I.

### AREAS OF FIGURES BOUNDED BY RIGHT LINES.

ART. 1. THE following definitions, which are nearly the same as in Euclid, are inserted here for the convenience of reference.

I. *Four-sided* figures have different names, according to the relative position and length of the sides. A *parallelogram* has its opposite sides equal and parallel; as ABCD. (Fig. 2.) A *rectangle*, or *right parallelogram*, has its opposite sides equal, and all its angles right angles; as AC. (Fig. 1.) A *square* has all its sides equal, and all its angles right angles; as ABGH. (Fig. 3.) A *rhombus* has all its sides equal, and its angles oblique; as ABCD. (Fig. 3.) A *rhomboid* has its opposite sides equal, and its angles oblique; as ABCD. (Fig. 2.) A *trapezoid* has only two of its sides parallel; as ABCD. (Fig. 4.) Any other four sided figure is called a *trapezium.*

II. A figure which has more than four sides is called a *polygon.* A *regular* polygon has all its sides equal, and all its angles equal.

III. The *height* of a *triangle* is the length of a perpendicular, drawn from one of the angles to the opposite side; as CP. (Fig. 5.) The *height* of a *four sided* figure is the perpendicular distance between two of its parallel sides; as GP. (Fig. 4.)

IV. The *area* or *superficial contents* of a figure is the *space* contained within the line or lines by which the figure is bounded.

2. In calculating areas, some particular portion of surface is fixed upon, as the *measuring unit*, with which the given figure is to be compared. This is commonly a *square;* as a square inch, a square foot, a square rod, &c. For this reason, determining the quantity of surface in a figure is called *squaring it*, or finding its *quadrature;* that is, finding a square or number of squares to which it is equal.

3. The *superficial* unit has generally the same name, as the *linear* unit which forms the side of the square.

The side of a square inch is a linear inch;
of a square foot, a linear foot;
of a square rod, a linear rod, &c.

There are some superficial measures, however, which have no corresponding denominations of length. The *acre*, for instance, is not a square which has a line of the same name for its side.

The following tables contain the linear measures in common use, with their corresponding square measures.

| *Linear Measures.* | | | | *Square Measures.* | | | |
|---|---|---|---|---|---|---|---|
| 12 | inches | =1 | foot. | 144 | inches | =1 | foot. |
| 3 | feet | =1 | yard. | 9 | feet | =1 | yard. |
| 6 | feet | =1 | fathom. | 36 | feet | =1 | fathom. |
| $16\frac{1}{2}$ | feet | =1 | rod. | $272\frac{1}{4}$ | feet | =1 | rod. |
| $5\frac{1}{2}$ | yards | =1 | rod. | $30\frac{1}{4}$ | yards | =1 | rod. |
| 4 | rods | =1 | chain. | 16 | rods | =1 | chain. |
| 40 | rods | =1 | furlong. | 1600 | rods | =1 | furlong. |
| 320 | rods | =1 | mile. | 102400 | rods | =1 | mile. |

An *acre* contains 160 square rods, or 10 square chains.

By reducing the denominations of square measure, it will be seen that

1 sq. mile=640 acres=102400 rods=27878400 feet=4014489600 inches.
1 acre=10 chains=160 rods=43560 feet=6272640 inches.

The fundamental problem in the mensuration of superficies is the very simple one of determining the area of a *right parallelogram.* The contents of other figures, particularly those which are rectilinear, may be obtained by finding parallelograms which are equal to them, according to the principles laid down in Euclid.

### PROBLEM I.

*To find the area of a* PARALLELOGRAM, *square, rhombus, or rhomboid.*

4. MULTIPLY THE LENGTH BY THE PERPENDICULAR HEIGHT OR BREADTH.

It is evident that the number of *square* inches in the parallelogram AC (Fig. 1.) is equal to the number of *linear* inches in the length AB, repeated as many times as there are inches in the breadth BC. For a more particular illustration of this, see Alg. 511—514.

The oblique parallelogram or rhomboid ABCD (Fig. 2.) is equal to the right parallelogram GHCD. (Euc. 36. 1.) The area, therefore, is equal to the length AB multiplied into the perpendicular height HC. And the rhombus ABCD (Fig. 3.) is equal to the *parallelogram* ABGH. As the sides of a *square* are all equal, its area is found, by *multiplying one of the sides into itself.*

Ex. 1. How many square feet are there in a floor $23\frac{1}{2}$ feet long, and 18 feet broad? Ans. $23\frac{1}{2}\times18=423$.

2. What are the contents of a piece of ground which is 66 feet square? Ans. 4356 sq. feet=16 sq. rods.

3. How many square feet are there in the four sides of a room which is 22 feet long, 17 feet broad, and 11 feet high? Ans. 858.

Art. 5. If the sides and angles of a parallelogram are given, the perpendicular height may be easily found by trigonometry. Thus CH (Fig. 2.) is the perpendicular of a right angled triangle, of which BC is the hypothenuse. Then (Trig. 134.)

$$R : BC :: \sin B : CH.$$

The area is obtained by multiplying CH thus found, into the length AB.

Or, to reduce the two operations to one,

As radius,
To the sine of any angle of a parallelogram;
So is the product of the sides including that angle,
To the area of the parallelogram.

For *the area*$=AB\times CH$, (Fig. 2.) But $CH=\dfrac{BC\times\sin B}{R}$.

Therefore,

*The area*$=\dfrac{AB\times BC\times\sin B}{R}$. Or, $R:\sin B::AB\times BC:$*the area.*

Ex. If the side AB be 58 rods, BC 42 rods, and the angle B 63°, what is the area of the parallelogram?

| | | |
|---|---|---|
| As radius | | 10.00000 |
| To the sine of B | 63° | 9.94988 |
| So is the product of AB | 58 | 1.76343 |
| Into BC (Trig. 39.) | 42 | 1.62325 |
| To the area | 2170.5 sq. rods | 3.33656 |

2. If the side of a rhombus is 67 feet, and one of the angles 73°, what is the area? Ans. 4292.7 feet.

6. When the dimensions are given in feet and inches, the multiplication may be conveniently performed by the arithmetical rule of *Duodecimals;* in which each inferior denomination is one twelfth of the next higher. Considering a foot as the measuring *unit*, a prime is the twelfth part of a foot; a second, the twelfth part of a prime, &c. It is to be observed, that, in measures of *length*, *inches* are *primes;* but in *superficial* measure they are *seconds.* In both, a prime is $\frac{1}{12}$ of a foot. But $\frac{1}{12}$ of a *square* foot is a parallelogram, a foot long and an inch broad. The twelfth part of this is a square inch, which is $\frac{1}{144}$ of a square foot.

Ex. 1. What is the surface of a board 9 feet 5 inches, by 2 feet 7 inches.

| F | | |
|---|---|---|
| 9 | 5′ | |
| 2 | 7 | |
| 18 | 10 | |
| 5 | 5 | 11 |
| 24 | 3 | 11″, or 24 feet 47 inches. |

2. How many feet of glass are there in a window 4 feet 11 inches high, and 3 feet 5 inches broad?

Ans. 16F. 9′ 7″, or 16 feet 115 inches.

7. If the area and one side of a parallelogram be given, the other side may be found by *dividing the area by the given side.* And if the area of a *square* be given, the side may be found by *extracting the square root of the area.* This is merely reversing the rule in art. 4. See Alg. 520, 521.

Ex. 1. What is the breadth of a piece of cloth which is 36 yds. long, and which contains 63 square yds. Ans. $1\frac{3}{4}$ yds.

2. What is the side of a square piece of land containing 289 square rods ?

3. How many yards of carpeting $1\frac{1}{4}$ yard wide, will cover a floor 30 feet long and $22\frac{1}{2}$ broad ?

Ans. $30 \times 22\frac{1}{2}$ feet$=10 \times 7\frac{1}{2}=75$ yds. And $75 \div 1\frac{1}{4}=60$.

4. What is the side of a square which is equal to a parallelogram 936 feet long and 104 broad?

5. How many panes of 8 by 10 glass are there, in a window 5 feet high, and 2 feet 8 inches broad ?

PROBLEM II.

*To find the area of a* TRIANGLE.

8. RULE I. MULTIPLY ONE SIDE BY HALF THE PERPENDICULAR FROM THE OPPOSITE ANGLE. Or, multiply half the side by the perpendicular. Or, multiply the whole side by the perpendicular, and take half the product.

The area of the triangle ABC (Fig. 5.) is equal to $\frac{1}{2}$ PC$\times$ AB, because a parallelogram of the same base and height is equal to PC$\times$AB, (Art. 4.) and by Euc. 41, 1, the triangle is half the parallelogram.

Ex. 1. If AB (Fig. 5.) be 65 feet, and PC 31.2, what is the area of the triangle? Ans. 1014 square feet.

2. What is the surface of a triangular board, whose base is 3 feet 2 inches, and perpendicular height 2 feet 9 inches ?

Ans. 4F. 4′ 3″, or 4 feet 51 inches.

9. If two sides of a triangle and the included angle, are given, the perpendicular on one of these sides may be easily found by rectangular trigonometry. And the area may be calculated in the same manner as the area of a parallelogram in art. 5. In the triangle ABC (Fig. 2.)

R : BC : : sin B : CH

And because the triangle is half the parallelogram of the same base and height,

As radius,
To the sine of any angle of a triangle ;
So is the product of the sides including that angle,
To twice the area of the triangle. (Art. 5.)

Ex. If AC (Fig. 5.) be 39 feet, AB 65 feet, and the angle at A 53° 7′ 48″, what is the area of the triangle?

Ans. 1014 square feet.

9. *b.* If *one side* and the *angles* are given; then

As the product of radius and the sine of the angle opposite the given side,

To the product of the sines of the two other angles;

So is the square of the given side,

To twice the area of the triangle.

If PC (Fig. 5.) be perpendicular to AB.

$$R : \sin B :: BC : CP$$

$$\sin ACB : \sin A :: AB : BC$$

Therefore (Alg. 390, 382.)

$$R \times \sin ACB : \sin A \times \sin B :: AB \times BC : CP \times BC :: \overline{AB}^2 : AB \times CP = \text{twice the area of the triangle.}$$

Ex. If one side of a triangle be 57 feet, and the angles at the ends of this side 50° and 60°, what is the area?

Ans. 1147 sq. feet.

10. If the *sides* only of a triangle are given, an angle may be found, by oblique trigonometry, Case IV, and then the perpendicular and the area may be calculated. But the area may be more directly obtained, by the following method.

RULE II. When the three sides are given, *from half their sum subtract each side severally, multiply together the half sum and the three remainders, and extract the square root of the product.*

If the sides of the triangle are $a$, $b$, and $c$, and if $h$=half their sum, then

$$\textit{The area} = \sqrt{h \times (h-a) \times (h-b) \times (h-c)}$$

For the demonstration of this rule, see Trigonometry, Art. 221.

If the calculation be made by *logarithms*, add the logarithms of the several factors, and half their sum will be the logarithm of the area. (Trig. 39, 47.)

Ex. 1. In the triangle ABC (Fig. 5.) given the sides $a$ 52 feet, $b$ 39, and $c$ 65; to find the side of a square which has the same area as the triangle.

$$\tfrac{1}{2}(a+b+c) = h = 78 \qquad h-b = 39$$

$$h-a = 26 \qquad h-c = 13$$

Then the area $= \sqrt{78 \times 26 \times 39 \times 13} = 1014$ square feet.

By logarithms.

| | | |
|---|---|---|
| The half sum | =78 | 1.89209 |
| First remainder | =26 | 1.41497 |
| Second do. | =39 | 1.59106 |
| Third do. | =13 | 1.11394 |
| | | 2)6.01206 |
| The area required | =1014 | 2)3.00603 |
| Side of the square | =31.843 (Trig. 47) | 1.50301 |

2. If the sides of a triangle are 134, 108, and 80 rods, what is the area? Ans. 4319.

3. What is the area of a triangle whose sides are 371, 264, and 225 feet?

11. In an *equilateral* triangle, one of whose sides is $a$, the expression for the area becomes

$$\sqrt{h\times(h-a)\times(h-a)\times(h-a)}$$

But as $h=\frac{3}{2}a$, and $h-a=\frac{3}{2}a-a=\frac{1}{2}a$, the area is

$$\sqrt{\tfrac{3}{2}a\times\tfrac{1}{2}a\times\tfrac{1}{2}a\times\tfrac{1}{2}a}=\sqrt{\tfrac{3}{16}a^4}=\tfrac{1}{4}a^2\sqrt{3} \text{ (Alg. 271.)}$$

That is, the area of an equilateral triangle is equal to $\frac{1}{4}$ the square of one of its sides, multiplied into the square root of 3, which is 1.732.

Ex. 1. What is the area of a triangle whose sides are each 34 feet? Ans. $500\frac{1}{2}$ feet.

2. If the sides of a triangular field are each 100 rods, how many acres does it contain?

### PROBLEM III.

*To find the area of a* TRAPEZOID.

21. MULTIPLY HALF THE SUM OF THE PARALLEL SIDES INTO THEIR PERPENDICULAR DISTANCE.

The area of the trapezoid ABCD (Fig. 4.) is equal to half the sum of the sides AB and CD, multiplied into the perpendicular distance PC or AH. For the whole figure is made up of the two triangles ABC and ADC; the area of the first of which is equal to the product of half the base AB into the perpendicular PC, (Art. 8.) and the area of the other is equal to the product of half the base DC into the perpendicular AH or PC.

**Ex. If AB (Fig. 4.) be 46 feet, BC 31, DC 38, and the angle B 70°, what is the area of the trapezoid?**

**R : BC : : sin B : PC=29.13. And 42×29.13=1223$\frac{1}{3}$.**

**2. What are the contents of a field which has two parallel sides 65 and 38 rods, distant from each other 27 rods?**

PROBLEM IV.

*To find the area of a* TRAPEZIUM, *or of an irregular* POLYGON.

13. DIVIDE THE WHOLE FIGURE INTO TRIANGLES, BY DRAWING DIAGONALS, AND FIND THE SUM OF THE AREAS OF THESE TRIANGLES. (Alg. 519.)

If the perpendiculars in two triangles fall upon the *same diagonal*, the area of the trapezium formed of the two triangles, is equal to half the product of the diagonal into the sum of the perpendiculars.

Thus the area of the trapezium ABCH (Fig. 6.) is

$$\tfrac{1}{2}BH\times AL+\tfrac{1}{2}BH\times CM=\tfrac{1}{2}BH\times(AL+CM.)$$

Ex. In the irregular polygon ABCDH (Fig. 6.)

if the diagonals $\left\{\begin{array}{l}BH=36,\\CH=32,\end{array}\right.$ and the perpendiculars $\left\{\begin{array}{l}AL=5.3\\CM=9.3\\DN=7.3\end{array}\right.$

The area=18×14.6+16×7.3=379.6.

14. If the diagonals of a *trapezium* are given, the area may be found, nearly in the same manner as the area of a parallelogram in Art. 5, and the area of a triangle in Art. 9.

In the trapezium ABCD (Fig. 8.) the sines of the four angles at N, the point of intersection of the diagonals, are all equal. For the two acute angles are *supplements* of the other two, and therefore have the same sine. (Trig. 90.) Putting, then, sin N for the sine of each of these angles, the areas of the four triangles of which the trapezium is composed, are given by the following proportions; (Art. 9.)

$$R : \sin N : : \left\{\begin{array}{l}BN\times AN : 2\,\textit{area}\,ABN\\BN\times CN : 2\,\textit{area}\,BCN\\DN\times CN : 2\,\textit{area}\,CDN\\DN\times AN : 2\,\textit{area}\,ADN\end{array}\right.$$

And by addition, (Alg. 388, Cor. 1.*)

R : sin N::BN×AN+BN×CN+DN×CN+DN×AN : 2 *area* ABCD.

The 3d term=(AN+CN)×(BN+DN)=AC×BD, by the figure.

Therefore, R : sin N::AC×BD : 2 *area* ABCD. That is,

As Radius,
To the sine of the angle at the intersection of the diagonals of a trapezium;
So is the product of the diagonals,
To twice the area of the trapezium.

It is evident that this rule is applicable to a parallelogram, as well as to a trapezium.

If the diagonals intersect at *right angles*, the sine of N is equal to radius; (Trig. 95.) and therefore the product of the diagonals is equal to twice the area. (Alg. 395.†)

Ex. 1. If the two diagonals of a trapezium are 37 and 62, and if they intersect at an angle of 54°, what is the area of the trapezium? Ans. 928.

2. If the diagonals are 85 and 93, and the angle of intersection 74°, what is the area of the trapezium?

14. *b.* When a trapezium can be *inscribed in a circle*, the area may be found by either of the following rules.

I. *Multiply together any two adjacent sides, and also the two other sides; then multiply half the sum of these products by the sine of the angle included by either of the pairs of sides multiplied together.*

Or,

II. *From half the sum of all the sides, subtract each side severally, multiply together the four remainders, and extract the square root of the product.*

If the sides are $a$, $b$, $c$, and $d$; and if $h$=half their sum;

$$\text{The area} = \sqrt{(h-a)\times(h-b)\times(h-c)\times(h-d)}$$

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* Euclid 2, 5. Cor. † Euclid 14. 5.

If the trapezium ABCD (Fig. 33.) can be inscribed in a circle, the sum of the opposite angles BAD and BCD is 180° (Euc. 22. 3.) Therefore the *sine* of BAD is equal to that of BCD or P′CD.

If $s$ = the sine of either of these angles, radius being 1, and if

$$AB=a, \quad BC=b, \quad CD=c, \quad AD=d;$$

The triangle $BAD=\frac{1}{2}ad\times s$, And $BCD=\frac{1}{2}bc\times s$; (Art. 9.)

Therefore,

1. *The area of* $ABCD=\frac{1}{2}(ad+bc)\times s$.

To obtain the value of $s$, in terms of the sides of the trapezium, draw DP and DP′ perpendicular to BA and BC.

Then Rad. : $s$ : : AD : DP : : CD : DP′.

Also $AP^2=AD^2-DP^2$, and $CP'^2=CD^2-DP'^2$.

So that $\begin{cases} DP=AD\times s=ds \\ DP'=CD\times s=cs \end{cases}$ And $\begin{cases} AP=\sqrt{d^2-d^2s^2}=d\sqrt{1-s^2} \\ CP'=\sqrt{c^2-c^2s^2}=c\sqrt{1-s^2} \end{cases}$

But by the figure $\begin{cases} BP=AB-AP=a-d\sqrt{1-s^2} \\ BP'=BC+CP'=b+c\sqrt{1-s^2} \end{cases}$

$$\text{And } \overline{BP}^2+\overline{DP}^2=\overline{DB}^2=\overline{BP'}^2+\overline{DP'}^2$$

$$\text{That is } a^2-2ad\sqrt{1-s^2}+d^2=b^2+2bc\sqrt{1-s^2}+c^2$$

Reducing the equation, we have

$$s^2=1-\frac{(b^2+c^2-a^2-d^2)^2}{(2ad+2bc)^2}, \text{ and}$$

$$s=\frac{\sqrt{(2ad+2bc)^2-(b^2+c^2-a^2-d^2)^2}}{2ad+2bc}$$

Substituting for $s$ in the first rule, the value here found, we have the area of the trapezium, equal to

$$\tfrac{1}{4}\sqrt{(2ad+2bc)^2-(b^2+c^2-a^2-d^2)^2}$$

The expression under the radical sign is the difference of two squares, and may be resolved, as in Trig. 221, into the factors

$$(\overline{b+c}^2-\overline{a-d}^2)\times(\overline{a+d}^2-\overline{b-c}^2)$$

and these again into

$$(a+b+c-d)(b+c+d-a)(a+b+d-c)(a+d+c-b)$$

If then $h$ = half the sum of the sides of the trapezium,

II. *The area*$=\sqrt{(h-a)\times(h-b)\times(h-c)\times(h-d)}$

If one of the sides, as $d$, is supposed to be diminished, till it is reduced to nothing; the figure becomes a *triangle*, and the expression for the area is the same as in art. 10. See Hutton's Mensuration.

PROBLEM V.

*To find the area of a* REGULAR POLYGON.

15. MULTIPLY ONE OF ITS SIDES INTO HALF ITS PERPENDICULAR DISTANCE FROM THE CENTER, AND THIS PRODUCT INTO THE NUMBER OF SIDES.

A regular polygon contains as many equal triangles as the figure has sides. Thus the hexagon ABDFGH (Fig. 7.) contains six triangles, each equal to ABC. The area of one of them is equal to the product of the side AB, into half the perpendicular CP. (Art. 8.) The area of the whole, therefore, is equal to this product multiplied into the *number* of sides.

Ex. 1. What is the area of a regular octagon, in which the length of a side is 60, and the perpendicular from the center 72.426? Ans. 17382.

2. What is the area of a regular decagon whose sides are 46 each, and the perpendicular 70.7867?

16. If only the length and number of sides of a regular polygon be given, the *perpendicular* from the center may be easily found by trigonometry. The periphery of the circle in which the polygon is inscribed, is divided into as many equal parts as the polygon has sides. (Euc. 16. 4. Schol.) The arc, of which one of the sides is a chord, is therefore known; and of course, the angle at the center subtended by this arc.

Let AB (Fig. 7.) be one side of a regular polygon, inscribed in the circle ABDG. The perpendicular CP bisects the line AB, and the angle ACB. (Euc. 3. 3.) Therefore BCP is the same part of 360°, which BP is of the perimeter of the polygon. Then, in the right angled triangle BCP, if BP be radius, (Trig. 122.)

R : BP : : cot BCP : CP. That is,

As Radius,
To half of one of the sides of the polygon ;
So is the cotangent of the opposite angle,
To the perpendicular from the center.

Ex. 1. If the side of a regular hexagon (Fig. 7.) be 38 inches, what is the area ?

The angle $BCP = \frac{1}{12}$ of $360° = 30°$. Then,

$R : 19 :: \cot 30° : 32.909 = CP$, the perpendicular.

And the area $= 19 \times 32.909 \times 6 = 3751.6$.

2. What is the area of a regular decagon whose sides are each 62 feet ? Ans. 29576.

17. From the proportion in the preceding article, a *table* of perpendiculars and areas may be easily formed, for a series of polygons, of which each side is a unit. Putting $R=1$, (Trig. 100.) and $n$=the number of sides, the proportion becomes

$$1 : \tfrac{1}{2} :: \cot \frac{360}{2n} : \textit{the perpendicular.}$$

$$\text{So that, } \textit{the perp.} = \tfrac{1}{2} \cot \frac{360}{2n}$$

And the *area* is equal to half the product of the perpendicular into the number of sides. (Art. 15.)

Thus, in the trigon, or equilateral triangle, the perpendicular $= \frac{1}{2} \cot \frac{360°}{6} = \frac{1}{2} \cot 60° = 0.2886752$.

And the area $= 0.4330127$.

In the tetragon, or square, the perpendicular $= \frac{1}{2} \cot \frac{360°}{8} = \frac{1}{2} \cot 45° = 0.5$. And the area $= 1$.

In this manner, the following table is formed, in which the side of each polygon is supposed to be a unit.

## A TABLE OF REGULAR POLYGONS.

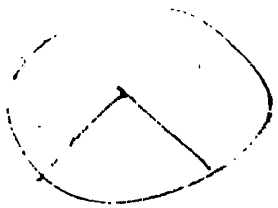
| Names. | Sides. | Angles. | Perpendiculars. | Areas. |
|---|---|---|---|---|
| Trigon, | 3 | 60° | 0.2886752 | 0.4330127 |
| Tetragon, | 4 | 45° | 0.5000000 | 1.0000000 |
| Pentagon, | 5 | 36° | 0.6881910 | 1.7204774 |
| Hexagon, | 6 | 30° | 0.8660254 | 2.5980762 |
| Heptagon, | 7 | $25\frac{5}{7}$ | 1.0382601 | 3.6339124 |
| Octagon, | 8 | $22\frac{1}{2}$ | 1.2071069 | 4.8284271 |
| Nonagon, | 9 | 20° | 1.3737385 | 6.1818242 |
| Decagon, | 10 | 18° | 1.5388418 | 7.6942088 |
| Undecagon, | 11 | $16\frac{4}{11}$ | 1.7028439 | 9.3656399 |
| Dodecagon, | 12 | 15° | 1.8660252 | 11.1961524 |

By this table may be calculated the area of any other regular polygon, of the same number of sides with one of these. For the areas of similar polygons are as the *squares* of their homologous sides. (Euc. 20, 6.)

To find, then, the area of a regular polygon, *multiply the square of one of its sides by the area of a similar polygon of which the side is a unit.*

Ex. 1. What is the area of a regular decagon whose sides are each 102 rods? Ans. 80050.5 rods.

2. What is the area of a regular dodecagon whose sides are each 87 feet?





## SECTION II.*

### THE QUADRATURE OF THE CIRCLE AND ITS PARTS.

ART. 18. *Definition* I. A CIRCLE is a plane bounded by a line which is equally distant in all its parts from a point within called the center. The bounding line is called the *circumference* or periphery. An *arc* is any portion of the circumference. A semi-circle is half, and a quadrant one fourth, of a circle.

II. A *Diameter* of a circle is a straight line drawn through the center, and terminated both ways by the circumference. A *Radius* is a straight line extending from the center to the circumference. A *Chord* is a straight line which joins the two extremities of an arc.

III. A Circular *Sector* is a space contained between an arc and the two radii drawn from the extremities of the arc. It may be *less* than a semi-circle, as ACBO, (Fig. 9.) or *greater*, as ACBD.

IV. A Circular *Segment* is the space contained between an arc and its chord, as ABO or ABD. (Fig. 9.) The chord is sometimes called the *base* of the segment. The *height* of a segment is the perpendicular from the middle of the base to the arc, as PO. (Fig. 9.)

V. A Circular *Zone* is the space between two parallel chords, as AGHB. (Fig. 15.) It is called the *middle* zone, when the two chords are equal.

VI. A Circular *Ring* is the space between the peripheries of two concentric circles, as AA′, BB′. (Fig. 13.)

VII. A *Lune* or Crescent is the space between two circular arcs which intersect each other, as ACBD. (Fig. 14.)

19. The *Squaring of the Circle* is a problem which has exercised the ingenuity of distinguished mathematicians for

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* Wallis's Algebra, Legendre's Geometry, Book IV, and Note IV. Hutton's Mensuration, Horseley's Trigonometry, Book I, Sec. 3; Introduction to Euler's Analysis of Infinites, London Phil. Trans. Vol. VI, No. 75, LXVI, p. 476, LXXXIV, p. 217, and Hutton's abridgment of do. Vol. II, p. 547.

many centuries. The result of their efforts has been only an *approximation* to the value of the area. This can be carried to a degree of exactness far beyond what is necessary for practical purposes.

20. If the *circumference* of a circle of given diameter were known, its area could be easily found. For the area is equal to the product of half the circumference into half the diameter. (Sup. Euc. 5, 1.*) But the circumference of a circle has never been exactly determined. The method of approximating to it is by inscribing and circumscribing *polygons*, or by some process of calculation which is, in principle, the same. The perimeters of the polygons can be easily and exactly determined. That which is circumscribed is *greater*, and that which is inscribed is *less*, than the periphery of the circle; and by increasing the number of sides, the difference of the two polygons may be made less than any given quantity. (Sup. Euc. 4, 1.)

21. The side of a *hexagon* inscribed in a circle, as AB, (Fig. 7.) is the chord of an arc of 60°, and therefore equal to the radius. (Trig. 95.) The chord of *half* this arc, as BO, is the side of a polygon of 12 equal sides. By repeatedly bisecting the arc, and finding the chord, we may obtain the side of a polygon of an immense number of sides. Or we may calculate the *sine*, which will be half the chord of double the arc, (Trig. 82, cor.); and the *tangent*, which will be half the side of a similar *circumscribed* polygon. Thus the sine AP (Fig. 7.) is half of AB, a side of the inscribed hexagon; and the tangent NO is half of NT, a side of the circumscribed hexagon. The difference between the sine and the arc AO is less, than the difference between the sine and the tangent. In the section on the computation of the canon, (Trig. 223.) by 12 successive bisections, beginning with 60 degrees, an arc is obtained which is the $\frac{1}{24576}$ of the whole circumference.

The *cosine* of this, if radius be 1, is found to be .99999996732
The *sine* is .00025566346

And the tangent $= \frac{\text{sine}}{\text{cosine}}$ (Trig. 228.) $=$.00025566347

The diff. between the sine and tangent is only .00000000001
And the difference between the sine and the *arc* is still less.

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* In this manner, the *Supplement* to *Playfair's Euclid* is referred to in this work.

Taking then .000255663465 for the length of the arc, multiplying by 24576, and retaining 8 places of decimals, we have 6.28318531 for the whole circumference, the radius being 1. Half of this,

3.14159265

is the circumference of a circle whose radius is $\frac{1}{2}$, and *diameter* 1.

22. If this be multiplied by 7, the product is 21.99+ or 22 nearly. So that,

Diam : Circum : : 7 : 22, nearly.

If 3.14159265 be multiplied by 113, the product is 354.9999+, or 355, very nearly. So that,

Diam : Circum : : 113 : 355, very nearly.

The first of these ratios was demonstrated by Archimedes.

There are various methods, principally by infinite series and fluxions, by which the labor of carrying on the approximation to the periphery of a circle may be very much abridged. The calculation has been extended to nearly 150 places of decimals.* But four or five places are sufficient for most practical purposes.

After determining the ratio between the diameter and the circumference of a circle, the following problems are easily solved.

PROBLEM I.

*To find the* CIRCUMFERENCE *of a circle from its diameter.*

23. MULTIPLY THE DIAMETER BY **3.14159.**†

Or,

*Multiply the diameter by* 22 *and divide the product by* 7. Or, multiply the diameter by 355, and divide the product by 113. (Art. 22.)

Ex. 1. If the diameter of the earth be 7930 miles, what is the circumference? Ans. 249128 miles.

2. How many miles does the earth move, in revolving round the sun; supposing the orbit to be a circle whose diameter is 190 million miles? Ans. 596,902,100.

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* See note A. † In many cases, 3.1416 will be sufficiently accurate.

3. What is the circumference of a circle whose diameter is 769843 rods?

PROBLEM II.

*To find the* DIAMETER *of a circle from its circumference.*

24. DIVIDE THE CIRCUMFERENCE BY **3.14159**.

Or,

*Multiply the circumference by* 7, *and divide the product by* 22. Or, multiply the circumference by 113, and divide the product by 355. (Art. 22.)

Ex. 1. If the circumference of the sun be 2,800,000 miles, what is his diameter? Ans. 891,267.

2. What is the diameter of a tree which is 5½ feet round?

25. As multiplication is more easily performed than division, there will be an advantage in exchanging the *divisor* 3.14159 for a *multiplier* which will give the same result. In the proportion 3.14159 : 1 : : Circum : Diam.
to find the fourth term, we may divide the second by the first, and multiply the quotient into the third. Now, $1 \div 3.14159 = 0.31831$. If then the circumference of a circle be multiplied by .31831, the product will be the diameter.*

Ex. 1. If the circumference of the moon be 6850 miles, what is her diameter? Ans. 2180.

2. If the whole extent of the orbit of Saturn be 5650 million miles, how far is he from the sun?

3. If the periphery of a wheel be 4 feet 7 inches, what is its diameter?

PROBLEM III.

*To find the length of an* ARC *of a circle.*

26. *As* 360°, *to the number of degrees in the arc;*
*So is the circumference of the circle, to the length of the arc.*

The circumference of a circle being divided into 360°, (Trig. 73.) it is evident that the length of an arc of any less number of degrees must be a proportional part of the whole.

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* See Note B.

Ex. What is the length of an arc of 16°, in a circle whose radius is 50 feet?

The circumference of the circle is 314.159 feet. (Art. 23.)

Then 360 : 16 :: 314.159 : 13.96 feet.

2. If we are 95 millions of miles from the sun, and if the earth revolves round it in 365¼ days, how far are we carried in 24 hours? Ans. 1 million 634 thousand miles.

27. The length of an arc may also be found, by multiplying the diameter into the number of degrees in the arc, and this product into .0087266, which is the length of *one* degree, in a circle whose diameter is 1. For 3.14159 ÷ 360 = 0.0087266. And in different circles, the circumferences, and of course the degrees, are as the diameters. (Sup. Euc. 8, 1.)

Ex. 1. What is the length of an arc of 10° 15′ in a circle whose radius is 68 rods? Ans. 12.165 rods.

2. If the circumference of the earth be 24913 miles, what is the length of a degree at the equator?

28. The length of an arc is frequently required, when the *number of degrees* is not given. But if the radius of the circle, and either the *chord* or the *height* of the arc, be known; the number of degrees may be easily found.

Let AB (Fig. 9.) be the chord, and PO the height, of the arc AOB. As the angles at P are right angles, and AP is equal to BP; (Art. 18. Def. 4.) AO is equal to BO. (Euc. 4, 1.) Then

BP is the *sine*, and CP the *cosine*, } of *half* the arc AOB.
OP the *versed sine*, and BO the *chord* }

And in the right angled triangle CBP,

$$CB : R :: \begin{cases} BP : \sin BCP \text{ or } BO \\ CP : \cos BCP \text{ or } BO \end{cases}$$

Ex. 1. If the radius CO (Fig. 9.) =25, and the chord AB=43.3; what is the length of the arc AOB?

CB : R :: BP : sin BCP or BO=60° very nearly.

The circumference of the circle =3.14159×50=157.08.

And 360°:60°::157.08:26.18=OB. Therefore AOB=52.36.

2. What is the length of an arc whose chord is 216½, in a circle whose radius is 125? Ans. 261.8.

29. If only the *chord* and the *height* of an arc be given, the radius of the circle may be found, and then the length of the arc.

If BA (Fig. 9.) be the chord, and PO the height of the arc AOB, then (Euc. 35. 3.)

$$DP=\frac{\overline{BP}^2}{OP}.\quad \text{And } DO=OP+DP=OP+\frac{\overline{BP}^2}{OP}.$$

That is, the *diameter* is equal to the height of the arc, + the square of half the chord divided by the height.

The diameter being found, the length of the arc may be calculated by the two preceding articles.

Ex. 1. If the chord of an arc be 173.2, and the height 50, what is the length of the arc?

The diameter $=50+\frac{\overline{86.6}^2}{50}=200$. The arc contains 120°, (Art. 28.) and its length is 209.44. (Art. 26.)

2. What is the length of an arc whose chord is 120, and height 45? Ans. 160.8.*

PROBLEM IV.

## *To find the* AREA *of a* CIRCLE.

30. MULTIPLY THE SQUARE OF THE DIAMETER BY THE DECIMALS .7854.

Or,

MULTIPLY HALF THE DIAMETER INTO HALF THE CIRCUMFERENCE. Or, multiply the whole diameter into the whole circumference, and take ¼ of the product.

The area of a circle is equal to the product of half the diameter into half the circumference; (Sup. Euc. 5, 1.) or which is the same thing, ¼ the product of the diameter and circumference. If the diameter be 1, the circumference is 3.14159; (Art. 23.) one fourth of which is 0.7854 nearly. But the areas of different circles are to each other, *as the squares of their diameters.* (Sup. Euc. 7, 1.)† The area of any circle, therefore, is equal to the product of the square

* See note C. † Euclid 2, 12.

of its diameter into 0.7854, which is the area of a circle whose diameter is 1.

Ex. 1. What is the area of a circle whose diameter is 623 feet? Ans. 304836 square feet.

2. How many acres are there in a circular island whose diameter is 124 rods? Ans. 75 acres, and 76 rods.

3. If the diameter of a circle be 113, and the circumference 355, what is the area? Ans. 10029.

4. How many square yards are there in a circle whose diameter is 7 feet?

31. If the *circumference* of a circle be given, the area may be obtained, by first finding the diameter; or, without finding the diameter, by multiplying the square of the circumference by .07958.

For, if the circumference of a circle be 1, the diameter $=1\div3.14159=0.31831$; and $\frac{1}{4}$ the product of this into the circumference is .07958 the area. But the areas of different circles, being as the squares of their diameters, are also as the squares of their *circumferences*. (Sup. Euc. 8, 1.)

Ex. 1. If the circumference of a circle be 136 feet, what is the area? Ans. 1472 feet.

2. What is the surface of a circular fish-pond, which is 10 rods in circumference?

32. If the area of a circle be *given*, the diameter may be found, by dividing the area by .7854, and extracting the square root of the quotient.

This is reversing the rule in art. 30.

Ex. 1. What is the diameter of a circle whose area is 380.1336 feet? Ans. $380.1336\div.7854=484$. And $\sqrt{484}=22$.

2. What is the diameter of a circle whose area is 19.635?

33. The area of a circle, is to the area of the *circumscribed square*; as .7854 to 1; and to that of the *inscribed* square as .7854 to $\frac{1}{2}$.

Let ABDF (Fig. 10.) be the inscribed square and LMNO the circumscribed square, of the circle ABDF. The area of the circle is equal to $\overline{AD}^2\times.7854$. (Art. 30.) But the area of the circumscribed square (Art. 4.) is equal to $\overline{ON}^2=AD^2$. And the smaller square is half of the larger one. For the latter contains 8 equal triangles, of which the former contains only 4.

Ex. What is the area of a square inscribed in a circle whose area is 159? Ans. .7854 : ½ : : 159 : 101.22.

### PROBLEM V.

*To find the area of a* SECTOR *of a circle.*

34. MULTIPLY THE RADIUS INTO HALF THE LENGTH OF THE ARC. Or,

As 360, TO THE NUMBER OF DEGREES IN THE ARC;
So IS THE AREA OF THE CIRCLE, TO THE AREA OF THE SECTOR.

It is evident, that the area of the sector has the same ratio to the area of the circle, which the length of the arc has to the length of the whole circumference; or which the number of *degrees* in the arc has to the number of degrees in the circumference.

Ex. 1. If the arc AOB (Fig. 9.) be 120°, and the diameter of the circle 226; what is the area of the sector AOBC?

The area of the whole circle is 40115. (Art. 30.)

And 360° : 120° : : 40115 : 13371⅔, the area of the sector.

2. What is the area of a quadrant whose radius is 621?

3. What is the area of a semi-circle whose diameter is 328?

4. What is the area of a sector which is less than a semicircle, if the radius be 15, and the chord of its arc 12?

Half the chord is the sine of 23° 34¾′ nearly. (Art. 28.)
The whole arc, then, is 47° 9½′
The area of the circle is 706.86
And 360° : 47° 9½′ : : 706.86 : 92.6 the area of the sector.

5. If the arc ADB (Fig. 9.) be 240 degrees, and the radius of the circle 113, what is the area of the sector ADBC?

### PROBLEM VI.

*To find the area of a* SEGMENT *of a circle.*

35. FIND THE AREA OF THE SECTOR WHICH HAS THE SAME ARC, AND ALSO THE AREA OF THE TRIANGLE FORMED BY THE CHORD OF THE SEGMENT AND THE RADII OF THE SECTOR.

THEN, IF THE SEGMENT BE LESS THAN A SEMI-CIRCLE, SUBTRACT THE AREA OF THE TRIANGLE FROM THE AREA OF THE SECTOR. BUT, IF THE SEGMENT BE GREATER THAN A SEMI-CIRCLE, ADD THE AREA OF THE TRIANGLE TO THE AREA OF THE SECTOR.

If the triangle ABC (Fig. 9.) be taken from the sector AOBC, it is evident the difference will be the segment AOBP, less than a semi-circle. And if the same triangle be added to the sector ADBC, the sum will be the segment ADBP, greater than a semi-circle.

The area of the triangle (Art. 8.) is equal to the product of half the chord AB into CP which is the difference between the radius and PO the height of the segment. Or CP is the *cosine* of half the arc BOA. If this cosine, and the chord of the segment are not given, they may be found from the arc and the radius.

Ex. 1. If the arc AOB (Fig. 9.) be 120°, and the radius of the circle be 113 feet, what is the area of the segment AOBP?

In the right angled triangle BCP,

R : BC : : sin BCO : BP=97.86, half the chord. (Art. 28.)

| | |
|---|---|
| The cosine PC=½CO (Trig. 96, Cor.) | =56.5 |
| The area of the sector AOBC (Art. 34.) | =13371.67 |
| The area of the triangle ABC=BP×PC | = 5528.97 |
| The area of the segment, therefore, | = 7842.7 |

2. If the base of a segment, less than a semi-circle, be 10 feet, and the radius of the circle 12 feet, what is the area of the segment?

| | | |
|---|---|---|
| The arc of the segment contains | 49¼ degrees. | (Art. 28.) |
| The area of the sector | =61.89 | (Art. 34.) |
| The area of the triangle | =54.54 | |
| And the area of the segment | = 7.35 square feet. | |

3. What is the area of a circular segment, whose height is 19.2 and base 70? Ans. 947.86.

4. What is the area of the segment ADBP, (Fig. 9.) if the base AB be 195.7, and the height PD 169.5?

Ans. 32272.*

36. The area of any figure which is bounded *partly* by arcs of circles, and partly by right lines, may be calculated, by finding the areas of the segments under the arcs, and then the area of the rectilinear space between the chords of the arcs and the other right lines.

Thus the Gothic arch ACB, (Fig. 11.) contains the two segments ACH, BCD, and the plane triangle ABC.

Ex. If AB (Fig. 11.) be 110, each of the lines AC and BC 100, and the height of each of the segments ACH, BCD 10.435; what is the area of the whole figure?

| | |
|---|---|
| The areas of the two segments are | 1404 |
| The area of the triangle ABC is | 4593.4 |
| And the whole figure is | 5997.4 |

PROBLEM VII.

*To find the area of a circular* ZONE.

37. FROM THE AREA OF THE WHOLE CIRCLE, SUBTRACT THE TWO SEGMENTS ON THE SIDES OF THE ZONE.

If from the whole circle (Fig. 12.) there be taken the two segments ABC and DFH, there will remain the zone ACDH.

Or, the area of the zone may be found, by subtracting the segment ABC from the segment HBD: Or, by adding the two small segments GAH and VDC to the trapezoid ACDH. See art. 36.

The latter method is rather the most expeditious in practice, as the two segments at the end of the zone are *equal.*

Ex. 1. What is the area of the zone ACDH, (Fig. 12.) if AC is 7.75, DH 6.93, and the diameter of the circle 8 ?

* For the method of finding the areas of segments by a *table*, see note D.

| | | |
|---|---|---|
| The area of the whole circle is | | 50.26 |
| of the segment ABC | 17.32 | |
| of the segment DFH | 9.82 | |
| of the zone ACDH | | 23.12 |

2. What is the area of a zone, one side of which is 23.25, and the other side 20.8, in a circle whose diameter is 24?

Ans. 208.

38. If the *diameter* of the circle is not given, it may be found from the sides and the breadth of the zone.

Let the center of the circle be at O. (Fig. 12.) Draw ON perpendicular to AH, NM perpendicular to LR, and HP perpendicular to AL. Then

$AN=\frac{1}{2}AH$, (Euc. 3. 3.) $\quad MN=\frac{1}{2}(LA+RH)$
$LM=\frac{1}{2}LR$, (Euc. 2. 6.) $\quad PA=LA-RH.$

The triangles APH and OMN are similar, because the sides of one are perpendicular to those of the other, each to each. Therefore

$$PH : PA :: MN : MO$$

MO being found, we have $ML-MO=OL$.

And the *radius* $CO=\sqrt{OL^2+CL^2}$. (Euc. 47. 1.)

Ex. If the breadth of the zone ACDH (Fig. 12.) be 6.4, and the sides 6.8 and 6; what is the radius of the circle?

$PA=3.4-3=0.4$. And $MN=\frac{1}{2}(3.4+3)=3.2$.

Then $6.4 : 0.4 :: 3.2 : 0.2=MO$. And $3.2-0.2=3=OL$

And the radius $CO=\sqrt{3^2+(3.4)^2}=4.534$.

PROBLEM VIII.

*To find the area of a* LUNE *or crescent.*

39. FIND THE DIFFERENCE OF THE TWO SEGMENTS WHICH ARE BETWEEN THE ARCS OF THE CRESCENT AND ITS CHORD.

If the segment ABC (Fig. 14.) be taken from the segment ABD; there will remain the lune or crescent ACBD.

Ex. If the chord AB be 88, the height CH 20, and the height DH 40; what is the area of the crescent ACBD?

| | |
|---|---|
| The area of the segment ABD is | 2698 |
| of the segment ABC | 1220 |
| of the crescent ACBD | 1478 |

PROBLEM IX.

*To find the area of a* RING, included between the peripheries of two concentric circles.

40. FIND THE DIFFERENCE OF THE AREAS OF THE TWO CIRCLES.

Or,

Multiply the product of the sum and difference of the two diameters by .7854.

The area of the ring (Fig. 13.) is evidently equal to the difference between the areas of the two circles AB and A'B'.

But the area of each circle is equal to the square of its diameter multiplied into .7854. (Art. 30.) And the *difference* of these squares is equal to the product of the sum and difference of the diameters. (Alg. 235.) Therefore the area of the ring is equal to the product of the sum and difference of the two diameters multiplied by .7854.

Ex. 1. If AB (Fig. 13.) be 221, and A'B' 106, what is the area of the ring?

Ans. $(\overline{221}^2 \times .7854) - (\overline{106}^2 \times .7854) = 29535$.

2. If the diameters of Saturn's larger ring be 205,000 and 190,000 miles, how many square miles are there on one side of the ring?

Ans. $395000 \times 15000 \times .7854 = 4,653,495,000$.

PROMISCUOUS EXAMPLES OF AREAS.

Ex. 1. What is the expense of paving a street 20 rods long, and 2 rods wide, at 5 cents for a square foot?

Ans. 544½ dollars.

2. If an equilateral triangle contains as many square feet as there are inches in one of its sides; what is the area of the triangle?

Let $x$=the number of square feet in the area.

Then $\frac{x}{12}$= the number of linear feet in one of the sides.

And (Art. 11.) $x=\frac{1}{4}\left(\frac{x}{12}\right)^2\times\sqrt{3}=\frac{x^2}{576}\times\sqrt{3}$.

Reducing the equation, $x=\frac{576}{\sqrt{3}}=332.55$ the area.

3. What is the side of a square whose area is equal to that of a circle 452 feet in diameter?

Ans. $\sqrt{(452)^2\times.7854}=400.574$. (Art. 30 and 7.)

4. What is the diameter of a circle which is equal to a square whose side is 36 feet?

Ans. $\sqrt{(36)^2\div0.7854}=40.6217$. (Art. 4. and 32.)

5. What is the area of a square inscribed in a circle whose diameter is 132 feet?

Ans. 8712 square feet. (Art. 33.)

6. How much carpeting, a yard wide, will be necessary to cover the floor of a room which is a regular octagon, the sides being 8 feet each? Ans. $34\frac{1}{3}$ yards.

7. If the diagonal of a square be 16 feet, what is the area?

Ans. 128 feet. (Art. 14.)

8. If a carriage wheel four feet in diameter revolve 300 times, in going round a circular green; what is the area of the green?

Ans. $4154\frac{1}{3}$ sq. rods, or 25 acres, 3 qrs. and $34\frac{1}{3}$ rods.

9. What will be the expense of papering the sides of a room, at 10 cents a square yard; if the room be 21 feet long, 18 feet broad, and 12 feet high; and if there be deducted 3 windows, each 5 feet by 3, two doors 8 feet by $4\frac{1}{2}$, and one fire-place 6 feet by $4\frac{1}{2}$? Ans. 8 dollars 80 cents.

10. If a circular pond of water 10 rods in diameter be surrounded by a gravelled walk $8\frac{1}{4}$ feet wide; what is the area of the walk? Ans. $16\frac{1}{2}$ sq. rods. (Art. 40.)

11. If CD (Fig. 17.) the base of the isosceles triangle VCD, be 60 feet, and the area 1200 feet; and if there be cut off, by the line LG parallel to CD, the triangle VLG, whose area is 432 feet; what are the sides of the latter triangle?

Ans. 30, 30, and 36 feet.

12. What is the area of an equilateral triangle inscribed in a circle whose diameter is 52 feet?

Ans. 878.15 sq. feet.

13. If a circular piece of land is enclosed by a fence, in which 10 rails make a rod in length; and if the field contains as many square rods, as there are rails in the fence; what is the value of the land at 120 dollars an acre?

Ans. 942.48 dollars.

14. If the area of the equilateral triangle ABD (Fig. 9.) be 219.5375 feet; what is the area of the circle OBDA, in which the triangle is inscribed?

The sides of the triangle are each 22.5167. (Art. 11.)
And the area of the circle is 530.93.

15. If 6 concentric circles are so drawn, that the space

| | |
|---|---|
| between the least or 1st, and the 2d is | 21.2058, |
| between the 2d and 3d | 35.343, |
| between the 3d and 4th | 49.4802, |
| between the 4th and 5th | 63.6174, |
| between the 5th and 6th | 77.7546; |

what are the several diameters, supposing the longest to be equal to 6 times the shortest?

Ans. 3, 6, 9, 12, 15, and 18.

16. If the area between two concentric circles be 1202.64 square inches, and the diameter of the lesser circle be 19 inches, what is the diameter of the other?

17. What is the area of a circular segment, whose height is 9, and base 24?

## SECTION III.

### SOLIDS BOUNDED BY PLANE SURFACES.

Art. 41. Definition I. A *prism* is a solid bounded by plane figures or faces, two of which are parallel, similar, and equal; and the others are parallelograms.

II. The parallel planes are sometimes called the *bases* or *ends;* and the other figures, the *sides* of the prism. The latter taken together constitute the *lateral surface.*

III. A prism is *right* or *oblique*, according as the sides are perpendicular or oblique to the bases.

IV. The *height* of a prism is the perpendicular distance between the planes of the bases. In a right prism, therefore, the height is equal to the length of one of the sides.

V. A *Parallelopiped* is a prism whose bases are parallelograms.

VI. A *Cube* is a solid bounded by six equal squares. It is a right prism whose sides and bases are all equal.

VII. A *Pyramid* is a solid bounded by a plane figure called the base, and several triangular planes, proceeding from the sides of the base, and all terminating in a single point. These triangles taken together constitute the *lateral surface.*

VIII. A pyramid is *regular*, if its base is a regular polygon, and if a line from the center of the base to the vertex of the pyramid is *perpendicular* to the base. This line is called the *axis* of the pyramid.

IX. The *height* of a pyramid is the perpendicular distance from the summit to the plane of the base. In a *regular* pyramid, it is the length of the *axis.*

X. The *slant-height* of a regular pyramid, is the distance from the summit to the middle of one of the sides of the base.

XI. A *frustum* or *trunk* of a pyramid is a portion of the solid next the base, cut off by a plane parallel to the base. The *height* of the frustum is the perpendicular distance of the two parallel planes. The *slant-height* of a frustum of a *regular* pyramid, is the distance from the middle of one of the sides of the base, to the middle of the corresponding side in the plane above. It is a line passing on the surface of the frustum, through the middle of one of its sides.

XII. A *Wedge* is a solid of five sides, viz. a rectangular base, two rhomboidal sides meeting in an edge, and two triangular ends; as ABHG. (Fig. 20.) The base is ABCD, the sides are ABHG and DCHG, meeting in the edge GH, and the ends are BCH and ADG. The *height* of the wedge is a perpendicular drawn from any point in the edge, to the plane of the base, as GP.

XIII. A *Prismoid* is a solid whose ends or bases are parallel, but not similar, and whose sides are quadrilateral. It differs from a prism or a frustum of a pyramid, in having its ends dissimilar. It is a *rectangular* prismoid, when its ends are right parallelograms.

XIV. A *linear side* or *edge* of a solid is the line of intersection of two of the planes which form the surface.

42. The common *measuring unit* of solids is a *cube*, whose sides are squares of the same name. The sides of a cubic inch are square inches; of a cubic foot, square feet, &c. Finding the *capacity*, *solidity*,* or *solid contents* of a body, is finding the number of cubic measures, of some given denomination contained in the body.

*In solid measure.*

| | | |
|---|---|---|
| 1728 | cubic inches | =1 cubic foot, |
| 27 | cubic feet | =1 cubic yard, |
| $4492\frac{1}{8}$ | cubic feet | =1 cubic rod, |
| 32768000 | cubic rods | =1 cubic mile, |
| 282 | cubic inches | =1 ale gallon, |
| 231 | cubic inches | =1 wine gallon, |
| 2150.42 | cubic inches | =1 bushel, |
| 1 | cubic foot of pure water weighs 1000 avoirdupois ounces, or $62\frac{1}{2}$ pounds. | |

* See note E.

### PROBLEM I.

*To find the* SOLIDITY *of a* PRISM.

43. MULTIPLY THE AREA OF THE BASE BY THE HEIGHT.

This is a general rule, applicable to parallelopipeds whether right or oblique, cubes, triangular prisms, &c.

As *surfaces* are measured, by comparing them with a right *parallelogram* (Art. 3.); so *solids* are measured, by comparing them with a right *parallelopiped.*

If ABCD (Fig. 1.) be the base of a right parallelopiped, as a stick of timber standing erect, it is evident that the number of *cubic* feet contained in *one* foot of the height, is equal to the number of *square* feet in the area of the base. And if the solid be of any other height, instead of one foot, the contents must have the same ratio. For parallelopipeds of the same base are to each other as their heights. (Sup. Euc. 9, 3.) The solidity of a right parallelopiped, therefore, is equal to the *product of its length, breadth, and thickness.* See Alg. 523.

And an *oblique* parallelopiped being equal to a right one of the same base and altitude, (Sup. Euc. 7, 3.) is equal to the area of the base multiplied into the perpendicular height. This is true also of *prisms*, whatever be the form of their bases. (Sup. Euc. 2. Cor. to 8, 3.)

44. As the sides of a *cube* are all *equal*, the solidity is found by *cubing one of its edges.* On the other hand, if the solid contents be given, the length of the edges may be found, by *extracting the cube root.*

45. When solid measure is cast by *Duodecimals*, it is to be observed that *inches* are not *primes* of feet, but *thirds.* If the unit is a cubic foot, a solid which is an inch thick and a foot square is a prime; a parallelopiped a foot long, an inch broad, and an inch thick is a second, or the twelfth part of a prime; and a cubic inch is a third, or a twelfth part of a second. A linear inch is $\frac{1}{12}$ of a foot, a square inch $\frac{1}{144}$ of a foot, and a cubic inch $\frac{1}{1728}$ of a foot.

Ex. 1. What are the solid contents of a stick of timber which is 31 feet long, 1 foot 3 inches broad, and 9 inches thick? Ans. 29 feet 9″, or 29 feet 108 inches.

2. What is the solidity of a wall which is 22 feet long, 12 feet high, and 2 feet 6 inches thick?

Ans. 660 cubic feet.

3. What is the capacity of a cubical vessel which is 2 feet 3 inches deep?

Ans. 11F. 4′ 8″ 3‴, or 11 feet 675 inches.

4. If the base of a prism be 108 square inches, and the height 36 feet, what are the solid contents?

Ans. 27 cubic feet.

5. If the height of a square prism be $2\frac{1}{4}$ feet, and each side of the base $10\frac{1}{3}$ feet, what is the solidity?

The area of the base $=10\frac{1}{3}\times10\frac{1}{3}=106\frac{7}{9}$ sq. feet.

And the solid contents $=106\frac{7}{9}\times2\frac{1}{4}=240\frac{1}{4}$ cubic feet.

6. If the height of a prism be 23 feet, and its base a regular pentagon, whose perimeter is 18 feet, what is the solidity?

Ans. 512.84 cubic feet.

46. The number of *gallons* or *bushels* which a vessel will contain may be found, by calculating the capacity in *inches*, and then dividing by the number of inches in 1 gallon or bushel.

The *weight of water* in a vessel of given dimensions is easily calculated; as it is found by experiment, that a cubic foot of pure water weighs 1000 ounces avoirdupois. For the weight in ounces, then, multiply the cubic feet by 1000; or for the weight in pounds, multiply by $62\frac{1}{2}$.

Ex. 1. How many ale gallons are there in a cistern which is 11 feet 9 inches deep, and whose base is 4 feet 2 inches square?

The cistern contains 352500 cubic inches;

And $352500\div282=1250$.

2. How many wine gallons will fill a ditch 3 feet 11 inches wide, 3 feet deep, and 462 feet long? Ans. 40608.

3. What weight of water can be put into a cubical vessel 4 feet deep? Ans. 4000 lbs.

PROBLEM II.

*To find the* LATERAL SURFACE *of a* RIGHT PRISM.

47. MULTIPLY THE LENGTH INTO THE PERIMETER OF THE BASE.

Each of the sides of the prism is a right parallelogram, whose area is the product of its length and breadth. But the breadth is one side of the base; and therefore, the sum of the breadths is equal to the perimeter of the base.

Ex. 1. If the base of a right prism be a regular hexagon whose sides are each 2 feet 3 inches, and if the height be 16 feet, what is the lateral surface? Ans. 216 square feet.

If the areas of the two ends be added to the lateral surface, the sum will be the whole surface of the prism. And the superficies of any solid bounded by planes, is evidently equal to the areas of all its sides.

Ex. 2. If the base of a prism be an equilateral triangle whose perimeter is 6 feet, and if the height be 17 feet, what is the surface?

The area of the triangle is 1.732. (Art. 11.)
And the whole surface is 105.464.

PROBLEM III.

*To find the* SOLIDITY *of a* PYRAMID.

48. MULTIPLY THE AREA OF THE BASE INTO $\frac{1}{3}$ OF THE HEIGHT.

The solidity of a *prism* is equal to the product of the area of the base into the height. (Art. 43.) And a pyramid is $\frac{1}{3}$ of a prism of the same base and altitude. (Sup. Euc. 15, 3. Cor. 1.) Therefore the solidity of a pyramid whether right or oblique, is equal to the product of the base into $\frac{1}{3}$ of the perpendicular height.

Ex. 1. What is the solidity of a triangular pyramid, whose height is 60, and each side of whose base is 4?

The area of the base is 6.928
And the solidity is 138.56.

2. Let ABC (Fig. 16.) be one side of an oblique pyramid whose base is 6 feet square; let BC be 20 feet, and make an angle of 70 degrees with the plane of the base; and let CP be perpendicular to this plane. What is the solidity of the pyramid?

In the right angled triangle BCP, (Trig. 134.)
R : BC : : sin B : : PC=18.79.
And the solidity of the pyramid is 225.48 feet.

3. What is the solidity of a pyramid whose perpendicular height is 72, and the sides of whose base are 67, 54, and 40?
Ans. 25920.

### PROBLEM IV.

*To find the* LATERAL SURFACE *of a* REGULAR PYRAMID.

49. MULTIPLY HALF THE SLANT-HEIGHT INTO THE PERIMETER OF THE BASE.

Let the triangle ABC (Fig. 18.) be one of the sides of a regular pyramid. As the sides AC and BC are equal, the angles A and B are equal. Therefore a line drawn from the vertex C to the middle of AB is *perpendicular* to AB. The area of the triangle is equal to the product of half this perpendicular into AB. (Art. 8.) The perimeter of the base is the sum of its sides, each of which is equal to AB. And the areas of all the equal triangles which constitute the lateral surface of the pyramid, are together equal to the product of the perimeter into half the slant-height CP.

The *slant-height* is the hypothenuse of a right angled triangle, whose legs are the axis of the pyramid, and the distance from the center of the base to the middle of one of the sides. See Def. 10.

Ex. 1. What is the lateral surface of a regular hexagonal pyramid, whose axis is 20 feet, and the sides of whose base are each 8 feet?

The square of the distance from the center of the base to one of the sides (Art. 16.) $=48$.
The slant-height (Euc. 47. 1.) $=\sqrt{48+(20)^2}=21.16$.
And the lateral surface $=21.16\times4\times6=507.84$ sq. feet.

2. What is the whole surface of a regular triangular pyramid whose axis is 8, and the sides of whose base are each 20.78?

| | |
|---|---|
| The lateral surface is | 312 |
| The area of the base is | 187 |
| And the whole surface is | 499 |

3. What is the lateral surface of a regular pyramid whose axis is 12 feet, and whose base is 18 feet square?

Ans. 540 square feet.

The lateral surface of an *oblique* pyramid may be found, by taking the sum of the areas of the unequal triangles which form its sides.

### PROBLEM V.

*To find the* SOLIDITY *of a* FRUSTUM *of a pyramid.*

50. ADD TOGETHER THE AREAS OF THE TWO ENDS, AND THE SQUARE ROOT OF THE PRODUCT OF THESE AREAS; AND MULTIPLY THE SUM BY $\frac{1}{3}$ OF THE PERPENDICULAR HEIGHT OF THE SOLID.

Let CDGL (Fig. 17.) be a vertical section, through the middle of a frustum of a right pyramid CDV whose base is a *square*.

Let $CD=a$, $LG=b$, $RN=h$.

By similar triangles, LG : CD::RV : NV.

Subtracting the antecedents, (Alg. 389.)
LG : CD−LG::RV : NV−RV=RN.

$$\text{Therefore } RV=\frac{RN\times LG}{CD-LG}=\frac{hb}{a-b}$$

The square of CD is the base of the pyramid CDV;
And the square of LG is the base of the small pyramid LGV.
Therefore, the solidity of the larger pyramid (Art. 48.) is

$$\overline{CD}^2\times\tfrac{1}{3}(RN+RV)=a^2\times\tfrac{1}{3}\left(h+\frac{hb}{a-b}\right)=\frac{ha^3}{3a-3b}$$

And the solidity of the smaller pyramid is equal to

$$\overline{LG}^2\times\tfrac{1}{3}RV=b^2\times\frac{hb}{3a-3b}=\frac{hb^3}{3a-3b}.$$

If the smaller pyramid be taken from the larger, there will remain the frustum CDLG, whose solidity is equal to

$$\frac{ha^3-hb^3}{3a-3b}=\tfrac{1}{3}h\times\frac{a^3-b^3}{a-b}=\tfrac{1}{3}h\times(a^2+ab+b^2) \qquad \text{(Alg. 466.)}$$

Or, because $\sqrt{a^2b^2}=ab$, (Alg. 259.)

$$\tfrac{1}{3}h\times(a^2+b^2+\sqrt{a^2b^2})$$

Here $h$, the height of the frustum, is multiplied into $a^2$ and $b^2$, the areas of the two ends, and into $\sqrt{a^2b^2}$ the square root of the products of these areas.

In this demonstration, the pyramid is supposed to be *square*. But the rule is equally applicable to a pyramid of any other form. For the solid contents of pyramids are equal, when they have equal heights and bases, whatever be the *figure* of their bases. (Sup. Euc. 14. 3.) And the sections parallel to the bases, and at equal distances, are equal to one another. (Sup. Euc. 12. 3. Cor. 2.)*

Ex. 1. If one end of the frustum of a pyramid be 9 feet square, the other end 6 feet square, and the height 36 feet, what is the solidity?

The areas of the two ends are 81 and 36.

The square root of their product is 54.

And the solidity of the frustum$=(81+36+54)\times 12=2052$.

2. If the height of a frustum of a pyramid be 24, and the areas of the two ends 441 and 121; what is the solidity?

Ans. 6344.

3. If the height of a frustum of a hexagonal pyramid be 48, each side of one end 26, and each side of the other end 16; what is the solidity? Ans. 56034.

### PROBLEM VI.

*To find the* LATERAL SURFACE *of a* FRUSTUM *of a regular pyramid.*

51. MULTIPLY HALF THE SLANT-HEIGHT BY THE SUM OF THE PERIMETERS OF THE TWO ENDS.

Each side of a frustum of a regular pyramid is a *trapezoid*, as ABCD. (Fig. 19.) The slant-height HP, (Def. 11.) though it is oblique to the base of the solid, is perpendicular to the line AB. The area of the trapezoid is equal to the product of half this perpendicular into the sum of the parallel sides AB and DC. (Art. 12.) Therefore the area of all the equal trapezoids which form the lateral surface of

* See note F.

the frustum, is equal to the product of half the slant-height into the sum of the perimeters of the ends.

Ex. If the slant-height of a frustum of a regular octagonal pyramid be 42 feet, the sides of one end 5 feet each, and the sides of the other end 3 feet each; what is the lateral surface?, Ans. 1344 square feet.

52. If the slant-height be not given, it may be obtained from the perpendicular height, and the dimensions of the two ends. Let GD (Fig. 17.) be the slant-height of the frustum CDGL, RN or GP the perpendicular height, ND and RG the radii of the circles inscribed in the perimeters of the two ends. Then PD is the difference of the two radii:

And the slant-height $GD = \sqrt{(\overline{GP}^2 + \overline{PD}^2)}$.

Ex. If the perpendicular height of a frustum of a regular hexagonal pyramid be 24, the sides of one end 13 each, and the sides of the other end 8 each; what is the whole surface?

$\sqrt{(\overline{BC}^2 - \overline{BP}^2)} = CP$, (Fig. 7.) that is, $\sqrt{(\overline{13}^2 - \overline{6.5}^2)} = 11.258$

And $\sqrt{8^2 - 4^2} = 6.928$

The difference of the two radii is, therefore, 4.33

The slant-height $= \sqrt{(\overline{24}^2 + \overline{4.33}^2)} = 24.3875$

The lateral surface is 1536.4

And the whole surface, 2141.75.

53. The height of the *whole pyramid* may be calculated from the dimensions of the frustum. Let VN (Fig. 17.) be the height of the pyramid, RN or GP the height of the frustum, ND and RG the radii of the circles inscribed in the perimeters of the ends of the frustum.

Then, in the similar triangles GPD and VND,

DP : GP : : DN : VN.

The height of the frustum subtracted from VN, gives VR the height of the small pyramid VLG. The *solidity* and *lateral surface* of the frustum may then be found, by subtracting from the whole pyramid, the part which is above the cutting plane. This method may serve to verify the calculations which are made by the rules in arts. 50 and 51.

Ex. If one end of the frustum CDGL (Fig. 17.) be 90 feet square, the other end 60 feet square, and the height RN 36 feet; what is the height of the whole pyramid VCD: and what are the solidity and lateral surface of the frustum?

DP=DN−GR=45−30=15. And GP=RN=36.

Then 15 : 36 : : 45 : 108=VN, the height of the whole pyramid.

And 108−36=72=VR, the height of the part VLG.

| | | |
|---|---|---|
| The solidity of the large pyramid is | 291600 | (Art. 48.) |
| of the small pyramid | 86400 | |
| of the frustum CDGL | 205200 | |
| The lateral surface of the large pyramid is | 21060 | (Art. 49.) |
| of the small pyramid | 9360 | |
| of the frustum | 11700 | |

PROBLEM VII.

*To find the* SOLIDITY *of a* WEDGE.

54. ADD THE LENGTH OF THE EDGE TO TWICE THE LENGTH OF THE BASE, AND MULTIPLY THE SUM BY $\frac{1}{6}$ OF THE PRODUCT OF THE HEIGHT OF THE WEDGE AND THE BREADTH OF THE BASE.

Let L=AB the length of the base. (Fig. 20.)
$l$=GH the length of the edge.
$b$=BC the breadth of the base.
$h$=PG the height of the wedge.
Then L − $l$=AB−GH=AM.

If the length of the base and the edge be *equal*, as BM and GH, (Fig 20.) the wedge MBHG is half a parallelopiped of the same base and height. And the solidity (Art. 43.) is equal to half the product of the height, into the length and breadth of the base; that is to $\frac{1}{2}$ *bhl*.

If the length of the base be *greater* than that of the edge, as ABGH; let a section be made by the plane GMN, parallel to HBC. This will divide the whole wedge into two parts MBHG and AMG. The latter is a pyramid, whose solidity (Art. 48.) is $\frac{1}{3}bh\times(L-l)$

The solidity of the parts together, is, therefore,

$$\tfrac{1}{2}bhl+\tfrac{1}{3}bh\times(L-l)=\tfrac{1}{6}bh3l+\tfrac{1}{6}bh2L-\tfrac{1}{6}bh2l=\tfrac{1}{6}bh\times(2L+l)$$

If the length of the base be *less* than that of the edge, it is evident that the pyramid is to be *subtracted* from half the parallelopiped, which is equal in height and breadth to the wedge, and equal in length to the edge.

The solidity of the wedge is, therefore,

$$\tfrac{1}{2}bhl-\tfrac{1}{3}bh\times(l-L)=\tfrac{1}{6}bh3l-\tfrac{1}{6}bh2l+\tfrac{1}{6}bh2L=\tfrac{1}{6}bh\times(2L+l)$$

Ex. 1. If the base of a wedge be 35 by 15, the edge 55, and the perpendicular height 12.4; what is the solidity?

$$\text{Ans. } (70+55)\times\frac{15\times12.4}{6}=3875.$$

2. If the base of a wedge be 27 by 8, the edge 36, and the perpendicular height 42; what is the solidity?

Ans. 5040.

### PROBLEM VIII.

*To find the* SOLIDITY *of a rectangular* PRISMOID.

55. TO THE AREAS OF THE TWO ENDS, ADD FOUR TIMES THE AREA OF A PARALLEL SECTION EQUALLY DISTANT FROM THE ENDS, AND MULTIPLY THE SUM BY $\frac{1}{6}$ OF THE HEIGHT.

Let L and B (Fig. 21.) be the length and breadth of one end,
$l$ and $b$ the length and breadth of the other end,
M and $m$ the length and breadth of the section in the middle.
and $h$ the height of the prismoid.

The solid may be divided into two wedges, whose bases are the ends of the prismoid, and whose edges are L and $l$. The solidity of the whole, by the preceding article, is

$$\tfrac{1}{6}Bh\times(2L+l)+\tfrac{1}{6}bh\times(2l+L)=\tfrac{1}{6}h(2BL+Bl+2bl+bL)$$

As M is equally distant from L and $l$,

$$2M=L+l, 2m=B+b, \text{ and } 4Mm=(L+l)(B+b)=BL+Bl+bL+lb.$$

Substituting $4Mm$ for its value, in the preceding expression for the solidity, we have

$$\frac{1}{6}h(BL+bl+4Mm)$$

That is, the solidity of the prismoid is equal to $\frac{1}{6}$ of the height, multiplied into the areas of the two ends, and 4 times the area of the section in the middle.

This rule may be applied to prismoids of other forms. For, whatever be the figure of the two ends, there may be drawn in each, such a number of small rectangles, that the sum of them shall differ less, than by any given quantity, from the figure in which they are contained. And the solids between these rectangles will be rectangular prismoids.

Ex. 1. If one end of a rectangular prismoid be 44 feet by 23, the other end 36 by 21, and the perpendicular height 72; what is the solidity?

The area of the larger end $=44\times23=1012$

of the smaller end $=36\times21=\ 756$

of the middle section $=40\times22=\ 880$

And the solidity $=(1012+756+4\times880)\times12=63456$ feet.

2. What is the solidity of a stick of hewn timber, whose ends are 30 inches by 27, and 24 by 18, and whose length is 48 feet? Ans. 204 feet.

Other solids not treated of in this section, if they be bounded by plane surfaces, may be measured by supposing them to be divided into prisms, pyramids, and wedges. And, indeed, every such solid may be considered as made up of triangular pyramids.

## THE FIVE REGULAR SOLIDS.

56. **A SOLID IS SAID TO BE REGULAR, WHEN ALL ITS SOLID ANGLES ARE EQUAL, AND ALL ITS SIDES ARE EQUAL AND REGULAR POLYGONS.**

The following figures are of this description;

| | | |
|---|---|---|
| 1. The *Tetraedron*, | whose sides are | four triangles; |
| 2. The *Hexaedron or cube*, | | six squares; |
| 3. The *Octaedron*, | | eight triangles; |
| 4. The *Dodecaedron*, | | twelve pentagons; |
| 5. The *Icosaedron*, | | twenty triangles.* |

Besides these five, there can be no other regular solids. The only plane figures which can form such solids, are triangles, squares, and pentagons. For the plane angles which contain any solid angle, are together less than four right angles or 360°. (Sup. Euc. 21. 2.) And the least number which can form a solid angle is three. (Sup. Euc. Def. 8. 2.) If they are angles of equilateral *triangles*, each is 60°. The sum of *three* of them is 180°, of *four* 240°, of *five* 300°, and of *six* 360°. The latter number is too great for a solid angle.

The angles of *squares* are 90° each. The sum of *three* of these is 270°, of four 360°, and of any other greater number, still more.

The angles of regular *pentagons* are 108° each. The sum of *three* of them is 324°; of four, or any other greater number, more than 360°. The angles of all other regular polygons are still greater.

In a regular solid, then, each solid angle must be contained by three, four, or five equilateral triangles, by three squares, or by three regular pentagons.

57. As the sides of a regular solid are similar and equal, and the angles are also alike; it is evident that the sides are all equally distant from a central point in the solid. If then, planes be supposed to proceed from the several edges to the center, they will divide the solid into as many equal *pyramids*, as it has sides. The base of each pyramid will be one of the sides; their common vertex will be the central point; and their height will be a perpendicular from the center to one of the sides.

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* For the geometrical construction of these solids, see Legendre's Geometry; Appendix to Books VI and VII.

### PROBLEM IX.

*To find the* SURFACE *of a* REGULAR SOLID.

58. MULTIPLY THE AREA OF ONE OF THE SIDES BY THE NUMBER OF SIDES.

Or,

MULTIPLY THE SQUARE OF ONE OF THE EDGES, BY THE SURFACE OF A SIMILAR SOLID WHOSE EDGES ARE 1.

As all the sides are *equal*, it is evident that the area of one of them, multiplied by the number of sides, will give the area of the whole.

Or, if a *table* is prepared, containing the surfaces of the several regular solids whose linear edges are *unity;* this may be used for other regular solids, upon the principle, that the areas of similar polygons are as the squares of their homologous sides. (Euc. 20. 6.) Such a table is easily formed, by multiplying the area of one of the sides, as given in art. 17, by the number of sides. Thus the area of an equilateral triangle whose side is 1, is 0.4330127. Therefore the surface,

Of a regular tetraedron $=.4330127\times4 \;=1.7320508.$
Of a regular octaedron $=.4330127\times8 \;=3.4641016.$
Of a regular icosaedron $=.4330127\times20=8.6602540.$

See the table in the following article.

Ex. 1. What is the surface of a regular dodecaedron whose edges are each 25 inches?

The area of one of the sides is 1075.3.
And the surface of the whole solid $=1075.3\times12=12903.6.$

2. What is the surface of a regular icosaedron whose edges are each 102? Ans. 90101.3.

### PROBLEM X.

*To find the* SOLIDITY *of a* REGULAR SOLID.

59. MULTIPLY THE SURFACE BY $\frac{1}{3}$ OF THE PERPENDICULAR DISTANCE FROM THE CENTER TO ONE OF THE SIDES.

Or,

MULTIPLY THE CUBE OF ONE OF THE EDGES, BY THE SOLIDITY OF A SIMILAR SOLID WHOSE EDGES ARE 1.

As the solid is made up of a number of equal pyramids, whose bases are the sides, and whose height is the perpendicular distance of the sides from the center (Art. 57.); the solidity of the whole must be equal to the areas of all the sides multiplied into $\frac{1}{3}$ of this perpendicular. (Art. 48.)

If the contents of the several regular solids whose edges are 1, be inserted in a *table*, this may be used to measure other similar solids. For two similar regular solids contain the same number of similar pyramids; and these are to each other as the *cubes* of their linear sides or edges. (Sup. Euc. 15. 3. Cor. 3.)

A TABLE OF REGULAR SOLIDS WHOSE EDGES ARE 1.

| *Names.* | *No. of sides* | *Surfaces.* | *Solidities.* |
|---|---|---|---|
| Tetraedron | 4 | 1.7320508 | 0.1178513 |
| Hexaedron | 6 | 6.0000000 | 1.0000000 |
| Octaedron | 8 | 3.4641016 | 0.4714045 |
| Dodecaedron | 12 | 20.6457288 | 7.6631189 |
| Icosaedron | 20 | 8.6602540 | 2.1816950 |

For the method of calculating the last column of this table, see Hutton's Mensuration, Part III. Sec. 2.

Ex. What is the solidity of a regular octaedron whose edges are each 32 inches? Ans. 15447 inches.

## SECTION IV.*

### THE CYLINDER, CONE, AND SPHERE.

ART. 61. DEFINITION I. A *right cylinder* is a solid described by the revolution of a rectangle about one of its sides. The *ends or bases* are evidently equal and parallel circles. And the *axis*, which is a line passing through the middle of the cylinder, is perpendicular to the bases.

The ends of an *oblique* cylinder are also equal and parallel circles; but they are not perpendicular to the axis. The *height* of a cylinder is the perpendicular distance from one base to the plane of the other. In a right cylinder, it is the length of the axis.

II. A *right cone* is a solid described by the revolution of a right angled triangle about one of the sides which contain the right angle. The *base* is a circle, and is perpendicular to the *axis*, which proceeds from the middle of the base to the vertex.

The base of an *oblique* cone is also a circle, but is not perpendicular to the axis. The *height* of a cone is the perpendicular distance from the vertex to the plane of the base. In a right cone, it is the length of the axis. The *slant-height* of a right cone is the distance from the vertex to the circumference of the base.

III. A *frustum* of a cone is a portion cut off by a plane parallel to the base. The *height* of the frustum is the perpendicular distance of the two ends. The *slant-height* of a frustum of a right cone, is the distance between the peripheries of the two ends, measured on the outside of the solid; as AD. (Fig. 23.)

IV. A *sphere* or *globe* is a solid which has a center equally distant from every part of the surface. It may be described by the revolution of a semicircle about a diameter. A *radius* of the sphere is a line drawn from the center to any

---

* Hutton's Mensuration, West's Mathematics, Legendre's, Clairaut's, and Camus's Geometry.

part of the surface. A *diameter* is a line passing through the center, and terminated at both ends by the surface. The *circumference* is the same as the circumference of a circle whose plane passes through the center of the sphere. Such a circle is called a *great circle*.

V. A *segment* of a sphere is a part cut off by any plane. The *height* of the segment is a perpendicular from the middle of the base to the convex surface, as LB. (Fig. 12.)

VI. A *spherical zone* or frustum is a part of the sphere included between two parallel planes. It is called the *middle zone*, if the planes are equally distant from the center. The *height* of a zone is the distance of the two planes, as LR. (Fig. 12.*)

VII. A *spherical sector* is a solid produced by a *circular* sector, revolving in the same manner as the semicircle which describes the whole sphere. Thus a spherical sector is described by the circular sector ACP (Fig. 15.) or GCE revolving on the axis CP.

VIII. A solid described by the revolution of any figure about a fixed axis, is called a *solid of revolution*.

### PROBLEM I.

*To find the* CONVEX SURFACE *of a* RIGHT CYLINDER.

62. MULTIPLY THE LENGTH INTO THE CIRCUMFERENCE OF THE BASE.

If a right cylinder be covered with a thin substance like paper, which can be spread out into a plane; it is evident that the plane will be a *parallelogram*, whose length and breadth will be equal to the length and circumference of the cylinder. The area must, therefore, be equal to the length multiplied into the circumference. (Art. 4.)

Ex. 1. What is the convex surface of a right cylinder which is 42 feet long, and 15 inches in diameter?

Ans. $42 \times 1.25 \times 3.14159 = 164.933$ sq. feet.

---

* According to some writers, a spherical *segment* is either a solid which is cut off from a sphere by a single plane, or one which is included between two planes: and a *zone* is the *surface* of either of these. In this sense, the term zone is commonly used in geography.

2. What is the whole surface of a right cylinder, which is 2 feet in diameter and 36 feet long?

| | |
|---|---|
| The convex surface is | 226.1945 |
| The area of the two ends (Art. 30.) is | 6.2832 |
| The whole surface is | 232.4777 |

3. What is the whole surface of a right cylinder whose axis is 82, and circumference 71? Ans. 6624.32.

63. It will be observed that the rules for the *prism* and *pyramid* in the preceding section, are substantially the same, as the rules for the *cylinder* and *cone* in this. There may be some advantage, however, in considering the latter by themselves.

In the base of a *cylinder*, there may be inscribed a polygon, which shall differ from it less than by any given space. (Sup. Euc. 6. 1. Cor.) If the polygon be the base of a *prism*, of the same height as the cylinder, the two solids may differ less than by any given quantity. In the same manner, the base of a *pyramid* may be a polygon of so many sides, as to differ less than by any given quantity, from the base of a *cone* in which it is inscribed. A cylinder is therefore considered, by many writers, as a prism of an infinite number of sides; and a cone, as a pyramid of an infinite number of sides. For the meaning of the term "infinite," when used in the mathematical sense, see Alg. Sec. xv.

## PROBLEM II.

### *To find the* SOLIDITY *of a* CYLINDER.

64. MULTIPLY THE AREA OF THE BASE BY THE HEIGHT.

The solidity of a *parallelopiped* is equal to the product of the base into the perpendicular altitude. (Art. 43.) And a parallelopiped and a cylinder which have equal bases and altitudes are equal to each other. (Sup. Euc. 17. 3.)

Ex. 1. What is the solidity of a cylinder, whose height is 121, and diameter 45.2?

Ans. $\overline{45.2}^2 \times .7854 \times 121 = 194156.6$.

2. What is the solidity of a cylinder, whose height is 424, and circumference 213? Ans. 1530837.

3. If the side AC of an oblique cylinder (Fig. 22.) be 27, and the area of the base 32.61, and if the side make an angle of 62° 44′ with the base, what is the solidity?

R : AC : : sin A : BC=24 the perpendicular height.

And the solidity is 782.64.

4. The Winchester bushel is a hollow cylinder, $18\frac{1}{2}$ inches in diameter, and 8 inches deep. What is its capacity?

The area of the base$=(18.5)^2\times.7853982=268.8025$.

And the capacity is 2150.42 cubic inches. See the table in Art. 42.

PROBLEM III.

*To find the* CONVEX SURFACE *of a* RIGHT CONE.

65. MULTIPLY HALF THE SLANT-HEIGHT INTO THE CIRCUMFERENCE OF THE BASE.

If the convex surface of a right cone be spread out into a plane, it will evidently form a *sector* of a circle whose radius is equal to the slant-height of the cone. But the area of the sector is equal to the product of half the radius into the length of the arc. (Art. 34.) Or if the cone be considered as a pyramid of an infinite number of sides, its lateral surface is equal to the product of half the slant-height into the perimeter of the base. (Art. 49.)

Ex. 1. If the slant-height of a right cone be 82 feet, and the diameter of the base 24, what is the convex surface?

Ans. $41\times24\times3.14159=3091.3$ square feet.

2. If the axis of a right cone be 48, and the diameter of the base 72, what is the whole surface?

The slant-height $=\sqrt{(\overline{36}^2+\overline{48}^2)}=60$. (Euc. 47. 1.)

| | |
|---|---|
| The convex surface is | 6786 |
| The area of the base | 4071.6 |
| And the whole surface | 10857.6 |

3. If the axis of a right cone be 16, and the circumference of the base 75.4; what is the whole surface?

Ans. 1206.4.

## PROBLEM IV.

### *To find the* SOLIDITY *of a* CONE.

66. MULTIPLY THE AREA OF THE BASE INTO $\frac{1}{3}$ OF THE HEIGHT.

The solidity of a *cylinder* is equal to the product of the base into the perpendicular height. (Art. 64.) And if a cone and a cylinder have the same base and altitude, the cone is $\frac{1}{3}$ of the cylinder. (Sup. Euc. 18. 3.) Or if a cone be considered as a pyramid of an infinite number of sides, the solidity is equal to the product of the base into $\frac{1}{3}$ of the height, by art. 48.

Ex. 1. What is the solidity of a right cone whose height is 663, and the diameter of whose base is 101?

Ans. $\overline{101}^2 \times .7854 \times 221 = 1770622$.

2. If the axis of an oblique cone be 738, and make an angle of 30° with the plane of the base; and if the circumference of the base be 355, what is the solidity?

Ans. 1233536.

## PROBLEM V.

### *To find the* CONVEX SURFACE *of a* FRUSTUM *of a right cone.*

67. MULTIPLY HALF THE SLANT-HEIGHT BY THE SUM OF THE PERIPHERIES OF THE TWO ENDS.

This is the rule for a frustum of a *pyramid*; (Art. 51.) and is equally applicable to a frustum of a *cone*, if a cone be considered as a pyramid of an infinite number of sides. (Art. 63.)

Or thus,

Let the sector ABV (Fig. 23.) represent the convex surface of a right cone, (Art. 65.) and DCV the surface of a portion of the cone, cut off by a plane parallel to the base. Then will ABCD be the surface of the frustum.

Let AB$=a$, DC$=b$, VD$=d$, AD$=h$.
Then the area ABV$=\frac{1}{2}a\times(h+d)=\frac{1}{2}ah+\frac{1}{2}ad$. (Art. 34.)
And the area DCV$=\frac{1}{2}bd$.

Subtracting the one from the other,
The area ABDC$=\frac{1}{2}ah+\frac{1}{2}ad-\frac{1}{2}bd$.
But $d:d+h::b:a$. (Sup.Euc.3.1.) Therefore $\frac{1}{2}ad-\frac{1}{2}bd=\frac{1}{2}bh$.

The surface of the frustum, then, is equal to
$\frac{1}{2}ah+\frac{1}{2}bh$. or $\frac{1}{2}h\times(a+b)$

Cor. The surface of the frustum is equal to the product of the slant-height into the circumference of a circle which is *equally distant* from the two ends. Thus the surface ABCD (Fig. 23.) is equal to the product of AD into MN. For MN is equal to half the sum of AB and DC.

Ex. 1. What is the convex surface of a frustum of a right cone, if the diameters of the two ends be 44 and 33, and the slant-height 84? Ans. 10159.8.

2. If the perpendicular height of a frustum of a right cone be 24, and the diameters of the two ends 80 and 44, what is the whole surface?

Half the difference of the diameters is 18.

And $\sqrt{\overline{18}^2+\overline{24}^2}=30$, the slant-height, (Art. 52.)

| | |
|---|---|
| The convex surface of the frustum is | 5843 |
| The sum of the areas of the two ends is | 6547 |
| And the whole surface is | 12390 |

PROBLEM VI.

*To find the* SOLIDITY *of a* FRUSTUM *of a cone.*

68. ADD TOGETHER THE AREAS OF THE TWO ENDS, AND THE SQUARE ROOT OF THE PRODUCT OF THESE AREAS; AND MULTIPLY THE SUM BY $\frac{1}{3}$ OF THE PERPENDICULAR HEIGHT.

This rule, which was given for the frustum of a *pyramid*, (Art. 50.) is equally applicable to the frustum of a cone; because a cone and a pyramid which have equal bases and altitudes are equal to each other.

Ex. 1. What is the solidity of a mast which is 72 feet long, 2 feet in diameter at one end, and 18 inches at the other? Ans. 174.36 cubic feet.

2. What is the capacity of a conical cistern which is 9 feet deep, 4 feet in diameter at the bottom, and 3 feet at the top? Ans. 87.18 cubic feet =652.15 wine gallons.

3. How many gallons of ale can be put into a vat in the form of a conic frustum, if the larger diameter be 7 feet, the smaller diameter 6 feet, and the depth 8 feet?

### PROBLEM VII.

*To find the* SURFACE *of a* SPHERE.

69. MULTIPLY THE DIAMETER BY THE CIRCUMFERENCE.

Let a hemisphere be described by the quadrant CPD, (Fig. 25.) revolving on the line CD. Let AB be a side of a regular polygon inscribed in the circle of which DBP is an arc. Draw AO and BN perpendicular to CD, and BH perpendicular to AO. Extend AB till it meets CD continued. The triangle AOV, revolving on OV as an axis, will describe a right *cone*. (Defin. 2.) AB will be the slant-height of a *frustum* of this cone extending from AO to BN. From G the middle of AB, draw GM parallel to AO. The surface of the frustum described by AB, (Art. 67. Cor.) is equal to

$$AB \times circ\ GM.*$$

From the center C draw CG, which will be perpendicular to AB, (Euc. 3. 3.) and the radius of a circle inscribed in the polygon. The triangles ABH and CGM are similar, because the sides are perpendicular, each to each. Therefore,

$$HB \text{ or } ON : AB :: GM : GC :: circ\ GM : circ\ GC.$$

So that $ON \times circ\ GC = AB \times circ\ GM$, that is, the surface of the frustum is equal to the product of ON the perpendicular height, into *circ* GC, the perpendicular distance from the center of the polygon to one of the sides.

* By *circ* GM is meant the circumference of a circle the radius of which is GM.

In the same manner it may be proved, that the surfaces produced by the revolution of the lines BD and AP about the axis DC, are equal to

$$ND \times circ\ GC, \qquad \text{and } CO \times circ\ GC.$$

The surface of the whole solid, therefore, (Euc. 1.2.) is equal to

$$CD \times circ\ GC.$$

The demonstration is applicable to a solid produced by the revolution of a polygon of *any* number of sides. But a polygon may be supposed which shall differ less than by any given quantity from the circle in which it is inscribed; (Sup. Euc. 4. 1.) and in which the perpendicular GC shall differ less than by any given quantity from the radius of the circle. Therefore the surface of a *hemisphere* is equal to the product of its radius into the circumference of its base; and *the surface of a sphere is equal to the product of its diameter into its circumference.*

Cor. 1. From this demonstration it follows, that the surface of any *segment* or *zone* of a sphere is equal to the product of the height of the segment or zone into the circumference of the sphere. The surface of the zone produced by the revolution of the arc AB about ON, is equal to $ON \times circ\ CP$. And the surface of the segment produced by the revolution of BD about DN is equal to $DN \times circ\ CP$.

Cor. 2. The surface of a sphere is equal to four times the area of a circle of the same diameter; and therefore, the convex surface of a hemisphere is equal to twice the area of its base. For the area of a circle is equal to the product of half the diameter into half the circumference; (Art. 30.) that is, to $\frac{1}{4}$ the product of the diameter and circumference.

Cor. 3. The surface of a sphere, or the convex surface of any spherical segment or zone, is equal to that of the circumscribing cylinder. A hemisphere described by the revolution of the arc DBP, is circumscribed by a cylinder produced by the revolution of the parallelogram D*d*CP. The convex surface of the cylinder is equal to its height multiplied by its circumference. (Art. 62.) And this is also the surface of the hemisphere.

So the surface produced by the revolution of AB is equal to that produced by the revolution of *ab*. And the surface produced by BD is equal to that produced by *bd*:

Ex. 1. Considering the earth as a sphere 7930 miles in diameter, how many square miles are there on its surface?
Ans. 197,558,500.

2. If the circumference of the sun be 2,800,000 miles, what is his surface? Ans. 2,495,547,600,000 sq. miles.

3. How many square feet of lead will it require, to cover a hemispherical dome whose base is 13 feet across?
Ans. 265½.

## PROBLEM VIII.

### *To find the* SOLIDITY *of a* SPHERE.

70. 1. MULTIPLY THE CUBE OF THE DIAMETER BY .5236.

Or,

2. MULTIPLY THE SQUARE OF THE DIAMETER BY $\frac{1}{6}$ OF THE CIRCUMFERENCE.

Or,

3. MULTIPLY THE SURFACE BY $\frac{1}{6}$ OF THE DIAMETER.

1. A sphere is *two thirds* of its circumscribing cylinder. (Sup. Euc. 21. 3.) The height and diameter of the cylinder are each equal to the diameter of the sphere. The solidity of the cylinder is equal to its height multiplied into the area of its base, (Art. 64.) that is putting D for the diameter,

$$D \times D^2 \times .7854 \quad \text{or} \quad D^3 \times .7854.$$

And the solidity of the *sphere*, being $\frac{2}{3}$ of this, is

$$D^3 \times .5236.$$

2. The base of the circumscribing cylinder is equal to half the circumference multiplied into half the diameter; (Art. 30.) that is, if C be put for the circumference,

$$\tfrac{1}{4}C \times D; \text{ and the solidity is } \tfrac{1}{4}C \times D^2.$$

Therefore the solidity of the sphere is

$$\tfrac{2}{3} \text{ of } \tfrac{1}{4}C \times D^2 = D^2 \times \tfrac{1}{6}C.$$

3. In the last expression, which is the same as $C \times D \times \frac{1}{6}D$, we may substitute S, the surface, for $C \times D$. (Art. 69.) We then have the solidity of the sphere equal to

$$S \times \tfrac{1}{6}D.$$

Or, the sphere may be supposed to be filled with small *pyramids*, standing on the surface of the sphere, and having their common vertex in the center. The number of these may be such, that the difference between their sum and the sphere shall be less than any given quantity. The solidity of each pyramid is equal to the product of its base into $\frac{1}{3}$ of its height. (Art. 48.) The solidity of the whole, therefore, is equal to the product of the surface of the sphere into $\frac{1}{3}$ of its radius, or $\frac{1}{6}$ of its diameter.

71. The numbers 3.14159, .7854, .5236, should be made perfectly familiar. The first expresses the ratio of the *circumference* of a circle to the *diameter;* (Art. 23.) the second, the ratio of the *area* of a circle to the square of the diameter (Art. 30.); and the third, the ratio of the *solidity* of a sphere to the *cube* of the diameter. The second is $\frac{1}{4}$ of the first, and the third is $\frac{1}{6}$ of the first.

As these numbers are frequently occurring in mathematical investigations, it is common to represent the first of them by the Greek letter $\pi$. According to this notation,

$$\pi = 3.14159, \qquad \tfrac{1}{4}\pi = .7854, \qquad \tfrac{1}{6}\pi = .5236.$$

If D = the *diameter*, and R = the *radius* of any circle or sphere;

$$\text{Then} \quad D = 2R \quad D^2 = 4R^2 \quad D^3 = 8R^3.$$

$$\left.\begin{array}{l}\text{And } \pi D \\ \text{Or } 2\pi R\end{array}\right\} = \text{the } periph. \quad \left.\begin{array}{l}\frac{1}{4}\pi D^2 \\ \text{or } \pi R^2\end{array}\right\} = \text{the } area \text{ of the circ.} \quad \left.\begin{array}{l}\frac{1}{6}\pi D^3 \\ \text{or } \frac{4}{3}\pi R^3\end{array}\right\} = \text{the}$$
*solidity* of the sphere.

Ex. 1. What is the solidity of the earth, if it be a sphere 7930 miles in diameter?

Ans. 261,107,000,000 cubic miles.

2. How many wine gallons will fill a hollow sphere 4 feet in diameter?

Ans. The capacity is 33.5104 feet = $250\frac{2}{3}$ gallons.

3. If the diameter of the moon be 2180 miles, what is its solidity? Ans. 5,424,600,000 miles.

72. If the solidity of a sphere be *given*, the diameter may be found by reversing the first rule in the preceding article; that is, *dividing by* .5236 and *extracting the cube root of the quotient.*

Ex. 1. What is the diameter of a sphere whose solidity is 65.45 cubic feet? Ans. 5 feet.

2. What must be the diameter of a globe to contain 16755 pounds of water? Ans. 8 feet.

### PROBLEM IX.

*To find the* CONVEX SURFACE *of a* SEGMENT *or* ZONE *of a sphere.*

73. MULTIPLY THE HEIGHT OF THE SEGMENT OR ZONE INTO THE CIRCUMFERENCE OF THE SPHERE.

For the demonstration of this rule, see art. 69.

Ex. 1. If the earth be considered a perfect sphere 7930 miles in diameter, and if the polar circle be 23° 28′ from the pole, how many square miles are there in one of the frigid zones?

If PQOE (Fig. 15.) be a meridian on the earth, ADB one of the polar circles, and P the pole; then the frigid zone is a spherical segment described by the revolution of the arc APB about PD. The angle ACD subtended by the arc AP is 23° 28′. And in the right angled triangle ACD,

$$R : AC :: \cos ACD : CD = 3637.$$

Then $CP - CD = 3965 - 3637 = 328 = PD$ the height of the segment.

And $328 \times 7930 \times 3.14159 = 8171400$ the surface.

2. If the diameter of the earth be 7930 miles, what is the surface of the torrid zone, extending 23° 28′ on each side of the equator?

If EQ (Fig. 15.) be the equator, and GH one of the tropics, then the angle ECG is 23° 28′. And in the right angled triangle GCM, R : CG : : sin ECG : GM=CN=1578.9 the height of half the zone.

The surface of the whole zone is 78669700.

3. What is the surface of each of the temperate zones?

The height DN=CP−CN−PD=2058.1

And the surface of the zone is 51273000.

| | |
|---|---|
| The surface of the two temperate zones is | 102,546,000 |
| of the two frigid zones | 16,342,800 |
| of the torrid zone | 78,669,700 |
| of the whole globe | 197,558,500 |

PROBLEM X.

*To find the* SOLIDITY *of a spherical* SECTOR.

74. MULTIPLY THE SPHERICAL SURFACE BY $\frac{1}{3}$ OF THE RADIUS OF THE SPHERE.

The spherical sector, (Fig. 24.) produced by the revolution of ACBD about CD, may be supposed to be filled with *small pyramids*, standing on the spherical surface ADB, and terminating in the point C. Their number may be so great, that the height of each shall differ less than by any given length from the radius CD, and the sum of their bases shall differ less than by any given quantity from the surface ABD. The solidity of each is equal to the product of its base into $\frac{1}{3}$ of the radius CD. (Art. 48.) Therefore, the solidity of all of them, that is, of the sector ADBC, is equal to the product of the spherical surface into $\frac{1}{3}$ of the radius.

Ex. Supposing the earth to be a sphere 7930 miles in diameter, and the polar circle ADB (Fig. 15.) to be 23° 28′ from the pole; what is the solidity of the spherical sector ACBP?

Ans. 10,799,867,000 miles.

PROBLEM XI.

*To find the* SOLIDITY *of a spherical* SEGMENT.

75. MULTIPLY HALF THE HEIGHT OF THE SEGMENT INTO THE AREA OF THE BASE, AND THE CUBE OF THE HEIGHT INTO .5236; AND ADD THE TWO PRODUCTS.

As the *circular* sector AOBC (Fig. 9.) consists of two parts, the segment AOBP and the triangle ABC; (Art. 35.) so the *spherical* sector produced by the revolution of AOC about OC consists of two parts, the *segment* produced by the revolution of AOP, and the *cone* produced by the revolution of ACP. If then the cone be subtracted from the sector, the remainder will be the segment.

Let CO=R, the radius of the sphere,
PB=$r$, the radius of the base of the segment,
PO=$h$, the height of the segment,
Then PC=R$-h$, the axis of the cone.

The sector$=2\pi R\times h\times\frac{1}{3}R$(Arts. 71, 73, 74.)$=\frac{2}{3}\pi hR^2$.
The cone$=\pi r^2\times\frac{1}{3}(R-h)$ (Arts. 71, 66.)$=\frac{1}{3}\pi r^2R-\frac{1}{3}\pi hr^2$.

Subtracting the one from the other,

The segment $=\frac{2}{3}\pi hR^2-\frac{1}{3}\pi r^2R+\frac{1}{3}\pi hr^2$.

But $DO\times PO=\overline{BO}^2$(Trig.97.*)$=\overline{PO}^2+\overline{PB}^2$ (Euc. 47. 1.)

That is, $2Rh=h^2+r^2$. So that, $R=\dfrac{h^2+r^2}{2h}$

$$\text{And } R^2=\left(\frac{h^2+r^2}{2h}\right)^2=\frac{h^4+2h^2r^2+r^4}{4h^2}$$

Substituting then, for R and $R^2$, their values, and multiplying the factors,

$$\text{The segment}=\tfrac{1}{6}\pi h^3+\tfrac{1}{3}\pi hr^2+\tfrac{1}{6}\frac{\pi r^4}{h}-\tfrac{1}{6}\pi hr^2-\tfrac{1}{6}\frac{\pi r^4}{h}+\tfrac{1}{3}\pi hr^2$$

which, by uniting the terms, becomes

$$\tfrac{1}{2}\pi hr^2+\tfrac{1}{6}\pi h^3.$$

---

* Euclid 31, 3, and 8, 6. Cor.

The first term here is $\frac{1}{2}h\times\pi r^2$, half the height of the segment multiplied into the area of the base; (Art. 71.) and the other $h^3\times\frac{1}{6}\pi$, the cube of the height multiplied into .5236.

If the segment be *greater* than a hemisphere, as ABD; (Fig. 9.) the cone ABC must be *added* to the sector ACBD.

Let PD$=h$ the height of the segment,
Then PC$=h-$R the axis of the cone.

$$\text{The sector ACBD}=\tfrac{2}{3}\pi h\text{R}^2$$

$$\text{The cone}=\pi r^2\times\tfrac{1}{3}(h-\text{R})=\tfrac{1}{3}\pi hr^2-\tfrac{1}{3}\pi r^2\text{R}$$

Adding them together, we have as before,

$$\text{The segment}=\tfrac{2}{3}\pi h\text{R}^2-\tfrac{1}{3}\pi r^2\text{R}+\tfrac{1}{3}\pi hr^2.$$

Cor. The solidity of a spherical segment is equal to half a cylinder of the same base and height + a sphere whose diameter is the height of the segment. For a cylinder is equal to its height multiplied into the area of its base; and a sphere is equal to the cube of its diameter multiplied by .5236.

Thus if O$y$ (Fig. 15.) be half O$x$, the spherical segment produced by the revolution of O$xt$ is equal to the cylinder produced by $tvyx$ + the sphere produced by O$yxz$; supposing each to revolve on the line O$x$.

Ex. 1. If the height of a spherical segment be 8 feet, and the diameter of its base 25 feet; what is the solidity?

Ans. $(25)^2\times.7854\times4+8^3\times.5236=2231.58$ feet.

2 If the earth be a sphere 7930 miles in diameter, and the polar circle 23° 28′ from the pole, what is the solidity of one of the frigid zones?

Ans. 1,303,000,000 miles.

PROBLEM XII.

*To find the* SOLIDITY *of a spherical* ZONE *or frustum.*

76. FROM THE SOLIDITY OF THE WHOLE SPHERE, SUBTRACT THE TWO SEGMENTS ON THE SIDES OF THE ZONE.

Or,

*Add together the squares of the radii of the two ends, and $\frac{1}{3}$ the square of their distance; and multiply the sum by three times this distance, and the product by* .5236.

If from the whole sphere, (Fig. 15.) there be taken the two segments ABP and GHO, there will remain the zone or frustum ABGH.

Or, the zone ABGH is equal to the difference between the segments GHP and ABP.

Let $\left.\begin{array}{l}NP=H\\DP=h,\end{array}\right\}$ the *heights* of the two segments.

$\left.\begin{array}{l}GN=R\\AD=r\end{array}\right\}$ the *radii* of their bases.

$DN=d=H-h$ the *distance* of the two bases, or the height of the zone.

Then the larger segment $=\frac{1}{2}\pi HR^2+\frac{1}{6}\pi H^3$
And the smaller segment $=\frac{1}{2}\pi hr^2+\frac{1}{6}\pi h^3$ } (Art. 75.)

Therefore the zone $ABGH=\frac{1}{6}\pi(3HR^2+H^3-3hr^2-h^3)$

By the properties of the circle, (Euc. 35, 3.)

$ON\times H=R^2$. Therefore $(ON+H)\times H=R^2+H^2$.

$$\text{Or } OP=\frac{R^2+H^2}{H}$$

In the same manner, $OP=\dfrac{r^2+h^2}{h}$

Therefore $3H\times(r^2+h^2)=3h\times(R^2+H^2)$.

Or $3Hr^2+3Hh^2-3hR^2-3hH^2=0$. (Alg. 178.)

8

To reduce the expression for the solidity of the zone to the required form, without altering its value, let these terms be added to it: and it will become

$$\tfrac{1}{6}\pi(3HR^2+3Hr^2-3hR^2-3hr^2+H^3-3H^2h+3Hh^2-h^3)$$

Which is equal to

$$\tfrac{1}{6}\pi\times3(H-h)\times(R^2+r^2+\tfrac{1}{3}(H-h)^2)$$

Or, as $\frac{1}{6}\pi$ equals .5236 (Art. 71.) and $H-h$ equals $d$,

$$\text{The zone}=.5236\times3d\times(R^2+r^2+\tfrac{1}{3}d^2)$$

Ex. 1. If the diameter of one end of a spherical zone is 24 feet, the diameter of the other end 20 feet, and the distance of the two ends, or the height of the zone 4 feet; what is the solidity? Ans. 1566.6 feet.

2. If the earth be a sphere 7930 miles in diameter, and the obliquity of the ecliptic 23° 28′; what is the solidity of one of the temperate zones?

Ans. 55,390,500,000 miles.

3. What is the solidity of the torrid zone?

Ans. 147,720,000,000 miles.

| | |
|---|---|
| The solidity of the two temperate zones is | 110,781,000,000 |
| of the two frigid zones | 2,606,000,000 |
| of the torrid zone | 147,720,000,000 |
| of the whole globe | 261,107,000,000 |

4. What is the convex surface of a spherical zone, whose breadth is 4 feet, on a sphere of 25 feet diameter?

5. What is the solidity of a spherical segment, whose height is 18 feet, and the diameter of its base 40 feet?

## PROMISCUOUS EXAMPLES OF SOLIDS.

Ex. 1. How much water can be put into a cubical vessel three feet deep, which has been previously filled with cannon balls of the same size, 2, 4, 6, or 9 inches in diameter, regularly arranged in tiers, one directly above another?

Ans. $96\frac{1}{8}$ wine gallons.

2. If a cone or pyramid, whose height is three feet, be divided into three equal portions, by sections parallel to the base; what will be the heights of the several parts?

Ans. 24.961, 6.488, and 4.551 inches.

3. What is the solidity of the greatest square prism which can be cut from a cylindrical stick of timber, 2 feet 6 inches in diameter and 56 feet long?*

Ans. 175 cubic feet.

4. How many such globes as the earth are equal in bulk to the sun; if the former is 7930 miles in diameter, and the latter 890,000?

Ans. 1,413,678.

5. How many cubic feet of wall are there in a conical tower, 66 feet high, if the diameter of the base be 20 feet from outside to outside, and the diameter of the top 8 feet; the thickness of the wall being 4 feet at the bottom, and decreasing regularly, so as to be only 2 feet at the top?

Ans. 7188.

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* The common rule for measuring *round timber* is to multiply the square of the *quarter-girt* by the length. The quarter-girt is one fourth of the circumference. This method does not give the whole solidity. It makes an allowance of about one-fifth, for waste in hewing, bark, &c. The solidity of a cylinder is equal to the product of the length into the area of the base.

If C=the circumference, and $\pi$=3.14159, then (Art. 81.)

$$\text{The area of the base} = \frac{C^2}{4\pi} = \left(\frac{C}{\sqrt{4\pi}}\right)^2 = \left(\frac{C}{3.545}\right)^2$$

If then the circumference were divided by 3.545, instead of 4, and the quotient squared, the area of the base would be correctly found. See note G.

6. If a metallic globe is filled with wine, which cost as much at 5 dollars a gallon, as the globe itself at 20 cents for every square inch of its surface; what is the diameter of the globe? Ans. 55.44 inches.

7. If the circumference of the earth be 25,000 miles, what must be the diameter of a metallic globe, which, when drawn into a wire $\frac{1}{20}$ of an inch in diameter, would reach round the earth? Ans. 15 feet and 1 inch.

8. If a conical cistern be 3 feet deep, $7\frac{1}{2}$ feet in diameter at the bottom, and 5 feet at the top; what will be the depth of a fluid occupying half its capacity?

Ans. 14.535 inches.

9. If a globe 20 inches in diameter be perforated by a cylinder 16 inches in diameter, the axis of the latter passing through the center of the former; what part of the solidity, and the surface of the globe will be cut away by the cylinder?

Ans. 3284 inches of the solidity, and 502,655 of the surface.

10. What is the solidity of the greatest cube which can be cut from a sphere three feet in diameter?

Ans. $5\frac{1}{5}$ feet.

11. What is the solidity of a conic frustum, the altitude of which is 36 feet, the greater diameter 16, and the lesser diameter 8?

12. What is the solidity of a spherical segment 4 feet high, cut from a sphere 16 feet in diameter?

# SECTION V.

## ISOPERIMETRY.*

Art. 77. It is often necessary to compare a number of different figures or solids, for the purpose of ascertaining which has the *greatest area*, within a given perimeter, or the *greatest capacity* under a given surface. We may have occasion to determine, for instance, what must be the form of a fort, to contain a given number of troops, with the least extent of wall; or what the shape of a metallic pipe to convey a given portion of water, or of a cistern to hold a given quantity of liquor, with the least expense of materials.

78. Figures which have equal perimeters are called *Isoperimeters*. When a quantity is *greater* than any other of the same class, it is called a *maximum*. A multitude of straight lines, of different lengths, may be drawn within a circle. But among them all, the *diameter* is a *maximum*. Of all *sines* of angles, which can be drawn in a circle, the sine of 90° is a *maximum*.

When a quantity is *less* than any other of the same class, it is called a *minimum*. Thus, of all straight lines drawn from a given point to a given straight line, that which is *perpendicular* to the given line is a *minimum*. Of all straight lines drawn from a given point in a circle, to the circumference, the *maximum* and *minimum* are the two parts of the diameter which pass through that point. (Euc. 7, 3.)

In isoperimetry, the object is to determine, on the one hand, in what cases the area is a *maximum*, within a given perimeter; or the capacity a *maximum*, within a given surface: and on the other hand, in what cases the perimeter is a *minimum* for a given area, or the surface a *minimum*, for a given capacity.

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* Emerson's, Simpson's, and Legendre's Geometry, Lhuillier, Fontenelle, Hutton's Mathematics, and Lond. Phil. Trans. Vol. 75.

PROPOSITION I.

79. *An* ISOSCELES TRIANGLE *has a greater area than any scalene triangle, of equal base and perimeter.*

If ABC (Fig. 26.) be an isosceles triangle whose equal sides are AC and BC; and if ABC′ be a scalene triangle on the same base AB, and having AC′+BC′=AC+BC; then the area of ABC is greater than that of ABC′.

Let perpendiculars be raised from each end of the base, extend AC to D, make C′D′ equal to AC′, join BD, and draw CH and C′H′ parallel to AB.

As the angle CAB=ABC, (Euc. 5, 1.) and ABD is a right angle, ABC+CBD=CAB+CDB=ABC+CDB. Therefore CBD=CDB, so that CD=CB; and by construction, C′D′=AC′. The perpendiculars of the equal right angled triangles CHD and CHB are equal; therefore, BH=$\frac{1}{2}$BD. In the same manner, AH′=$\frac{1}{2}$AD′. The line AD=AC+BC =AC′+BC′=D′C′+BC′. But D′C′+BC′>BD′. (Euc. 20, 1.) Therefore, AD>BD′; BD>AD′, (Euc. 47, 1.) and $\frac{1}{2}$BD>$\frac{1}{2}$AD′. But $\frac{1}{2}$BD, or BH, is the height of the isosceles triangle; (Art. 1.) and $\frac{1}{2}$AD′ or AH′, the height of the scalene triangle; and the areas of two triangles which have the same base are as their heights. (Art. 8.) Therefore the area of ABC is greater than that of ABC′. Among all triangles, then, of a given perimeter, and upon a given base, the isosceles triangle is a *maximum.*

Cor. The isosceles triangle has a *less perimeter* than any scalene triangle of the same base and area. The triangle ABC′ being less than ABC, it is evident the perimeter of the former must be enlarged, to make its area equal to the area of the latter.

PROPOSITION II.

80. *A triangle in which two given sides make a* RIGHT ANGLE, *has a greater area than any triangle in which the same sides make an oblique angle.*

If BC, BC′, and BC″ (Fig. 27.) be equal, and if BC be perpendicular to AB; then the right angled triangle ABC, has a greater area than the acute angled triangle ABC′, or the oblique angled triangle ABC″.

Let P′C′ and PC″ be perpendicular to AP. Then, as the three triangles have the same base AB, their areas are as their heights; that is, as the perpendiculars BC, P′C′, and PC″. But BC is equal to BC′, and therefore greater than P′C′. (Euc. 47, 1.) BC is also equal to BC″, and therefore greater than PC″.

### PROPOSITION III.

81. *If all the sides* EXCEPT ONE *of a polygon be given, the area will be the greatest, when the given sides are so disposed, that the figure may be* INSCRIBED IN A SEMICIRCLE, *of which the undetermined side is the diameter.*

If the sides AB, BC, CD, DE, (Fig. 28.) be given, and if their position be such that the area, included between these and another side whose length is not determined, is a *maximum;* the figure may be inscribed in a semicircle, of which the undetermined side AE is the diameter.

Draw the lines AD, AC, EB, EC. By varying the angle at D, the triangle ADE may be enlarged or diminished, without affecting the area of the other parts of the figure. The whole area, therefore, cannot be a *maximum*, unless this triangle be a *maximum*, while the sides AD and ED are given. But if the triangle ADE be a *maximum*, under these conditions, the angle ADE is a right angle; (Art. 80.) and therefore the point D is in the circumference of a circle, of which AE is the diameter. (Euc. 31, 3.) In the same manner it may be proved, that the angles ACE and ABE are right angles, and therefore that the points C and B are in the circumference of the same circle.

The term *polygon* is used in this section to include *triangles*, and *four-sided* figures, as well as other right-lined figures.

82. The area of a polygon, inscribed in a semicircle, in the manner stated above, will not be altered by varying the *order* of the given sides.

The sides AB, BC, CD, DE, (Fig. 28.) are the *chords* of so many arcs. The sum of these arcs, in whatever order they are arranged, will evidently be equal to the semicircumference. And the *segments* between the given sides and the arcs will be the same, in whatever part of the circle they are situated. But the area of the polygon is equal to the area of the semicircle, diminished by the sum of these segments.

83. If a polygon, of which all the sides except one are given, be inscribed in a semicircle whose diameter is the undetermined side; a polygon having the same given sides, cannot be inscribed in any *other* semicircle which is either greater or less than this, and whose diameter is the undetermined side.

The given sides AB, BC, CD, DE, (Fig. 28.) are the chords of arcs whose sum is 180 degrees. But in a larger circle, each would be the chord of a less number of degrees, and therefore the sum of the arcs would be less than 180°: and in a smaller circle, each would be the chord of a greater number of degrees, and the sum of the arcs would be greater than 180°.

## PROPOSITION IV.

84. *A polygon* INSCRIBED IN A CIRCLE *has a greater area, than any polygon of equal perimeter, and the same number of sides, which cannot be inscribed in a circle.*

If in the circle ACHF, (Fig. 30.) there be inscribed a polygon ABCDEFG; and if another polygon *abcdefg* (Fig. 31.) be formed of sides which are the same in number and length, but which are so disposed, that the figure cannot be inscribed in a circle; the area of the former polygon is greater than that of the latter.

Draw the diameter AH, and the chords DH and EH. Upon *de* make the triangle *deh* equal and similar to DEH, and join *ah*. The line *ah* divides the figure *abcdhefg* into two parts, of which *one at least* cannot, by supposition, be inscribed in a semicircle of which the diameter is AH, nor in any other semicircle of which the diameter is the undetermined side. (Art. 83.) It is therefore less than the corresponding part of the figure ABCDHEFG. (Art. 81.) And the other part of *abcdhefg* is not greater than the corresponding part of ABCDHEFG. Therefore the whole figure ABCDHEFG is greater than the whole figure *abcdhefg*. If from these there be taken the equal triangles DEH and *deh*, there will remain the polygon ABCDEFG greater than the polygon *abcdefg*.

85. A polygon of which all the sides are given in number and length, can not be inscribed in circles of different diameters. (Art. 83.) And the area of the polygon will not be altered, by changing the *order* of the sides. (Art. 82.)

## PROPOSITION V.

86. *When a polygon has a greater area than any other, of the same number of sides, and of equal perimeter, the sides are* EQUAL.

The polygon ABCDF (Fig. 29.) cannot be a *maximum*, among all polygons of the same number of sides, and of equal perimeters, unless it be equilateral. For if any two of the sides, as CD and FD, are unequal, let CH and FH be equal, and their sum the same as the sum of CD and FD. The isosceles triangle CHF is greater than the scalene triangle CDF (Art. 79.); and therefore the polygon ABCHF is greater than the polygon ABCDF; so that the latter is not a *maximum*.

## PROPOSITION VI.

87. *A* REGULAR POLYGON *has a greater area than any other polygon of equal perimeter, and of the same number of sides.*

For, by the preceding article, the polygon which is a *maximum* among others of equal perimeters, and the same number of sides, is *equilateral*, and by art. 84, it may be *inscribed in a circle*. But if a polygon inscribed in a circle is equilateral, as ABDFGH (Fig. 7.) it is also *equiangular*. For the sides of the polygon are the bases of so many isosceles triangles, whose common vertex is the center C. The angles at these bases are all equal; and two of them, as AHC and GHC, are equal to AHG one of the angles of the polygon. The polygon, then, being equiangular, as well as equilateral, is a *regular* polygon. (Art. 1. Def. 2.)

Thus an *equilateral triangle* has a greater area, than any other triangle of equal perimeter. And a *square* has a greater area, than any other four-sided figure of equal perimeter.

Cor. A regular polygon has a *less perimeter* than any other polygon of equal area, and the same number of sides.

For if, with a given perimeter, the regular polygon is greater than one which is not regular; it is evident the perimeter of the former must be diminished, to make its area equal to that of the latter.

PROPOSITION VII.

88. *If a polygon be* DESCRIBED ABOUT A CIRCLE, *the areas of the two figures are as their perimeters.*

Let ST (Fig. 32.) be one of the sides of a polygon, either regular or not, which is described about the circle LNR. Join OS and OT, and to the point of contact M draw the radius OM, which will be perpendicular to ST. (Euc. 18, 3.) The triangle OST is equal to half the base ST multiplied into the radius OM. (Art. 8.) And if lines be drawn, in the same manner, from the center of the circle, to the extremities of the several sides of the circumscribed polygon, each of the triangles thus formed will be equal to half its base multiplied into the radius of the circle. Therefore the area of the whole polygon is equal to half its perimeter multiplied into the radius: and the area of the circle is equal to half its circumference multiplied into the radius. (Art. 30.) So that the two areas are to each other as their perimeters.

Cor. 1. If different polygons are described about the same circle, their areas are to each other as their perimeters. For the area of each is equal to half its perimeter, multiplied into the radius of the inscribed circle.

Cor. 2. The *tangent* of an arc is always greater than the arc itself. The triangle OMT (Fig. 32.) is to OMN, as MT to MN. But OMT is greater than OMN, because the former includes the latter. Therefore the tangent MT is greater than the arc MN.

### PROPOSITION VIII.

89. *A* CIRCLE *has a greater area than any p*[illegible] *equal perimeter.*

If a circle and a regular polygon have the sa[illegible] and equal perimeters; each of the sides of the pol[illegible] fall partly *within* the circle. For the area of a *circ*[illegible] polygon is greater than the area of the circle, a[illegible] cludes the other: and therefore, by the precedin[illegible] *perimeter* of the former is greater than that of t[illegible]

Let AD then (Fig. 32.) be one side of a regu[illegible] whose perimeter is equal to the circumference [illegible] RLN. As this falls partly within the circle, the [illegible] lar OP is less than the radius OR. But the area [illegible] gon is equal to half its perimeter multiplied into [illegible] dicular (Art. 15.); and the area of the circle is [illegible] its circumference multiplied into the radius. (Ar[illegible] circle then is greater than the given regular po[illegible] therefore greater than any other polygon of equal [illegible] (Art. 87.)

Cor. 1. A circle has a *less perimeter*, than any polyg[illegible] equal area.

Cor. 2. Among regular polygons of a given perimeter, that which has the *greatest number of sides*, has also the *greatest area.* For the greater the number of sides, the more nearly does the perimeter of the polygon approach to a coincidence with the circumference of a circle.*

### PROPOSITION IX.

90. *A right* PRISM *whose bases are* REGULAR POLYGONS, *has a less surface than any other right prism of the same solidity, the same altitude, and the same number of sides.*

If the altitude of a prism is given, the area of the base is as the solidity (Art. 43.); and if the number of sides is also given, the perimeter is a *minimum* when the base is a regular

* For a rigorous demonstration of this, see Legendre's Geometry, Appendix to Book iv.

Cor. *A* (Art. 87. Cor.) But the lateral surface is as the polygon -. (Art. 47.) Of two right prisms, then, which have For if, altitude, the same solidity, and the same number of greater t.t whose bases are regular polygons has the least rimeter orface, while the areas of the ends are equal.

equal to t right prism whose bases are regular polygons has *olidity*, than any other right prism of the same same altitude, and the same number of sides.

88. *If a p*

*of the two f*

PROPOSITION X.

ht CYLINDER *has a less surface, than any right*

Let ST *same altitude and solidity.*

regular or

Join OS an prism and cylinder have the same altitude and radius OM, areas of their bases are equal. (Art. 64.) But The triang r of the cylinder is less, than that of the prism into the ra r. 1.); and therefore its lateral surface is less, same man reas of the ends are equal.

ties of t

of A right cylinder has a *greater solidity*, than any right n of the same altitude and surface.

PROPOSITION XI.

92. A CUBE *has a less surface than any other right parallelopiped of the same solidity.*

A parallelopiped is a prism, any one of whose faces may be considered a base. (Art. 41. Def. I. and V.) If these are not all *squares*, let one which is not a square be taken for a base. The perimeter of this may be diminished, without altering its area (Art. 87. Cor.); and therefore the surface of the solid may be diminished, without altering its altitude or solidity. (Art. 43, 47.) The same may be proved of each of the other faces which are not squares. The surface is therefore a *minimum*, when *all* the faces are squares, that is, when the solid is a *cube*.

Cor. A cube has a *greater solidity* than any other right parallelopiped of the same surface.

PROPOSITION XII.

93. *A* CUBE *has a greater solidity, than any other right parallelopiped, the sum of whose length, breadth, and depth is equal to the sum of the corresponding dimensions of the cube.*

The solidity is equal to the product of the length, breadth, and depth. If the length and breadth are unequal, the solidity may be increased, without altering the sum of the three dimensions. For the product of two factors whose sum is given, is the greatest when the factors are equal. (Euc. 27. 6.) In the same manner, if the breadth and depth are unequal, the solidity may be increased, without altering the sum of the three dimensions. Therefore, the solid can not be a *maximum*, unless its length, breadth, and depth are equal.

PROPOSITION XIII.

94. *If a* PRISM BE DESCRIBED ABOUT A CYLINDER, *the capacities of the two solids are as their surfaces.*

The capacities of the solids are as the *areas* of their bases, that is, as the *perimeters* of their bases. (Art. 88.) But the lateral surfaces are also as the perimeters of the bases. Therefore the *whole* surfaces are as the solidities.

Cor. The capacities of different prisms, described about the same right cylinder, are to each other as their surfaces.

PROPOSITION XIV.

95. *A right cylinder* WHOSE HEIGHT IS EQUAL TO THE DIAMETER OF ITS BASE *has a greater solidity than any other right cylinder of equal surface.*

Let C be a right cylinder whose height is equal to the diameter of its base; and C′ another right cylinder having the same surface, but a different altitude. If a square prism P be described about the former, it will be a *cube*. But a square prism P′ described about the latter will not be a cube.

Then the surfaces of C and P are as their bases (Arts. 47 and 88.); which are as the bases of C′ and P′ (Sup. Euc. 7, 1.); so that,

$surf C : surf P :: base C : base P :: base C' : base P' :: surf C' : surf P'$

But the surface of C is, by supposition, equal to the surface of C′. Therefore, (Alg. 395.) the surface of P is equal to the surface of P′. And by the preceding article,

$solid P : solid C :: surf P : surf C :: surf P' : surf C' :: solid P' : solid C'$

But the solidity of P is greater than that of P′. (Art. 92. Cor.) Therefore the solidity of C is greater than that of C′.

Schol. A right cylinder whose height is equal to the diameter of its base, is that which *circumscribes a sphere.* It is also called *Archimedes' cylinder;* as he discovered the ratio of a sphere to its circumscribing cylinder; and these are the figures which were put upon his tomb.

Cor. Archimedes' cylinder has a *less surface*, than any other right cylinder of the same capacity.

PROPOSITION XV.

96. *If a* SPHERE BE CIRCUMSCRIBED *by a solid bounded by plane surfaces; the capacities of the two solids are as their surfaces.*

If planes be supposed to be drawn from the center of the sphere, to each of the edges of the circumscribing solid, they will divide it into as many pyramids as the solid has faces. The base of each pyramid will be one of the faces; and the height will be the radius of the sphere. The capacity of the pyramid will be equal, therefore, to its base multiplied into $\frac{1}{3}$ of the radius (Art. 48.); and the capacity of the whole circumscribing solid, must be equal to its whole surface multiplied into $\frac{1}{3}$ of the radius. But the capacity of the sphere is also equal to its surface multiplied into $\frac{1}{3}$ of its radius. (Art. 70.)

Cor. The capacities of different solids circumscribing the same sphere, are as their surfaces.

### PROPOSITION XVI.

97. *A* SPHERE *has a greater solidity than any regular polyedron of equal surface.*

If a sphere and a regular polyedron have the same center, and equal surfaces; each of the faces of the polyedron must fall partly *within* the sphere. For the solidity of a *circumscribing* solid is greater than the solidity of the sphere, as the one includes the other: and therefore, by the preceding article, the *surface* of the former is greater than that of the latter.

But if the faces of the polyedron fall partly within the sphere, their perpendicular distance from the center must be less than the radius. And therefore, if the surface of the polyedron be only equal to that of the sphere, its solidity must be less. For the solidity of the polyedron is equal to its surface multiplied into $\frac{1}{3}$ of the distance from the center. (Art. 59.) And the solidity of the sphere is equal to its surface multiplied into $\frac{1}{3}$ of the radius.

Cor. A sphere has a *less surface* than any regular polyedron of the same capacity.

For other cases of Isoperimetry, see Fluxions.

# APPENDIX.—PART I.

*Containing rules, without demonstrations, for the mensuration of the Conic Sections, and other figures not treated of in the Elements of Euclid.**

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PROBLEM I.

*To find the area of an* ELLIPSE.

101. Multiply the product of the transverse and conjugate axes into .7854.

Ex. What is the area of an ellipse whose transverse axis is 36 feet, and conjugate 28? Ans. 791.68 feet.

PROBLEM II.

*To find the area of a* SEGMENT *of an ellipse*, cut off by a line perpendicular to either axis.

102. If either axis of an ellipse be made the diameter of a circle; and if a line perpendicular to this axis cut off a segment from the ellipse, and from the circle;
The diameter of the circle, is to the other axis of the ellipse;
As the circular segment, to the elliptic segment.

---

* For demonstrations of these rules, see Conic Sections, Spherical Trigonometry, and Fluxions, or Hutton's Mensuration.

Ex. What is the area of a segment cut off from an ellipse whose transverse axis is 415 feet, and conjugate 332; if the height of the segment is 96 feet, and its base is perpendicular to the transverse axis?

The circular segment is 23680 feet.
And the elliptic segment 18944

PROBLEM III.

*To find the area of a conic* PARABOLA.

103. Multiply the base by $\frac{2}{3}$ of the height.

Ex. If the base of a parabola is 26 inches, and the height 9 feet; what is the area? Ans. 13 feet.

PROBLEM IV.

*To find the area of a* FRUSTUM *of a parabola,* cut off by a line parallel to the base.

104. Divide the difference of the cubes of the diameters of the two ends, by the difference of their squares; and multiply the quotient by $\frac{2}{3}$ of the perpendicular height.

Ex. What is the area of a parabolic frustum, whose height is 12 feet, and the diameters of its ends 20 and 12 feet?

Ans. 196 feet.

PROBLEM V.

*To find the area of a conic* HYPERBOLA.

105. Multiply the base by $\frac{2}{3}$ of the height; and correct the product by subtracting from it the series

$$2bh \times \left(\frac{z}{1.3.5}+\frac{z^2}{3.5.7}+\frac{z^3}{5.7.9}+\frac{z^4}{7.9.11}+\&c.\right)$$

In which $\begin{cases} b = \text{the base or double ordinate,} \\ h = \text{the height or abscissa,} \\ z = \text{the height divided by the sum of the height and transverse axis.} \end{cases}$

The series converges so rapidly, that a few of the first terms will generally give the correction with sufficient exactness. This correction is the difference between the hyperbola, and a parabola of the same base and height.

Ex. If the base of a hyperbola be 24 feet, the height 10 and the transverse axis 30; what is the area?

| | |
|---|---|
| The base $\times$ $\frac{2}{3}$ the height is | 160. |
| The first term of the series is | 0.016666 |
| The second | 0.000592 |
| The third | 0.000049 |
| The fourth | 0.000006 |
| Their sum | 0.017313 |
| This into $2bh$ is | 8.31 |
| And the area corrected is | 151.69 |

PROBLEM VI.

*To find the area of a* SPHERICAL TRIANGLE formed by three arcs of great circles of a sphere.

106. As 8 right angles or 720°,
To the excess of the 3 given angles above 180°;
So is the whole surface of the sphere,
To the area of the spherical triangle.

Ex. What is the area of a spherical triangle, on a sphere whose diameter is 30 feet, if the angles are 130°, 102°, and 68°? Ans. 471.24 feet.

PROBLEM VII.

*To find the area of a* SPHERICAL POLYGON formed by arcs of great circles.

107. As 8 right angles, or 720°,
To the excess of all the given angles above the product of the number of angles $-2$ into 180°;
So is the whole surface of the sphere,
To the area of the spherical polygon.

Ex. What is the area of a spherical polygon of seven sides, on a sphere whose diameter is 17 inches; if the sum of all the angles is 1080°? Ans. 227 inches.

### PROBLEM VIII.

*To find the lunar surface included between two great circles of a sphere.*

108. As 360°, to the angle made by the given circles;
So is the whole surface of the sphere, to the surface between the circles.

Or,

The lunar surface is equal to the breadth of the middle part of it, multiplied into the diameter of the sphere.

Ex. If the earth be 7930 miles in diameter, what is the surface of that part of it which is included between the 65th and 83d degree of longitude?

Ans. 9,878,000 square miles.

### PROBLEM IX.

*To find the solidity of a* SPHEROID, formed by the revolution of an ellipse about either axis.

109. Multiply the product of the fixed axis and the square of the revolving axis, into .5236.

Ex. 1. What is the solidity of an oblong spheroid, whose longest and shortest diameters are 40 and 30 feet?

Ans. $40 \times \overline{30}^2 \times .5236 = 18850$ feet.

2. If the earth be an oblate spheroid, whose polar and equatorial diameters are 7930 and 7960 miles; what is its solidity? Ans. 263,000,000,000 miles.

### PROBLEM X.

*To find the solidity of the* MIDDLE FRUSTUM *of a spheroid,* included between two planes which are perpendicular to the axis, and equally distant from the center.

110. Add together the square of the diameter of one end, and twice the square of the middle diameter; multiply the sum by $\frac{1}{3}$ of the height, and the product by .7854.

If D and $d$ = the two diameters, and $h$ = the height;

$$\text{The solidity} = (2D^2+d^2)\times\tfrac{1}{3}h\times.7854.$$

Ex. If the diameter of one end of a middle frustum of a spheroid be 8 inches, the middle diameter 10, and the height 30, what is the solidity?

Ans. 2073.4 inches.

Cor. *Half* the middle frustum is equal to a frustum of which one of the ends passes through the center.

If then D and $d$= the diameters of the two ends, and $h$= the height,

$$\text{The solidity} = (2D^2+d^2)\times\tfrac{1}{3}h\times.7854.$$

### PROBLEM XI.

*To find the solidity of a* PARABOLOID.

111. Multiply the area of the base by half the height.

Ex. If the diameter of the base of a paraboloid be 12 feet, and the height 22 feet, what is the solidity?

Ans. 1243 feet.

### PROBLEM XII.

*To find the solidity of a* FRUSTUM *of a paraboloid.*

112. Multiply the sum of the areas of the two ends by half their distance.

Ex. If the diameter of one end of a frustum of a paraboloid be 8 feet, the diameter of the other end 6 feet, and the length 24 feet; what is the solidity?

Ans. $942\frac{1}{2}$ feet.

Cor. If a cask be in the form of *two equal* frustums of a paraboloid; and

If D=the middle diam. $d$=the end diam. and $h$=the length;

$$\text{The solidity}=(D^2+d^2)\times\tfrac{1}{2}h\times.7854.$$

### PROBLEM XIII.

*To find the solidity of a* HYPERBOLOID, produced by the revolution of a hyperbola on its axis.

113. Add together the square of the radius of the base, and the square of the diameter of a section which is equally distant from the base and the vertex; multiply the sum by the height, and the product by .5236.

If R=the radius of the base, D=the middle diameter, and $h$=the height;

$$\text{The solidity}=(R^2+D^2)\times h\times.5236.$$

Ex. If the diameter of the base of a hyperboloid be 24, the square of the middle diameter 252, and the height 10, what is the solidity? Ans. 2073.4.

### PROBLEM XIV.

*To find the solidity of a* FRUSTUM *of a hyperboloid.*

114. Add together the squares of the radii of the two ends, and the square of the middle diameter; multiply the sum by the height, and the product by .5236.

If R and $r$=the two radii, D=the middle diameter, and $h$=the height;

$$\text{The solidity}=(R^2+r^2+D^2)\times h\times.5236.$$

Ex. If the diameter of one end of a frustum of a hyperboloid be 32, the diameter of the other end 24, the square of the middle diameter $793\frac{3}{5}$, and the length 20, what is the solidity? Ans. 12499.3.

### PROBLEM XV.

*To find the solidity of a* CIRCULAR SPINDLE, produced by the revolution of a circular segment about its base or chord as an axis.

115. From $\frac{1}{3}$ of the cube of half the axis, subtract the product of the central distance into half the revolving circular segment, and multiply the remainder by four times 3.14159.

If $a$=the area of the revolving circular segment,
$l$=half the length or axis of the spindle,
$c$=the distance of this axis from the center of the circle to which the revolving segment belongs;
The solidity$=(\frac{1}{3}l^3-\frac{1}{2}ac)\times 4\times 3.14159$.

Ex. Let a circular spindle be produced by the revolution of the segment ABO (Fig. 9.) about AB. If the axis AB be 140, and OP half the middle diameter of the spindle be 38.4; what is the solidity?

| | |
|---|---|
| The area of the revolving segment is | 3791 |
| The central distance PC | 44.6 |
| The solidity of the spindle | 374402 |

### PROBLEM XVI.

*To find the solidity of the* MIDDLE FRUSTUM *of a circular spindle.*

116. From the square of half the axis of the whole spindle, subtract $\frac{1}{3}$ of the square of half the length of the frustum; multiply the remainder by this half length; from the product subtract the product of the revolving area into the central distance; and multiply the remainder by twice 3.14159.

If L=half the length or axis of the whole spindle,
$l$=half the length of the middle frustum,
$c$=the distance of the axis from the center of the circle,
$a$=the area of the figure which, by revolving, produces the frustum;
The solidity$=(\overline{L^2-\frac{1}{3}l^2}\times l-ac)\times 2\times 3.14159$.

Ex. If the diameter of each end of a frustum of a circular spindle be 21.6, the middle diameter 60, and the length 70; what is the solidity?

| | |
|---|---|
| The length of the whole spindle is | 79.75 |
| The central distance | 11.5 |
| The revolving area | 1703.8 |
| The solidity | 136751.5 |

PROBLEM XVII.

*To find the solidity of a* PARABOLIC SPINDLE, produced by the revolution of a parabola about a double ordinate or base.

117. Multiply the square of the middle diameter by $\frac{8}{15}$ of the axis, and the product by .7854.

Ex. If the axis of a parabolic spindle be 30, and the middle diameter 17, what is the solidity?

Ans. 3631.7.

PROBLEM XVIII.

*To find the solidity of the* MIDDLE FRUSTUM *of a parabolic spindle.*

118. Add together the square of the end diameter, and twice the square of the middle diameter; from the sum subtract $\frac{2}{5}$ of the square of the difference of the diameters, and multiply the remainder by $\frac{1}{3}$ of the length, and the product by .7854.

If D and $d$ = the two diameters, and $l$ = the length;

The solidity $= (2D^2+d^2-\frac{2}{5}(D-d)^2)\times\frac{1}{3}l\times.7854$.

Ex. If the end diameters of a frustum of a parabolic spindle be each 12 inches, the middle diameter 16, and the length 30; what is the solidity? Ans. 5102 inches.

# APPENDIX.—PART II.

## GAUGING OF CASKS.

Art. 119. GAUGING is a practical art, which does not admit of being treated in a very scientific manner. Casks are not commonly constructed in exact conformity with any regular mathematical figure. By most writers on the subject, however, they are considered as nearly coinciding with one of the following forms;

1. The middle frustum of a spheroid,
2. The middle frustum of a parabolic spindle.
3. Two equal frustums of a paraboloid,
4. Two equal frustums of a cone.

The *second* of these varieties agrees more nearly than any of the others, with the forms of casks, as they are commonly made. The first is too much curved, the third too little, and the fourth not at all, from the head to the bung.

120. Rules have already been given, for finding the capacity of each of the four varieties of casks. (Arts. 68, 110, 112, 118.) As the dimensions are taken in *inches*, these rules will give the contents in cubic inches. To abridge the computation, and adapt it to the particular measures used in gauging, the factor .7854 is divided by 282 or 231; and the quotient is used instead of .7854, for finding the capacity in ale gallons or wine gallons.

$$\text{Now}\ \frac{.7854}{282}=.002785,\ \text{or .0028 nearly;}$$

$$\text{And}\ \frac{.7854}{231}=.0034.$$

If then .0028 and 0034 be substituted for .7854, in the rules referred to above; the contents of the cask will be given in ale gallons and wine gallons. These numbers are to each other nearly as 9 to 11.

PROBLEM I.

*To calculate the contents of a cask, in the form of the middle frustum of a* SPHEROID.

121. Add together the square of the head diameter, and twice the square of the bung diameter; multiply the sum by $\frac{1}{3}$ of the length, and the product by .0028 for ale gallons, or by .0034 for wine gallons.

If D and $d$=the two diameters, and $l$=the length;
The capacity in inches$=(2D^2+d^2)\times\frac{1}{3}l\times.7854$. (Art. 110.)

And by substituting .0028 or .0034 for .7854, we have the capacity in ale gallons or wine gallons.

Ex. What is the capacity of a cask of the first form, whose length is 30 inches, its head diameter 18, and its bung diameter 24?

Ans. 41.3 ale gallons, or 50.2 wine gallons.

PROBLEM II.

*To calculate the contents of a cask, in the form of the middle frustum of a* PARABOLIC SPINDLE.

122. Add together the square of the head diameter, and twice the square of the bung diameter, and from the sum subtract $\frac{2}{5}$ of the square of the difference of the diameters; multiply the remainder by $\frac{1}{3}$ of the length, and the product by .0028 for ale gallons, or .0034 for wine gallons.

The capacity in inches $=(2D^2+d^2-\frac{2}{5}(D-d)^2)\times\frac{1}{3}l\times .7854$. (Art. 118.)

Ex. What is the capacity of a cask of the second form, whose length is 30 inches, its head diameter 18, and its bung diameter 24?

Ans. 40.9 ale gallons, or 49.7 wine gallons.

### PROBLEM III.

*To calculate the contents of a cask, in the form of two equal frustums of a* PARABOLOID.

123. Add together the square of the head diameter, and the square of the bung diameter; multiply the sum by half the length, and the product by .0028 for ale gallons, or .0034 for wine gallons.

The capacity in inches$=(D^2+d^2)\times\frac{1}{2}l\times.7854$. (Art. 112. Cor.)

Ex. What is the capacity of a cask of the third form, whose dimensions are, as before, 30, 18, and 24?

Ans. 37.8 ale gallons, or 45.9 wine gallons.

### PROBLEM IV.

*To calculate the contents of a cask, in the form of two equal frustums of a* CONE.

124. Add together the square of the head diameter, the square of the bung diameter, and the product of the two diameters; multiply the sum by $\frac{1}{3}$ of the length, and the product by .0028 for ale gallons, or .0034 for wine gallons.

The capacity in inches$=(D^2+d^2+Dd)\times\frac{1}{3}l\times.7854$.(Art.68.)

Ex. What is the capacity of a cask of the fourth form, whose length is 30, and its diameters 18 and 24?

Ans. 37.3 ale gallons, or 45.3 wine gallons.

125. The preceding rules, though correct in theory, are not very well adapted to practice, as they suppose the form of the cask to be *known*. The two following rules, taken from Hutton's Mensuration, may be used for casks of the usual forms. For the first, *three* dimensions are required, the length, the head diameter, and the bung diameter. It is evident that no allowance is made by this, for different degrees of curvature from the head to the bung. If the cask is more or less curved than usual, the following rule is to be preferred, for which *four* dimensions are required, the head and

bung diameters, and a third diameter taken in the middle between the bung and the head. For the demonstration of these rules, see Hutton's Mensuration, Part v. Sec. 2. Ch. 5. and 7.

### PROBLEM V.

*To calculate the contents of any common cask from* THREE *dimensions.*

126. Add together
25 times the square of the head diameter,
39 times the square of the bung diameter, and
26 times the product of the two diameters;
Multiply the sum by the length, divide the product by 90, and multiply the quotient by .0028 for ale gallons, or .0034 for wine gallons.

The capacity in inches $=(39D^2+25d^2+26Dd)\times\frac{l}{90}\times.7854$.

Ex. What is the capacity of a cask whose length is 30 inches, the head diameter 18, and the bung diameter 24?

Ans. 39 ale gallons, or $47\frac{1}{3}$ wine gallons.

### PROBLEM VI.

*To calculate the contents of a cask from* FOUR *dimensions,* the length, the head and bung diameters, and a diameter taken in the middle between the head and the bung.

127. Add together the squares of the head diameter, of the bung diameter, and of double the middle diameter; multiply the sum by $\frac{1}{6}$ of the length, and the product by .0028 for ale gallons, or .0034 for wine gallons.

If D=the bung diameter, $d$=the head diameter, $m$=the middle diameter, and $l$=the length;

The capacity in inches $=(D^2+d^2+\overline{2m}^2)\times\frac{1}{6}l\times.7854$.

Ex. What is the capacity of a cask, whose length is 30 inches, the head diameter 18, the bung diameter 24, and the middle diameter $22\frac{1}{2}$?

Ans. 41 ale gallons, or $49\frac{2}{3}$ wine gallons.

128. In making the calculations in gauging, according to the preceding rules, the multiplications and divisions are frequently performed by means of a *Sliding Rule*, on which are placed a number of logarithmic lines, similar to those on Gunter's Scale. See Trigonom. Sec. vi. and Note G. p. 141.

Another instrument commonly used in gauging is the *Diagonal Rod*. By this, the capacity of a cask is very expeditiously found, from a single dimension, the distance from the bung to the intersection of the opposite stave with the head. The measure is taken by extending the rod through the cask, from the bung to the most distant part of the head. The number of gallons corresponding to the length of the line thus found, is marked on the rod. The *logarithmic* lines on the gauging rod are to be used in the same manner, as on the sliding rule.

## ULLAGE OF CASKS.

129. When a cask is *partly* filled, the whole capacity is divided, by the surface of the liquor, into two portions; the *least* of which, whether full or empty, is called the *ullage*. In finding the ullage, the cask is supposed to be in one of two positions; either *standing*, with its axis perpendicular to the horizon; or *lying*, with its axis parallel to the horizon. The rules for ullage which are *exact*, particularly those for lying casks, are too complicated for common use. The following are considered as sufficiently near approximations. See Hutton's Mensuration.

### PROBLEM VII.

*To calculate the ullage of a* STANDING *cask.*

130. Add together the squares of the diameter at the surface of the liquor, of the diameter of the nearest end, and of double the diameter in the middle between the other two; multiply the sum by $\frac{1}{6}$ of the distance between the surface and the nearest end, and the product by .0028 for ale gallons, or .0034 for wine gallons.

If $D$=the diameter of the surface of the liquor,
$d$=the diameter of the nearest end,
$m$=the middle diameter, and
$l$=the distance between the surface and the nearest end;
The ullage in inches$=(D^2+d^2+\overline{2m^2})\times\frac{1}{6}l\times.7854$.

Ex. If the diameter at the surface of the liquor, in a standing cask, be 32 inches, the diameter of the nearest end 24, the middle diameter 29, and the distance between the surface of the liquor and the nearest end 12; what is the ullage?

Ans. $27\frac{4}{5}$ ale gallons, or $33\frac{3}{4}$ wine gallons.

## PROBLEM VIII.

### *To calculate the ullage of a* LYING *cask.*

131. Divide the distance from the bung to the surface of the liquor, by the whole bung diameter, find the quotient in the column of heights or versed sines in a table of circular segments, take out the corresponding segment, and multiply it by the whole capacity of the cask, and the product by $1\frac{1}{4}$ for the part which is empty.

If the cask be not half full, divide the depth of the liquor by the whole bung diameter, take out the segment, multiply, &c. for the contents of the part which is full.

Ex. If the whole capacity of a lying cask be 41 ale gallons, or $49\frac{2}{3}$ wine gallons, the bung diameter 24 inches and the distance from the bung to the surface of the liquor 6 inches; what is the ullage?

Ans. $7\frac{3}{4}$ ale gallons, or $9\frac{1}{2}$ wine gallons.

# NOTES.

## Note A. p. 16.

One of the earliest approximations to the ratio of the circumference of a circle to its diameter, was that of *Archimedes.* He demonstrated that the ratio of the perimeter of a regular inscribed polygon of 96 sides, to the diameter of the circle, is greater than $3\frac{10}{71} : 1$; and that the ratio of the perimeter of a circumscribed polygon of 192 sides, to the diameter, is less than $3\frac{10}{70} : 1$, that is, than $22 : 7$.

*Metius* gave the ratio of 355 : 113, which is more accurate than any other expressed in small numbers. This was confirmed by *Vieta*, who by inscribed and circumscribed polygons of 393216 sides, carried the approximation to ten places of figures, viz.

3.141592653.

*Van Ceulen* of Leyden afterwards extended it, by the laborious process of repeated bisections of an arc, to 36 places. This calculation was deemed of so much consequence at the time, that the numbers are said to have been put upon his tomb.

But since the invention of *fluxions*, methods much more expeditious have been devised, for approximating to the required ratio. These principally consist in finding the sum of a series, in which the length of an arc is expressed in terms of its *tangent.*

If $t$=the tangent of an arc, the radius being 1,

The arc $= t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t^9}{9} -$ &c. See Fluxions.

This series is in itself very simple. Nothing more is necessary to make it answer the purpose in practice, than that the arc be *small*, so as to render the series sufficiently converging, and that the tangent be expressed in such simple numbers, as can easily be raised to the several powers. The given series will be expressed in the most simple numbers, when the arc is 45°, whose tangent is equal to radius. If the radius be 1,

The arc of $45° = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \&c.$ And this multiplied by 8 gives the length of the whole circumference.

But a series in which the tangent is smaller, though it be less simple than this, is to be preferred, for the rapidity with which it converges. As the tangent of $30° = \sqrt{\frac{1}{3}}$, if the radius be 1,

$$\text{The arc of } 30° = \sqrt{\tfrac{1}{3}} \times \left(1 - \frac{1}{3.3} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \frac{1}{9.3^4} - \&c.\right)$$

And this multiplied into 12 will give the whole circumference.

This was the series used by Dr. Halley. By this also, Mr. *Abraham Sharp* of Yorkshire computed the circumference to 72 places of figures, Mr. *John Machin*, Professor of Astronomy in Gresham college, to 100 places, and M. De Lagny to 128 places. Several expedients have been devised, by Machin, Euler, Dr. Hutton, and others, to reduce the labor of summing the terms of the series. See Euler's Analysis of Infinites, Hutton's Mensuration, Appendix to Maseres on the Negative Sign, and Lond. Phil. Trans. for 1776. For a demonstration that the diameter and the circumference of a circle are incommensurable, see Legendre's Geometry, Note IV.

The circumference of a circle whose diameter is 1, is

**3.1415926535, 8979323846, 2643383279,**
**5028841971, 6939937510, 5820974944,**
**5923078164, 0628620899, 8628034825,**
**3421170679, 8214808651, 3272306647,**
**0938446 + or 7 −.**

## NOTE B. p. 17.

The following multipliers may frequently be useful;

The diam'r of a circle
- ×.8862 = the side of an equal square.
- ×.707 = the side of an ins'bed sq're.
- ×.866 = the side of an inscribed equilateral triangle.

The circumf.
- ×.2821 = the side of an equal square.
- ×.2251 = the side of an inscribed square.
- ×.2756 = the side of an ins'bed eq'lat. triang.

The side of a sq.
- ×1.128 = the diameter of an equal circle.
- ×3.545 = the circumf. of an equal circle.
- ×1.414 = the diam. of the circums. circle.
- ×4.443 = the cir. of the circumsc. circle.

## NOTE C. p. 19.

The following approximating rules may be used for finding the arc of a circle.

1. The arc of a circle is nearly equal to $\frac{1}{3}$ of the difference between the chord of the whole arc, and 8 times the chord of half the arc.

2. If $h$ = the *height* of an arc, and $d$ = the diameter of the circle;

$$\text{The arc} = 2d\sqrt{\frac{3h}{3d-h}} \qquad \text{Or,}$$

3. $$\text{The arc} = 2\sqrt{dh}\times\left(1+\frac{h}{2.3d}+\frac{3h^2}{2.4.5d^2}+\frac{3.5h^3}{2.4.6.7d^3}\ \&c.\right) \text{ Or,}$$

4. $$\text{The arc} = \tfrac{2}{9}\left(5d\sqrt{\frac{5h}{5d-3h}}+4\sqrt{dh}\right) \text{ very nearly.}$$

5. If $s$ = the *sine* of an arc, and $r$ = the radius of the circle;

$$\text{The arc} = s\times\left(1+\frac{s^2}{2.3r^2}+\frac{3s^4}{5.2.4r^4}+\frac{3.5s^6}{7.2.4.6r^6}\ \&c.\right)$$

See Hutton's Mensuration.

## Note D. p. 23.

To expedite the calculation of the areas of circular segments, a *table* is provided, which contains the areas of segments in a circle whose diameter is 1. See the table at the end of the book, in which the diameter is supposed to be divided into 1000 equal parts. By this may be found the areas of segments of other circles. For the heights of similar segments of different circles are as the diameters. If then the height of any given segment be divided by the diameter of the circle, the quotient will be the height of a similar segment in a circle whose diameter is 1. The area of the latter is found in the table; and from the properties of similar figures, the two segments are to each other, as the squares of the diameters of the circles. We have then the following rule:

### *To find the area of a circular* SEGMENT *by the* TABLE.

*Divide the height of the segment by the diameter of the circle; look for the quotient in the column of heights in the table; take out the corresponding number in the column of areas; and multiply it by the square of the diameter.*

It is to be observed, that the figures in each of the columns in the table are *decimals.*

If accuracy is required, and the quotient of the height divided by the diameter, is *between* two numbers in the column of heights; allowance may be made for a *proportional part* of the difference of the corresponding numbers in the column of areas; in the same manner, as in taking out logarithms.

Segments *greater than a semicircle* are not contained in the table. If the area of such a segment is required, as ABD (Fig. 9.), find the area of the segment ABO, and subtract this from the area of the whole circle.

Or,

Divide the height of the given segment by the diameter, subtract the quotient from 1, find the remainder in the column of heights, subtract the corresponding area from .7854, and multiply this remainder by the square of the diameter.

Ex. 1. What is the area of a segment whose height is 16, the diameter of the circle being 48? Ans. 528.

2. What is the area of a segment whose height is 32, the diameter being 48? Ans. 1281.55.

The following rules may also be used for a circular segment.

1. To the chord of the whole arc, add $\frac{4}{3}$ of the chord of half the arc, and multiply the sum by $\frac{2}{5}$ of the height.

If C and $c$=the two chords, and $h$=the height;

The segment $=(C+\frac{4}{3}c)\frac{2}{5}h$ nearly.

2. If $h$= the height of the segment, and $d$=the diameter of the circle;

$$\text{The segment}=2h\sqrt{dh}\times\left(\frac{2}{3}-\frac{h}{5d}-\frac{h^2}{28d^2}-\frac{h^3}{72d^3}\ \&c.\right)$$

## Note E. p. 29.

The term *solidity* is used here in the customary sense, to express the magnitude of any geometrical quantity of three dimensions, length, breadth, and thickness; whether it be a solid body, or a fluid, or even a portion of empty space. This use of the word, however, is not altogether free from objection. The same term is applied to one of the general properties of matter; and also to that peculiar quality by which certain substances are distinguished from *fluids*. There seems to be an impropriety in speaking of the *solidity* of a body of *water*, or of a vessel which is *empty*. Some writers have therefore substituted the word *volume* for solidity. But the latter term, if it be properly defined, may be retained without danger of leading to mistake.

## Note F. p. 35.

The *geometrical* demonstration of the rule for finding the solidity of a frustum of a pyramid, depends on the following proposition:

*A frustum of a triangular pyramid is equal to three pyramids; the greatest and least of which are equal in height to the frustum, and have the two ends of the frustum for their bases; and the third is a mean proportional between the other two.*

Let ABCDFG (Fig. 34.) be a frustum of a triangular pyramid. If a plane be supposed to pass through the points AFC, it will cut off the pyramid ABCF. The height of this is evidently equal to the height of the frustum, and its base is ACB, the greater end of the frustum.

Let another plane pass through the points AFD. This will divide the remaining part of the figure into two triangular pyramids AFDG and AFDC. The height of the former is equal to the height of the frustum, and its base is DFG, the smaller end of the frustum.

To find the magnitude of the third pyramid AFDC, let F be now considered as the vertex of this, and of the second pyramid AFDG. Their bases will then be the triangles ADC and ADG. As these are in the same plane, the two pyramids have the same altitude, and are to each other as their bases. But these triangular bases, being between the same parallels, are as the lines AC and DG. Therefore the pyramid AFDC is to the pyramid AFDG as AC to DG; and $\overline{AFDC}^2 : \overline{AFDG}^2 :: \overline{AC}^2 : \overline{DG}^2$. (Alg. 391.) But the pyramids ABCF and AFDG, having the same altitude, are as their bases ABC and DFG, that is, as $\overline{AC}^2$ and $\overline{DG}^2$. (Euc. 19, 6.) We have then

$$\left.\begin{array}{l}\overline{AFDC}^2 : \overline{AFDG}^2 :: \overline{AC}^2 : \overline{DG}^2 \\ ABCF : AFDG :: \overline{AC}^2 : \overline{DG}^2\end{array}\right\}$$

$$\text{Therefore } \overline{AFDC}^2 : \overline{AFDG}^2 :: ABCF : AFDG.$$

$$\text{And } \overline{AFDC}^2 = AFDG \times ABCF.$$

That is, the pyramid AFDC is a mean proportional between AFDG and ABCF.

Hence, the solidity of a frustum of a triangular pyramid is equal to $\frac{1}{3}$ of the height, multiplied into the sum of the areas of the two ends and the square root of the product of these areas. This is true also of a frustum of any other pyramid. (Sup. Euc. 12, 3. Cor. 2.)

If the smaller end of a frustum of a pyramid be enlarged, till it is made equal to the other end; the frustum will become a *prism*, which may be divided into three *equal* pyramids. (Sup. Euc. 15, 3.)

## Note G. p. 59.

The following simple rule for the solidity of round timber, or of any cylinder, is nearly exact:

*Multiply the length into twice the square of $\frac{1}{5}$ of the circumference.*

If C=the circumference of a cylinder;

The area of the base$=\frac{C^2}{4\pi}=\frac{C^2}{12.566}$ But $2\left(\frac{C}{5}\right)^2=\frac{C^2}{12.5}$

It is common to measure *hewn* timber, by multiplying the length into the square of the *quarter-girt*. This gives exactly the solidity of a parallelopiped, if the ends are *squares*. But if the ends are parallelograms, the area of each is *less* than the square of the quarter-girt. (Euc. 27, 6.)

Timber which is *tapering* may be exactly measured by the rule for the frustum of a pyramid or cone (Art. 50, 68.); or, if the ends are not similar figures, by the rule for a prismoid. (Art. 55.) But for common purposes, it will be sufficient to multiply the length by the area of a section *in the middle* between the two ends.

# A TABLE

OF THE SEGMENTS OF A CIRCLE, WHOSE DIAMETER IS 1, AND IS SUPPOSED TO BE DIVIDED INTO 1000 EQUAL PARTS.

| Height. | Area Seg. | Height. | Area Seg. | Height. | Area Seg. |
|---|---|---|---|---|---|
| .001 | .000042 | .034 | .008273 | .067 | .022652 |
| 002 | 000119 | 035 | 008638 | 068 | 023154 |
| 003 | 000219 | 036 | 009008 | 069 | 023659 |
| 004 | 000337 | 037 | 009383 | 070 | 024168 |
| 005 | 000471 | 038 | 009763 | 071 | 024680 |
| 006 | 000618 | 039 | 010148 | 072 | 025195 |
| 007 | 000779 | 040 | 010537 | 073 | 025714 |
| 008 | 000952 | 041 | 010932 | 074 | 026236 |
| 009 | 001135 | 042 | 011331 | 075 | 026761 |
| 010 | 001329 | 043 | 011734 | 076 | 027289 |
| 011 | 001533 | 044 | 012142 | 077 | 027821 |
| 012 | 001746 | 045 | 012554 | 078 | 028356 |
| 013 | 001968 | 046 | 012971 | 079 | 028894 |
| 014 | 002199 | 047 | 013392 | 080 | 029435 |
| 015 | 002438 | 048 | 013818 | 081 | 029979 |
| 016 | 002685 | 049 | 014247 | 082 | 030526 |
| 017 | 002940 | 050 | 014681 | 083 | 031076 |
| 018 | 003202 | 051 | 015119 | 084 | 031629 |
| 019 | 003472 | 052 | 015561 | 085 | 032186 |
| 020 | 003748 | 053 | 016007 | 086 | 032745 |
| 021 | 004032 | 054 | 016457 | 087 | 033307 |
| 022 | 004322 | 055 | 016911 | 088 | 033872 |
| 023 | 004618 | 056 | 017369 | 089 | 034441 |
| 024 | 004921 | 057 | 017831 | 090 | 035011 |
| 025 | 005231 | 058 | 018296 | 091 | 035585 |
| 026 | 005546 | 059 | 018766 | 092 | 036162 |
| 027 | 005867 | 060 | 019239 | 093 | 036741 |
| 028 | 006194 | 061 | 019716 | 094 | 037323 |
| 029 | 006527 | 062 | 020206 | 095 | 037909 |
| 030 | 006865 | 063 | 020690 | 096 | 038496 |
| 031 | 007209 | 064 | 021178 | 097 | 039087 |
| 032 | 007558 | 065 | 021659 | 098 | 039680 |
| .033 | .007913 | .066 | .022154 | .099 | .040276 |

| Height. | Area Seg. | Height. | Area Seg. | Height. | Area Seg. |
|---|---|---|---|---|---|
| .100 | .040875 | .144 | .069625 | .188 | .102334 |
| 101 | 041476 | 145 | 070328 | 189 | 103116 |
| 102 | 042080 | 146 | 071033 | 190 | 103900 |
| 103 | 042687 | 147 | 071741 | 191 | 104685 |
| 104 | 043296 | 148 | 072450 | 192 | 105472 |
| 105 | 043908 | 149 | 073161 | 193 | 106261 |
| 106 | 044522 | 150 | 073874 | 194 | 107051 |
| 107 | 045139 | 151 | 074589 | 195 | 107842 |
| 108 | 045759 | 152 | 075306 | 196 | 108636 |
| 109 | 046381 | 153 | 076026 | 197 | 109430 |
| 110 | 047005 | 154 | 076747 | 198 | 110226 |
| 111 | 047632 | 155 | 077469 | 199 | 111024 |
| 112 | 048262 | 156 | 078194 | 200 | 111823 |
| 113 | 048894 | 157 | 078921 | 201 | 112624 |
| 114 | 049528 | 158 | 079649 | 202 | 113426 |
| 115 | 050165 | 159 | 080380 | 203 | 114230 |
| 116 | 050804 | 160 | 081112 | 204 | 115035 |
| 117 | 051446 | 161 | 081846 | 205 | 115842 |
| 118 | 052090 | 162 | 082582 | 206 | 116650 |
| 119 | 052736 | 163 | 083320 | 207 | 117460 |
| 120 | 053385 | 164 | 084059 | 208 | 118271 |
| 121 | 054036 | 165 | 084801 | 209 | 119083 |
| 122 | 054689 | 166 | 085544 | 210 | 119897 |
| 123 | 055345 | 167 | 086289 | 211 | 120712 |
| 124 | 056003 | 168 | 087036 | 212 | 121529 |
| 125 | 056663 | 169 | 087785 | 213 | 122347 |
| 126 | 057326 | 170 | 088535 | 214 | 123167 |
| 127 | 057991 | 171 | 089287 | 215 | 123988 |
| 128 | 058658 | 172 | 090041 | 216 | 124810 |
| 129 | 059327 | 173 | 090797 | 217 | 125634 |
| 130 | 059999 | 174 | 091554 | 218 | 126459 |
| 131 | 060672 | 175 | 092313 | 219 | 127285 |
| 132 | 061348 | 176 | 093074 | 220 | 128113 |
| 133 | 062026 | 177 | 093836 | 221 | 128942 |
| 134 | 062707 | 178 | 094601 | 222 | 129773 |
| 135 | 063389 | 179 | 095366 | 223 | 130605 |
| 136 | 064074 | 180 | 096134 | 224 | 131438 |
| 137 | 064760 | 181 | 096903 | 225 | 132272 |
| 138 | 065449 | 182 | 097674 | 226 | 133108 |
| 139 | 066140 | 183 | 098447 | 227 | 133945 |
| 140 | 066833 | 184 | 099221 | 228 | 134784 |
| 141 | 067528 | 185 | 099997 | 229 | 135624 |
| 142 | 068225 | 186 | 100774 | 230 | 136465 |
| .143 | .068924 | .187 | .101553 | .231 | .137307 |

| Height. | Area Seg. | Height. | Area Seg. | Height. | Area Seg. |
|---|---|---|---|---|---|
| .232 | .138150 | .277 | .177330 | .322 | .218533 |
| 233 | 138995 | 278 | 178225 | 323 | 219468 |
| 234 | 139841 | 279 | 179122 | 324 | 220404 |
| 235 | 140688 | 280 | 180019 | 325 | 221340 |
| 236 | 141537 | 281 | 180918 | 326 | 222277 |
| 237 | 142387 | 282 | 181817 | 327 | 223215 |
| 238 | 143238 | 283 | 182718 | 328 | 224154 |
| 239 | 144091 | 284 | 183619 | 329 | 225093 |
| 240 | 144944 | 285 | 184521 | 330 | 226033 |
| 241 | 145799 | 286 | 185425 | 331 | 226974 |
| 242 | 146655 | 287 | 186329 | 332 | 227915 |
| 243 | 147512 | 288 | 187234 | 333 | 228858 |
| 244 | 148371 | 289 | 188140 | 334 | 229801 |
| 245 | 149230 | 290 | 189047 | 335 | 230745 |
| 246 | 150091 | 291 | 189955 | 336 | 231689 |
| 247 | 150953 | 292 | 190864 | 337 | 232634 |
| 248 | 151816 | 293 | 191775 | 338 | 233580 |
| 249 | 152680 | 294 | 192684 | 339 | 234526 |
| 250 | 153546 | 295 | 193596 | 340 | 235473 |
| 251 | 154412 | 296 | 194509 | 341 | 236421 |
| 252 | 155280 | 297 | 195422 | 342 | 237369 |
| 253 | 156149 | 298 | 196337 | 343 | 238318 |
| 254 | 157019 | 299 | 197252 | 344 | 239268 |
| 255 | 157890 | 300 | 198168 | 345 | 240218 |
| 256 | 158762 | 301 | 199085 | 346 | 241169 |
| 257 | 159636 | 302 | 200003 | 347 | 242121 |
| 258 | 160510 | 303 | 200922 | 348 | 243074 |
| 259 | 161386 | 304 | 201841 | 349 | 244026 |
| 260 | 162263 | 305 | 202761 | 350 | 244980 |
| 261 | 163140 | 306 | 203683 | 351 | 245934 |
| 262 | 164019 | 307 | 204605 | 352 | 246889 |
| 263 | 164899 | 308 | 205527 | 353 | 247845 |
| 264 | 165780 | 309 | 206451 | 354 | 248801 |
| 265 | 166663 | 310 | 207376 | 355 | 249757 |
| 266 | 167546 | 311 | 208301 | 356 | 250715 |
| 267 | 168430 | 312 | 209227 | 357 | 251673 |
| 268 | 169315 | 313 | 210154 | 358 | 252631 |
| 269 | 170202 | 314 | 211082 | 359 | 253590 |
| 270 | 171089 | 315 | 212011 | 360 | 254550 |
| 271 | 171978 | 316 | 212940 | 361 | 255510 |
| 272 | 172867 | 317 | 213871 | 362 | 256471 |
| 273 | 173758 | 318 | 214802 | 363 | 257433 |
| 274 | 174649 | 319 | 215733 | 364 | 258395 |
| 275 | 175542 | 320 | 216666 | 365 | 259357 |
| .276 | .176435 | .321 | .217599 | .366 | .260320 |

| Height. | Area Seg. | Height. | Area Seg. | Height. | Area Seg. |
|---|---|---|---|---|---|
| .367 | .261284 | .412 | .305155 | .457 | .349752 |
| 368 | 262248 | 413 | 306140 | 458 | 350748 |
| 369 | 263213 | 414 | 307125 | 459 | 351745 |
| 370 | 264178 | 415 | 308110 | 460 | 352742 |
| 371 | 265144 | 416 | 309095 | 461 | 353739 |
| 372 | 266111 | 417 | 310081 | 462 | 354736 |
| 373 | 267078 | 418 | 311068 | 463 | 355732 |
| 374 | 268045 | 419 | 312054 | 464 | 356730 |
| 375 | 269013 | 420 | 313041 | 465 | 357727 |
| 376 | 269982 | 421 | 314029 | 466 | 358725 |
| 377 | 270951 | 422 | 315016 | 467 | 359723 |
| 378 | 271920 | 423 | 316004 | 468 | 360721 |
| 379 | 272890 | 424 | 316992 | 469 | 361719 |
| 380 | 273861 | 425 | 317981 | 470 | 362717 |
| 381 | 274832 | 426 | 318970 | 471 | 363715 |
| 382 | 275803 | 427 | 319959 | 472 | 364713 |
| 383 | 276775 | 428 | 320948 | 473 | 365712 |
| 384 | 277748 | 429 | 321938 | 474 | 366710 |
| 385 | 278721 | 430 | 322928 | 475 | 367709 |
| 386 | 279694 | 431 | 323918 | 476 | 368708 |
| 387 | 280668 | 432 | 324909 | 477 | 369707 |
| 388 | 281642 | 433 | 325900 | 478 | 370706 |
| 389 | 282617 | 434 | 326892 | 479 | 371705 |
| 390 | 283592 | 435 | 327882 | 480 | 372704 |
| 391 | 284568 | 436 | 328874 | 481 | 373703 |
| 392 | 285544 | 437 | 329866 | 482 | 374702 |
| 393 | 286521 | 438 | 330858 | 483 | 375702 |
| 394 | 287498 | 439 | 331850 | 484 | 376702 |
| 395 | 288476 | 440 | 332843 | 485 | 377701 |
| 396 | 289454 | 441 | 333836 | 486 | 378701 |
| 397 | 290432 | 442 | 334829 | 487 | 379700 |
| 398 | 291411 | 443 | 335822 | 488 | 380700 |
| 399 | 292390 | 444 | 336816 | 489 | 381699 |
| 400 | 293369 | 445 | 337810 | 490 | 382699 |
| 401 | 294349 | 446 | 338804 | 491 | 383699 |
| 402 | 295330 | 447 | 339798 | 492 | 384699 |
| 403 | 296311 | 448 | 340793 | 493 | 385699 |
| 404 | 297292 | 449 | 341787 | 494 | 386669 |
| 405 | 298273 | 450 | 342782 | 495 | 387699 |
| 406 | 299255 | 451 | 343777 | 496 | 388699 |
| 407 | 300238 | 452 | 344772 | 497 | 389699 |
| 408 | 301220 | 453 | 345768 | 498 | 390699 |
| 409 | 302203 | 454 | 346764 | 499 | 391699 |
| 410 | 303187 | 455 | 347759 | .500 | .392699 |
| .411 | .304171 | .456 | .348755 | | |

THE

MATHEMATICAL PRINCIPLES

OF

# NAVIGATION AND SURVEYING,

WITH THE

## MENSURATION

OF

## HEIGHTS AND DISTANCES.

BEING

THE FOURTH PART

OF

## A COURSE OF MATHEMATICS.

ADAPTED TO THE METHOD OF INSTRUCTION IN THE

AMERICAN COLLEGES.

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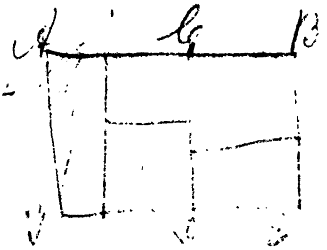

As the following treatise has been prepared for the use of a class in College, it does not contain all the details which would be requisite for a practical navigator or surveyor. The object of a scientific education is rather to teach *principles*, than the minute rules which are called for in professional practice. The principles should indeed be accompanied with such illustrations and examples as will render it easy for the student to make the applications for himself, whenever occasion shall require. But a collection of rules merely, would be learned, only to be forgotten, except by a few who might have use for them in the course of their business. There are many things belonging to the art of navigation, which are not comprehended in the mathematical part of the subject. Seamen will of course make use of the valuable system of Mackay, or the still more complete work of Bowditch.

The student is supposed to be familiar with the principles of Geometry and Trigonometry, before he enters upon the present number, which contains little more than the application of those principles to some of the most simple problems in heights and distances, navigation, and surveying.

# CONTENTS.

# HEIGHTS AND DISTANCES.

ART. 1. THE most direct and obvious method of determining the distance or height of any object, is to apply to it some known measure of length, as a foot, a yard, or a rod. In this manner, the height of a room is found, by a joiner's rule; or the side of a field by a surveyor's chain. But in many instances, the object, or a part, at least, of the line which is to be measured is *inaccessible.* We may wish to determine the breadth of a river, the height of a cloud, or the distances of the heavenly bodies. In such cases it is necessary to measure some *other* line; from which the required line may be obtained, by geometrical construction, or more exactly, by trigonometrical calculation. The line first measured is frequently called a *base* line.

2. In measuring *angles*, some instrument is used which contains a portion of a graduated circle divided into degrees and minutes. For the proper measure of an angle is an arc of a circle, whose center is the angular point. (Trig. 74.) The instruments used for this purpose are made in different forms, and with various appendages. The essential parts are a graduated circle, and an index with sight-holes, for taking the directions of the lines which include the angles.

3. Angles of *elevation*, and of *depression* are in a plane perpendicular to the horizon, which is called a *vertical plane.* An angle of *elevation* is contained between a parallel to the horizon, and an ascending line, as BAC (Fig. 2.) An angle of *depression* is contained between a parallel to the horizon, and a descending line, as DCA. The *complement* of this is the angle ACB.

4. The instrument by which angles of elevation, and of depression, are commonly measured, is called a *Quadrant.* In its most simple form, it is a portion of a circular board

ABC, (Fig. 1.) on which is a graduated arc of 90 degrees, AB, a plumb line CP, suspended from the central point C, and two sight-holes D and E, for taking the direction of the object.

To measure an angle of *elevation* with this, hold the plane of the instrument perpendicular to the horizon, bring the center C to the angular point, and direct the edge AC in such a manner, that the object G may be seen through the two sight-holes. Then the arc BO measures the angle BCO, which is equal to the angle of elevation FCG. For as the plumb-line is perpendicular to the horizon, the angle FCO is a right angle, and therefore equal to BCG. Taking from these the common angle BCF, there will remain the angle BCO=FCG.

In taking an angle of *depression*, as HCL (Fig. 1.) the eye is placed at C, so as to view the object at L, through the sight-holes D and E.

5. In treating of the mensuration of heights and distances, no *new principles* are to be brought into view. We have only to make an application of the rules for the solution of triangles, to the particular circumstances in which the observer may be placed, with respect to the line to be measured. These are so numerous, that the subject may be divided into a great number of distinct cases. But as they are all solved upon the same general principles, it will not be necessary to give examples under each. The following problems may serve as a specimen of those which most frequently occur in practice.

## Problem I.

### TO FIND THE PERPENDICULAR HEIGHT OF AN ACCESSIBLE OBJECT STANDING ON A HORIZONTAL PLANE.

6. MEASURE FROM THE OBJECT TO A CONVENIENT STATION, AND THERE TAKE THE ANGLE OF ELEVATION SUBTENDED BY THE OBJECT.

If the distance AB (Fig. 2.) be measured, and the angle of elevation BAC; there will be given in the right angled triangle ABC, the base and the angles, to find the perpendicular. (Trig. 137.)

As the instrument by which the angle at A is measured, is commonly raised a few feet above the ground; a point B must be taken in the object, so that AB shall be parallel to the horizon. The part BP, may afterwards be added to the height BC, found by trigonometrical calculation.

Ex. 1. What is the height of a tower BC, (Fig. 2.) if the distance AB, on a horizontal plane, be 98 feet; and the angle BAC $35\frac{1}{2}$ degrees?

Making the hypothenuse radius, (Trig. 121.)

Cos. BAC : AB : : Sin. BAC : BC=69.9 feet.

For the *geometrical construction* of the problem, see Trig. 169.

2. What is the height of the perpendicular sheet of water at the falls of Niagara, if it subtends an angle of 40 degrees, at the distance of 163 feet from the bottom, measured on a horizontal plane? Ans. $136\frac{3}{4}$ feet.

7. If the height of the object be *known*, its *distance* may be found by the angle of elevation. In this case the angles, and the perpendicular of the triangle are given, to find the base.

Ex. A person on shore, taking an observation of a ship's mast which is known to be 99 feet high, finds the angle of elevation $3\frac{1}{2}$ degrees. What is the distance of the ship from the observer? Ans. 98 rods.

8. If the observer be stationed at the *top* of the perpendicular BC, (Fig. 2.) whose height is known; he may find the length of the base line AB, by measuring the angle of *depression* ACD, which is equal to BAC.

Ex. A seaman at the top of a mast 66 feet high, looking at another ship, finds the angle of depression 10 degrees. What is the distance of the two vessels from each other?

Ans $22\frac{2}{3}$ rods.

We may find the distance between *two objects* which are in the same vertical plane with the perpendicular, by calculating the distance of each from the perpendicular. Thus AG (Fig. 2.) is equal to the difference between AB and GB.

## Problem II.

### TO FIND THE HEIGHT OF AN ACCESSIBLE OBJECT STANDING ON AN INCLINED PLANE.

9. MEASURE THE DISTANCE FROM THE OBJECT TO A CONVENIENT STATION, AND TAKE THE ANGLES WHICH THIS BASE MAKES WITH LINES DRAWN FROM ITS TWO ENDS TO THE TOP OF THE OBJECT.

If the base AB (Fig. 3.) be measured and the angles BAC and ABC; there will be given, in the oblique angled triangle ABC, the side AB, and the angles, to find BC. (Trig. 150.)

Or the height BC may be found by measuring the distances BA, AD, and taking the angles, BAC and BDC. There will then be given in the triangle ADC, the angles and the side AD, to find AC; and consequently, in the triangle ABC, the sides AB and AC with the angle BAC, to find BC.

Ex. If AB (Fig. 3.) be 76 feet, the angle B 101° 25′ and the angle A 44° 42′; what is the height of the tree BC?

Sin. C : AB: :Sin. A: :BC=95.9 feet.

For the *geometrical construction* of the problem, see Trig. 169.

10. The following are some of the methods by which the height of an object may be found, without measuring the angle of elevation.

1. *By shadows*. Let the staff *bc* (Fig. 4.) be parallel to an object BC whose height is required. If the shadow of BC extend to A, and that of *bc* to *a*; the rays of light CA and *ca* coming from the sun may be considered parallel; and therefore the triangles ABC and *abc* are similar; so that

*ab* : *bc*: :AB : BC.

Ex. If *ab* be 3 feet, *bc* 5 feet, and AB 69 feet, what is the height of BC? Ans. 115 feet.

2. *By parallel rods*. If two poles *am* and *cn* (Fig. 5.) be placed parallel to the object BC, and at such distances as to bring the points C, *c*, *a* in a line, and if *ab* be made parallel to AB; the triangles ABC, and *abc* will be similar; and we shall have

*ab* : *bc*: :AB : BC.

*One pole* will be sufficient, if the observer can place his eye at the point A, so as to bring A, *a*, and C in a line.

3. *By a mirror*. Let the smooth surface of a body of water at A (Fig. 6.) or any plane mirror parallel to the horizon, be so situated, that the eye of the observer at *c* may view the top of the object C reflected from the mirror. By a law of Optics, the angle BAC is equal to *bAc*; and if *bc*

be made parallel to BC, the triangle $bAc$ will be similar to BAC; so that

$$Ab : bc :: AB : BC.$$

## Problem III.

### TO FIND THE HEIGHT OF AN INACCESSIBLE OBJECT ABOVE A HORIZONTAL PLANE.

11. **Take two stations in a vertical plane passing through the top of the object, measure the distance from one station to the other, and the angle of elevation at each.**

If the base AB (Fig. 7.) be measured, with the angles CBP and CAB; as ABC is the supplement of CBP, there will be given, in the oblique angled triangle ABC, the side AB and the angles, to find BC; and then, in the right angled triangle BCP, the hypothenuse and the angles, to find the perpendicular CP.

Ex. 1. If C (Fig. 7.) be the top of a spire, the horizontal base line AB 100 feet, the angle of elevation BAC 40°, and the angle PBC 60°; what is the perpendicular height of the spire?

The difference between the angles PBC and BAC is equal to ACB. (Euc. 32. 1.)

Then Sin ACB : AB :: Sin BAC : BC=187.9

And R : BC :: Sin PBC : CP=162$\frac{3}{4}$ feet.

2. If two persons 120 rods from each other, are standing on a horizontal plane, and also in a vertical plane passing through a *cloud*, both being on the same side of the cloud: and if they find the angles of elevation at the two stations to be 68° and 76°; what is the height of the cloud?

Ans. 2 miles 135.7 rods.

12. The preceding problems are useful in particular cases. But the following is a *general* rule, which may be used for finding the height of any object whatever, within moderate distances.

## Problem IV.

### TO FIND THE HEIGHT OF ANY OBJECT, BY OBSERVATIONS AT TWO STATIONS.

13. **Measure the base line between the two stations, the angles between this base and lines drawn from each of the stations to each end of the object, and the angle subtended by the object, at one of the stations.**

If BC (Fig. 8.) be the object whose height is required, and if the distance between the stations A and D be measured, with the angles ADC, DAC, ADB, DAB, and BAC; there will be given, in the triangle ADC, the side AD and the angles, to find AC; in the triangle ADB, the side AD and the angles, to find AB; and then, in the triangle BAC, the sides AB and AC with the included angle, to find the required height BC.

If the two stations A and D be in the *same plane* with BC, the angle BAC will be equal to the difference between BAD and CAD. In this case it will not be necessary to measure BAC.

Ex. If AD=83 feet, (Fig. 8.) $\begin{cases} ADC=51° \\ DAC=95° \end{cases}$ $\begin{cases} ADB=33° \\ DAB=121° \end{cases}$ BAC=26°,

What is the height of the object BC?

Sin ACD : AD::ADC : AC=115.3

Sin ABD : AD::ADB : AB=103.1

(AC+AB) : (AC−AB)::Tan ½(ABC+ACB) : Tan ½(ABC −ACB)=13° 38′

Sin ACB : AB::Sin BAC : BC=50.57 feet.

If the object BC be perpendicular to the horizon, its height, after obtaining AB and AC as before, may be found by taking the *angles* of *elevation* BAP and CAP. The difference of the perpendiculars in the right angled triangles ABP and ACP, will be the height required.

## Problem V.

### TO FIND THE DISTANCE OF AN INACCESSIBLE OBJECT.

14. **Measure a base line between two stations, and the angles between this and lines drawn from each of the stations to the object.**

If C (Fig. 9.) be the object, and if the distance between the stations A and B be measured, with the angles at B and A; there will be given, in the oblique angled triangle ABC, the side AB and the angles, to find AC and BC, the distances of the object from the two stations.

For the geometrical construction, see Trig. 169.

Ex. 1. What are the distances of the two stations A and B (Fig. 9.) from the house C, on the opposite side of a river; if AB be 26.6 rods, B 92° 46′, and A 38° 40′?

The angle $C=180-(A+B)=48° \; 34'$. Then

$$\text{Sin } C : AB :: \begin{cases} \text{Sin } A : BC=22.17 \\ \text{Sin } B : AC=35.44. \end{cases}$$

2. Two ships in a harbor, wishing to ascertain how far they are from a fort on shore, find that their mutual distance is 90 rods, and that the angles formed between a line from one to the other, and lines drawn from each to the fort are 45° and 56° 15′. What are their respective distances from the fort? Ans. 76.3 and 64.9 rods.

15. The *perpendicular* distance of the object from the line joining the two stations may be easily found, after the distance from one of the stations is obtained. The perpendicular distance PC (Fig. 9.) is one of the sides of the right angled triangle BCP. Therefore

$$R : BC :: \text{Sin } B : PC.$$

PROBLEM VI.

TO FIND THE DISTANCE BETWEEN TWO OBJECTS, WHEN THE PASSAGE FROM ONE TO THE OTHER, IN A STRAIGHT LINE IS OBSTRUCTED.

16. MEASURE THE RIGHT LINES FROM ONE STATION TO EACH OF THE OBJECTS, AND THE ANGLE INCLUDED BETWEEN THESE LINES.

If A and B (Fig. 10.) be the two objects, and if the distances BC and AC be measured, with the angle at C; there will be given, in the oblique angled triangle ABC, two sides and the included angle, to find the other two angles, and the remaining side. (Trig. 153.)

Ex. The passage between the two objects A and B (Fig. 10.) being obstructed by a morass, the line BC was measured and found to be 109 rods, the line AC 76 rods, and the angle at C 101° 30′. What is the distance AB?

Ans. 144.7 rods.

## Problem VII.

### TO FIND THE DISTANCE BETWEEN TWO INACCESSIBLE OBJECTS.

17. Measure a base line between two stations and the angles between this base and lines drawn from each of the stations to each of the objects.

If A and B (Fig. 11.) be the two objects, and if the distance between the stations C and D be measured, with the angles BDC, BCD, ADC, and ACD; the lines AC and BC may be found as in Problem V, and then the distance AB as in Problem VI.

This rule is substantially the same as that in art. 13. The two stations are supposed to be in the *same plane* with the objects. If they are not, it will be necessary to measure the angle ACB.

18. The same process by which we obtain the distance of *two* objects from each other, will enable us to find the distance between one of these and a third, between that and a fourth, and so on, till a connection is formed between a great number of remote points. This is the plan of the great *Trigonometrical Surveys*, which have been lately carried on, with surprising exactness, particularly in England and France. See Surveying, Section II.

19. In the preceding problems for determining altitudes, the objects are supposed to be at such moderate distances, that the observations are not sensibly affected by the *spherical figure of the earth.* The height of an object is measured from an *horizontal plane*, passing through the station at which the angle of elevation is taken. But in an extent of several miles, the figure of the earth ought to be taken into account.

Let AB (Fig. 12.) be a portion of the earth's surface, H an object above it, and AT a tangent at the point A, or a horizontal line passing through A. Then HT, the oblique height of the object above the horizon of A, is only a *part* of the height above the surface of the earth, or the level of the ocean. To obtain the true altitude, it is necessary to add BT to the height HT found by observation. The height BT may be calculated, if the diameter of the earth and the distance AT be previously known. Or if the height BT be first determined from observation, with the distance AT; the diameter of the earth may be thence deduced.

## Problem VIII.

TO FIND THE DIAMETER OF THE EARTH, FROM THE KNOWN HEIGHT OF A DISTANT MOUNTAIN, WHOSE SUMMIT IS JUST VISIBLE IN THE HORIZON.

20. **From the square of the distance divided by the height, subtract the height.**

If BT (Fig. 12.) be a mountain whose height is known, with the distance AT; and if the summit T be just visible in the horizon at A; then AT is a *tangent* at the point A.

Let $2BC=D$, the diameter of the earth,

$AT=d$, the distance of the mountain,

$BT=h$, its height.

Then considering AT as a straight line, and the earth as a sphere, we have (Euc. 36.3.)

$$(2BC+BT)\times BT=\overline{AT}^2\text{; that is, } (D+h)\times h=d^2,$$

and reducing the equation,

$$D=\frac{d^2}{h}-h.$$

Ex. The highest point of the Andes is about 4 miles above the level of the ocean. If a straight line from this touch the surface of the water at the distance of $178\frac{1}{4}$ miles; what is the diameter of the earth? Ans. 7940 miles.

21. If the distance AT (Fig. 12.) be *unknown*, it may be found by measuring with a quadrant the angle ATC. Draw BG perpendicular to BC, and join CG. The triangles ACG and BCG are equal, because each has a right angle, the sides AC and BC are equal, and the hypothenuse CG is common. Therefore BG and AG are equal. In the right angled triangle BGT, the angle BTG is given, and the perpendicular BT. From these may be found BG and TG, whose sum is equal to AT, the distance required.*

22. In the common measurement of angles, the light is supposed to come from the object to the eye in a *straight line*. But this is not strictly true. The direction of the light is affected by the *refraction of the atmosphere*. If the object be near, the deviation is very inconsiderable. But in an ex-

* This method of determining the diameter of the earth is not as accurate as that by measuring a degree of Latitude. See Surveying, Sec. II.

tent of several miles, and particularly in such nice observations as determining the height of distant mountains, and the diameter of the earth, it is necessary to make allowance for the refraction.*

## Problem IX.

### TO FIND THE GREATEST DISTANCE AT WHICH A GIVEN OBJECT CAN BE SEEN ON THE SURFACE OF THE EARTH.

23. TO THE PRODUCT OF THE HEIGHT OF THE OBJECT INTO THE DIAMETER OF THE EARTH, ADD THE SQUARE OF THE HEIGHT; AND EXTRACT THE SQUARE ROOT OF THE SUM.

Let $2BC=D$, the diameter of the earth, (Fig. 12.)
$BT=h$, the height of the object,
$AT=d$, the distance required.
Then $(D+h)\times h=d^2$. And $d=\sqrt{Dh+h^2}$.

Ex. If the diameter of the earth be 7940 miles, and Mount Ætna 2 miles high; how far can its summit be seen at sea?
Ans. 126 miles.

The actual distance at which an object can be seen, is increased by the refraction of the air.*

24. In this problem, the eye is supposed to be placed at the level of the ocean. But if the observer be elevated above the surface, as on the deck of a ship, he can see to a greater distance. If BT (Fig. 13.) be the height of the object, and B'T' the height of the eye above the level of the ocean; the distance at which the object can be seen, is evidently equal to the *sum* of the tangents AT and AT'.

Ex. The top of a ship's mast 132 feet high is just visible in the horizon, to an observer whose eye is 33 feet above the surface of the water. What is the distance of the ship?
Ans. $21\frac{1}{8}$ miles.

25. The distance to which a person can see the smooth surface of the ocean, if no allowance be made for refraction, is equal to a tangent to the earth drawn from his eye, as T'A. (Fig. 13.)

Ex. If a man standing on the level of the ocean, has his eye raised $5\frac{1}{2}$ feet above the water: to what distance can he see the surface?
Ans. $2\frac{7}{8}$ miles.

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* See Note A.

26. If the distance AT, (Fig. 12.) with the diameter of the earth be given, and the *height* BT be required; the equation in Art. 23 gives

$$h=\sqrt{\tfrac{1}{4}D^2+d^2}-\tfrac{1}{2}D$$

See Surveying, Section IV, on *Leveling*.

27. When the diameter of the earth is ascertained, this may be made a *base line* for determining the distances of the *heavenly bodies*. A right angled triangle may be formed, the perpendicular sides of which shall be the distance required, and the semi-diameter of the earth. If then one of the angles be found by observation, the required side may be easily calculated.

Let AC (Fig. 14.) be the semi-diameter of the earth, AH the sensible horizon at A, and CM the rational horizon parallel to AH, passing through the moon M. The angle HAM may be found by astronomical observation. This angle, which is called the *Horizontal Parallax*, is equal to AMC, the angle at the moon subtended by the semi-diameter of the earth. (Euc. 29. 1.)

## Problem X.

### TO FIND THE DISTANCE OF ANY HEAVENLY BODY WHOSE HORIZONTAL PARALLAX IS KNOWN.

28. As RADIUS, TO THE SEMI-DIAMETER OF THE EARTH; SO IS THE CO-TANGENT OF THE HORIZONTAL PARALLAX, TO THE DISTANCE.

In the right angled triangle ACM, (Fig. 14.) if AC be made radius;

$$R : AC :: \text{Cot. } AMC : CM.$$

Ex. If the horizontal parallax of the moon be 0° 57′, and the diameter of the earth 7940 miles; what is the distance of the moon from the center of the earth?

Ans. 239,414 miles.

29. The *fixed stars* are too far distant to have any sensible horizontal parallax. But from late observations it would seem, that some of them are near enough, to suffer a small apparent change of place, from the revolution of the earth round the sun. The distance of the sun, then, which is the semi-diameter of the earth's orbit, may be taken as a *base line*, for finding the distance of the stars.

We thus proceed by degrees from measuring a line on the surface of the earth, to calculate the distances of the heavenly bodies. From a base line on a plane, is determined the height of a mountain; from the height of a mountain, the diameter of the earth; from the diameter of the earth, the distance of the sun, and from the distance of the sun the distance of the stars.

30. After finding the distance of a heavenly body, its *magnitude* is easily ascertained; if it have an apparent diameter, sufficiently large to be measured by the instruments which are used for taking angles.

Let AEB (Fig. 15.) be the angle which a heavenly body subtends at the eye. Half this angle, if C be the center of the body, is AEC; the line EA is a tangent to the surface, and therefore EAC is a right angle. Then making the distance EC radius,

R : EC : : Sin. AEC : AC.

That is, radius is to the distance, as the sine of half the angle which the body subtends, to its semi-diameter.

Ex. If the sun subtends an angle of 32′ 2″, and if his distance from the earth be 95 million miles; what is his diameter? Ans. 885 thousand miles.

## Promiscuous Examples.

1. On the bank of a river, the angle of elevation of a tree on the opposite side is found to be 46°; and at another station 100 feet directly back on the same level, 31°. What is the height of the tree? Ans. 143 feet.

2. On a horizontal plane, observations were taken of a tower standing on the top of a hill. At one station the angle of elevation of the top of the tower was found to be 50°; that at the bottom 39°; and at another station 150 feet directly back, the angle of elevation of the top of the tower was 32°. What are the heights of the hill and the tower?
Ans. The hill is 134 feet high; the tower 63.

3. What is the altitude of the sun, when the shadow of a tree, cast on a horizontal plane, is to the height of the tree as 4 to 3? Ans. 36° 52′ 12″.

4. If a straight line from the top of the White Mountains in New Hampshire touch the ocean at the distance of $103\frac{1}{3}$ miles? what is the height of the mountains?

Ans. 7100 feet.

5. From the top of a perpendicular rock 55 yards high, the angle of depression of the nearest bank of a river is found to be 55° 54′, that of the opposite bank 33° 20′. Required the breadth of the river, and the distance of its nearest bank from the bottom of the rock.

The breadth of the river is 46.4 yards;
Its distance from the rock 37.2.

6. If the moon subtend an angle of 31′ 14″, when her distance is 240,000 miles; what is her diameter?

Ans. 2180 miles.

7. Observations are made on the altitude of a balloon, by two persons standing on the same side of the balloon, and in a vertical plane passing through it. The distance of the stations is half a mile. At one, the angle of elevation is 30° 58′, at the other 36° 52′. What is the height of the balloon above the ground? Ans. $1\frac{1}{2}$ mile.

8. The shadow of the top of a mountain, when the altitude of the sun on the meridian is 32°, strikes a certain point on a level plain below; but when the meridian altitude of the sun is 67°, the shadow strikes half a mile farther south, on the same plain. What is the height of the mountain above the plain? Ans. 2245 feet.

# NAVIGATION.

## SECTION I.

### PLANE SAILING.

ART. 33. NAVIGATION is the art of conducting a ship on the ocean. The most accurate method of ascertaining the situation of a vessel at sea is to find, by astronomical observations, her *latitude* and *longitude*. But this requires a view of the heavenly bodies; and these are often obscured by intervening clouds. The mariner must therefore have recourse to other means for determining the progress which he has made, and the particular part of the ocean through which he is at any time making his way. The common method is to measure the rate of the ship's going by a *log-line*, and to find the direction in which she sails by a *mariner's compass*. From these data, the difference of latitude, the departure, and the difference of longitude, may be calculated. The two first may be found by plane sailing; the last by middle latitude sailing, or more correctly by Mercator's sailing. See Sec. II. and III.

34. The *log-line* is a cord which is wound round a reel, one end being attached to a piece of wood called a *log*. It is used to determine the distance which a ship runs in an hour, by measuring the distance which she runs in half a minute. The *log* is commonly a small piece of board, in the form of a quadrant of a circle. The arc is loaded with a quantity of lead sufficient to give the board a perpendicular position, when thrown upon the water. This will prevent it from moving forward toward the vessel, while the line is running off the reel. So that the length of line drawn off

by the log in half a minute, is equal to the distance which the vessel moves through the water in that time.

The log-line, which is a hundred fathoms or more, is divided into equal portions called *knots*. Each of these has the same ratio to a nautical mile, which half a minute has to an hour. That is, a knot is the 120th part of a mile. If therefore the motion of the ship is uniform, she sails as many miles in an hour, as she does knots in a half a minute.

The *time* is measured by a *half minute-glass*, constructed like an hour glass. This is turned when the log is thrown upon the water; and the knots drawn from the reel, while the sands are running, give the rate of the ship. The log is thrown either every hour, or once in two hours.

35. The *Mariner's compass* is a circular card, attached to a magnetic needle, which is balanced on an upright pin, so as to move freely in any direction. The ends of the needle turn towards the northern and southern points of the horizon. It places itself in the magnetic meridian, which nearly coincides with the astronomical meridian, or a north and south line.* Directly over the needle, a line is drawn on a card, one end of which is marked N, and the other S. The whole circumference is divided into equal parts by 32 *points*. Four of these, the N, S, E, and W, are called *cardinal* points. The interval between two adjacent points is 11° 15′, which is the quotient of 360° divided by 32. The card and the needle are inclosed in a circular box, on the inside of which a black *mark* is drawn perpendicular to the horizon. When the compass is placed in the vessel, a line passing from this mark through the centre of the card should be parallel to the keel. The part of the circumference which coincides with the mark will then shew the point of compass to which the keel is directed. To prevent the needle from being affected by the motion of the vessel, the box, and brass ring by which it is surrounded, have four points of suspension so contrived as to keep the card nearly parallel to the horizon.

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* For the *variation* of the needle, see SURVEYING, Sec. V.

The following is a table of the number of degrees and minutes corresponding to each point and quarter point of the compass. See Fig. 16.

| North East Quadrant. | South-East Quadrant. | Points. | D. | M. | S. | South West Quadrant. | North-West Quadrant. |
|---|---|---|---|---|---|---|---|
| *North.* | *South.* | 0 0 | 0 | 0 | 0 | *South.* | *North.* |
| N$\frac{1}{4}$E | S$\frac{1}{4}$E | 0 $\frac{1}{4}$ | 2 | 48 | 45 | S$\frac{1}{4}$W | N$\frac{1}{4}$W |
| N$\frac{1}{2}$E | S$\frac{1}{2}$E | 0 $\frac{1}{2}$ | 5 | 37 | 30 | S$\frac{1}{2}$W | N$\frac{1}{2}$W |
| N$\frac{3}{4}$E | S$\frac{3}{4}$E | 0 $\frac{3}{4}$ | 8 | 26 | 15 | S$\frac{3}{4}$W | N$\frac{3}{4}$W |
| N*b*E | S*b*E | 1 0 | 11 | 15 | 0 | S*b*W | N*b*W |
| N*b*E$\frac{1}{4}$E | S*b*E$\frac{1}{4}$E | 1 $\frac{1}{4}$ | 14 | 3 | 45 | S*b*W$\frac{1}{4}$W | N*b*W$\frac{1}{4}$W |
| N*b*E$\frac{1}{2}$E | S*b*E$\frac{1}{2}$E | 1 $\frac{1}{2}$ | 16 | 52 | 30 | S*b*W$\frac{1}{2}$W | N*b*W$\frac{1}{2}$W |
| N*b*E$\frac{3}{4}$E | S*b*E$\frac{3}{4}$E | 1 $\frac{3}{4}$ | 19 | 41 | 15 | S*b*W$\frac{3}{4}$W | N*b*W$\frac{3}{4}$W |
| NNE | SSE | 2 0 | 22 | 30 | 0 | SSW | NNW |
| NNE$\frac{1}{4}$E | SSE$\frac{1}{4}$E | 2 $\frac{1}{4}$ | 25 | 18 | 45 | SSW$\frac{1}{4}$W | NNW$\frac{1}{4}$W |
| NNE$\frac{1}{2}$E | SSE$\frac{1}{2}$E | 2 $\frac{1}{2}$ | 28 | 7 | 30 | SSW$\frac{1}{2}$W | NNW$\frac{1}{2}$W |
| NNE$\frac{3}{4}$E | SSE$\frac{3}{4}$E | 2 $\frac{3}{4}$ | 30 | 56 | 15 | SSW$\frac{3}{4}$W | NNW$\frac{3}{4}$W |
| NE*b*N | SE*b*S | 3 0 | 33 | 45 | 0 | SW*b*S | NW*b*N |
| NE$\frac{3}{4}$N | SE$\frac{3}{4}$S | 3 $\frac{1}{4}$ | 36 | 33 | 45 | SW$\frac{3}{4}$S | NW$\frac{3}{4}$N |
| NE$\frac{1}{2}$N | SE$\frac{1}{2}$S | 3 $\frac{1}{2}$ | 39 | 22 | 30 | SW$\frac{1}{2}$S | NW$\frac{1}{2}$N |
| NE$\frac{1}{4}$N | SE$\frac{1}{4}$S | 3 $\frac{3}{4}$ | 42 | 11 | 15 | SW$\frac{1}{4}$S | NW$\frac{1}{4}$N |
| NE | SE | 4 0 | 45 | 0 | 0 | SW | NW |
| NE$\frac{1}{4}$E | SE$\frac{1}{4}$E | 4 $\frac{1}{4}$ | 47 | 48 | 45 | SW$\frac{1}{4}$W | NW$\frac{1}{4}$W |
| NE$\frac{1}{2}$E | SE$\frac{1}{2}$E | 4 $\frac{1}{2}$ | 50 | 37 | 30 | SW$\frac{1}{2}$W | NW$\frac{1}{2}$W |
| NE$\frac{3}{4}$E | SE$\frac{3}{4}$E | 4 $\frac{3}{4}$ | 53 | 26 | 15 | SW$\frac{3}{4}$W | NW$\frac{3}{4}$W |
| NE*b*E | SE*b*E | 5 0 | 56 | 15 | 0 | SW*b*W | NW*b*W |
| NE*b*E$\frac{1}{4}$E | SE*b*E$\frac{1}{4}$E | 5 $\frac{1}{4}$ | 59 | 3 | 45 | SW*b*W$\frac{1}{4}$W | NW*b*W$\frac{1}{4}$W |
| NE*b*E$\frac{1}{2}$E | SE*b*E$\frac{1}{2}$E | 5 $\frac{1}{2}$ | 61 | 52 | 30 | SW*b*W$\frac{1}{2}$W | NW*b*W$\frac{1}{2}$W |
| NE*b*E$\frac{3}{4}$E | SE*b*E$\frac{3}{4}$E | 5 $\frac{3}{4}$ | 64 | 41 | 15 | SW*bW*$\frac{3}{4}$W | NW*b*W$\frac{3}{4}$W |
| ENE | ESE | 6 0 | 67 | 30 | 0 | WSW | WNW |
| E*b*N$\frac{3}{4}$N | E*b*S$\frac{3}{4}$S | 6 $\frac{1}{4}$ | 70 | 18 | 45 | W*b*S$\frac{3}{4}$S | W*b*N$\frac{3}{4}$N |
| E*b*N$\frac{1}{2}$N | E*b*S$\frac{1}{2}$S | 6 $\frac{1}{2}$ | 73 | 7 | 30 | W*b*S$\frac{1}{2}$S | W*b*N$\frac{1}{2}$N |
| E*b*N$\frac{1}{4}$N | E*b*S$\frac{1}{4}$S | 6 $\frac{3}{4}$ | 75 | 56 | 15 | W*b*S$\frac{1}{4}$S | W*b*N$\frac{1}{4}$N |
| E*b*N | E*b*S | 7 0 | 78 | 45 | 0 | W*b*S | W*b*N |
| E$\frac{3}{4}$N | E$\frac{3}{4}$S | 7 $\frac{1}{4}$ | 81 | 33 | 45 | W$\frac{3}{4}$S | W$\frac{3}{4}$N |
| E$\frac{1}{2}$N | E$\frac{1}{2}$S | 7 $\frac{1}{2}$ | 84 | 22 | 30 | W$\frac{1}{2}$S | W$\frac{1}{2}$N |
| E$\frac{1}{4}$N | E$\frac{1}{4}$S | 7 $\frac{3}{4}$ | 87 | 11 | 15 | W$\frac{1}{4}$S | W$\frac{1}{4}$N |
| *East.* | *East.* | 8 0 | 90 | 0 | 0 | *West.* | *West.* |

36. Plane Sailing is the method of calculating the situation and progress of a ship by means of a plane triangle. Though the surface of the ocean, conforming to the general figure of the earth, is nearly *spherical*;* yet the quantities which are the objects of inquiry in plane sailing, have the same relations to each other, as the sides and angles of a rectilinear triangle. The particulars which are either given or required are *four*, viz.

1. The Course,
2. The Distance,
3. The Difference of Latitude,
4. The Departure.

37. The *Course* is the angle between a meridian line passing through the ship, and the direction in which she sails. It is described by saying that it is so many points or degrees east or west from a north or south line. Thus if the vessel steers NE by E, the course is said to be N 5 points E, or N 56° 15′ E : if SSW, it is said to S 2 points W, or S 22½°W.

A ship is said to continue on the *same course*, when she cuts every meridian which she crosses at the *same angle*. She is steered in any required direction, by causing the keel to make a constant angle with the needle. The line thus described is not a straight line, nor an arc of a circle, but a peculiar kind of curve called the *Loxodromic*† *spiral* or *Rhumb-line*.

38. The *Distance* is the length of the line which the vessel describes in the given time.

39. *Difference of Latitude* is the distance between two parallels of latitude, measured on a meridian. It is also called *Northing* or *Southing*.

40. *Departure is the deviation of a ship east or west from a meridian.* If she sails on a parallel of latitude, her departure is the length of that portion of the parallel over which she passes. But if her course is *oblique*, she is continually chan-

* The true figure of the earth is nearer a *spheroid* than a sphere. But the difference is too inconsiderable to be taken into account in any calculations for which the lines and angles are given from the log and the compass. In this and the following sections, therefore, the earth will be considered as a sphere.

† From Λοξος and δρομος, an oblique course.

ging her latitude; and her departure for each instant ought to be considered as measured on the parallel which she is then crossing. The measure will not be correct, if it be taken wholly on the parallel which the ship has left, or on that upon which she has arrived. Suppose she proceeds from A to C. (Fig. 18.) Let the whole distance be divided into indefinitely small portions A*m*, *mn*, *n*C. Draw the meridians PM, PM′, PM″, PM‴; and the parallels AD, *om*, *yn*, BC. The departure for the first portion is *om*, for the second *sn*, for the third *t*C. And the whole departure is $om+sn+tC$; which, on account of the obliquity of the meridians, is *less* than $Bv+vt+tC=BC$ the meridian distance measured on the parallel upon which the ship has arrived, but *greater* than AD the meridian distance on the parallel which she has left.

41. The distance, departure, and difference of latitude, are measured in *geographical miles* or *minutes;* one of which is equal to the 60th part of a degree at the equator. As the circumference of the earth is about 25 thousand English miles, a degree is nearly $69\frac{1}{2}$ miles. So that a geographical or nautical mile is nearly $\frac{1}{6}$ greater than the common English mile. A *league* is three miles.

42. The peculiar nature of the *Rhumb-line* gives this important advantage in calculation, that the distance, departure, and difference of latitude, though they are curved lines, may be exactly given in length by the sides of a *right angled plane triangle*, in which one of the angles is equal to the course. Suppose a ship proceeds from A to C, (Fig. 18.) describing the rhumb-line A*mn*C, on which the angles MA*m*, M′*mn*, M″*n*C are equal. Let the whole distance be divided into portions so small, that the triangles A*mo*, *mns*, *n*C*t*, shall not differ sensibly from plane triangles. The meridians and parallels being drawn, the several differences of latitude are A*o*, *ms*, *nt;* and the departures *om*, *sn*, *t*C. (Art. 40.)

In the straight line A′C′ (Fig. 19.) make $A'm'=Am$, (Fig. 18.) $m'n'=mn$, $n'C'=nC$, and the angle $C'A'B'=mAo$. Draw *m′v′* and *n′t′* parallel to A′B′; and *m′o′*, *n′y′*, and C′B′ perpendicular to A′B′. Then the triangles A*mo*, A′*m′o′*, *mns*, *m′n′s′*, C*tn*, and C′*t′n′* are all similar to A′B′C′. The difference of latitude is

$$AB=Ao+ms+nt=A'o'+m's'+n't'=A'B'.$$

And the departure is

$$om+sn+tC=o'm'+s'n'+t'C'=B'C'.$$

43. In plane sailing, then, the process of calculation is as accurate,* and as simple, as if the surface of the ocean were a plane. Let NS (Fig. 20.) be a meridian line. If a ship sails from A to C, and BC is perpendicular to NS; then

The *Course* is the *angle* at A; and the complement of the course, the angle at C;

The *Distance* is the *hypothenuse* AC;

The *Departure* is the *base* BC, which is always opposite to the course; and

The *Difference of Latitude* is the *perpendicular* AB, which is opposite to the complement of the course.

Of these four quantities, any *two* being given, the others may be found by rectangular trigonometry. (Trig. 116.) The parts given may be

1. The course and distance; or
2. The course and departure; or
3. The course and difference of latitude; or
4. The distance and departure; or
5. The distance and difference of latitude; or
6. The departure and difference of latitude.

The solutions may be made by arithmetical computation, by Gunter's scale or sliding rule, or by geometrical construction. (Trig. Sec. III, V, VI.) The first method is by far the most accurate. As the student is supposed to be already familiar with trigonometry, the operations will not be repeated here. In the geometrical construction, it will be proper to consider the upper side of the paper as *north*, and the lower side *south*. The right hand will then be east, and the left hand west.

CASE I.

44. Given $\left\{\begin{array}{l}\text{The course,}\\ \text{And distance;}\end{array}\right\}$ to find $\left\{\begin{array}{l}\text{The departure and}\\ \text{Difference of latitude.}\end{array}\right.$

Here we have the hypothenuse and angles given, to find the base and perpendicular. (Trig. 134.)

Making then the distance radius,

$$\text{Rad.} : \text{Dist.} :: \left\{\begin{array}{l}\text{Sin. Course} : \text{Departure.}\\ \text{Cos. Course} : \text{Diff. Lat.}\end{array}\right.$$

*Example* 1.

A ship sails from A (Fig. 20.) SW. by S., 38 miles to C. Required her departure and difference of latitude?

* See Note B.

The course is 3 points, or 33° 45′ (Art. 35.)

$$R : 38 :: \begin{cases} \text{Sin. } 33° 45' : 21.1 = \text{Depart.} \\ \text{Cos. } 33° 45' : 31.6 = \text{Diff. Lat.} \end{cases}$$

*Example* 2.

A ship sails S. 29° E., 34 leagues. Her departure and difference of latitude are required.

Ans. 16.5 and 29.7 leagues.

The proportions in this and the following cases may be varied, by making different sides radius, as in Trigonometry Sec. III.

CASE II.

45. Given { The course / And departure; } to find { The distance, and / Difference of lat.

Making the distance radius, (Trig. 137.)

$$\text{Sin. Course} : \text{Depart} :: \begin{cases} \text{Rad.} : \text{Distance} \\ \text{Cos. Course} : \text{Diff. Lat.} \end{cases}$$

*Example* 1.

A ship leaving a port in latitude 42° N. has sailed S. 37° W. till she finds her departure 62 miles. What distance has she run, and in what latitude has she arrived?

$$\text{Sin. } 37° : 62 :: \begin{cases} \text{Rad.} : 103 = \text{Distance.} \\ \text{Cos. } 37° : 82.3 = \text{Diff. of latitude.} \end{cases}$$

The difference of latitude is 82.3 miles, or 1° 12′.3. (Art. 41.) This is to be *subtracted* from the original latitude of the ship, because her course was *towards* the equator. The remainder is 40° 47′.7, the latitude on which she has arrived.

*Example* 2.

A ship leaves a port in latitude 63° S., and runs N. 54° E. till she makes a harbor where her departure is found to be 74 miles; how great is the distance of the two places, and what is the latitude of the latter?

The distance is 91½ miles; and the latitude of the latter place is 62° 06′.2.

CASE III.

46. Given { The course, and / Diff. of latitude; } to find { The distance, / And departure.

Making the distance radius,

$$\text{Cos. Course} : \text{Diff. Lat.} :: \begin{cases} \text{Rad.} : \text{Distance.} \\ \text{Sin. Course} : \text{Departure.} \end{cases}$$

*Example.*

A ship sails S. 50° E. from latitude 7° N., to latitude 4° S. Required her distance and departure.

As the two latitudes are on different sides of the equator, the distance of the parallels is evidently equal to the *sum* of the given latitudes. This is 11°, or 660 miles. The distance is 1026.8 miles, and the departure 786½.

CASE IV.

47. Given { The distance, And departure; } to find { The course, and Diff. of latitude.

Making the distance radius, (Trig. 135.)

Dist. : Rad. : : Depart : Sin. Course,

Rad. : Dist. : : Cos. Course : Diff. lat.

*Example.*

A ship having left a port in Lat. 3° N., and sailing between S. and E. 400 miles, finds her departure 180 miles. What course has she steered, and what is her latitude?

Her latitude is 2° 57½′ S., and her course S. 26° 44½′ E.

CASE V.

48. Given { The distance, and Diff. of latitude; } to find { The course, And departure.

Making the distance radius,

Dist. : Rad. : : Diff. Lat. : Cos. Course,

Rad. : Dist. : : Sin. Course : Departure.

*Example.*

A vessel sails between N. and E. 66 miles, from Lat. 34° 50′ to Lat. 35° 40′. Required her course and departure.

The course is N. 40° 45′ E., and the departure 43.08 miles.

CASE VI.

49. Given { The departure, and Diff. of latitude; } to find { The course, And distance.

Making the difference of latitude radius, (Trig. 139.)

Diff. Lat. : Rad. : : Depart : Tan. Course,

Rad. : Diff. Lat. : : Sec. Course : Distance.

*Example.*

A ship sails from the equator between S. and W., till her latitude is 5° 52′, and her departure 264 miles. Required her course and distance.

The course is S. 36° $52\frac{1}{5}'$ W, and the distance is 440 miles.

*Examples for practice.*

1. Given a ship's course S. 46° E., and departure 59 miles; to find the distance and difference of latitude.
2. Given the distance 68 miles, and departure 47; to find the course and difference of latitude.
3. Given the course SSE., and the distance 57 leagues; to find the departure and difference of latitude.
4. Given the course NW. by N., and the difference of latitude 2° 36′; to find the distance and departure.
5. Given the departure 92, and the difference of latitude 86; to find the course and distance.
6. Given the distance 123, and the difference of latitude 96; to find the course and departure.

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## THE TRAVERSE TABLE.

50. To save the labor of calculation, tables have been prepared, in which are given the departure and difference of latitude, for every degree of the quadrant, or for every quarter of a degree. These are called *Traverse tables*, or tables of *Departure and Latitude*. The distance is placed in the left hand column, the departure and difference of latitude directly opposite, and the degrees if less than 45° or 4 points, at the top of the page, but if more than 45°, at the bottom. The titles at the top of the columns correspond to the courses at the top; and the titles at the bottom, to the courses at the bottom; the difference of latitude for a course *greater* than 45°, being the same as the departure for one which is as much *less* than 45°. See Trig. 104.

If the given distance is greater than any contained in the table, it may be *divided into parts*, and the departure and difference of latitude found for each of the parts. The *sums* of the numbers thus found will be the numbers required.

The departure and difference of latitude for *decimal parts* may be found in the same manner as for whole numbers, by supposing the decimal point in each of the columns to be moved to the left, as the case requires.

With the aid of a traverse table, all the cases of plane sailing may be easily solved by inspection.

Ex. 1. Given the course 33° 45′; and the distance 38 miles; to find the departure and difference of latitude.

Under 33¾°, and opposite 38, will be found the difference of latitude 31.6, and the departure 21.11; the same as in page 21.

2. Given the course 57°, and the distance 163.

| | | | |
|---|---|---|---|
| The departure and diff. of lat. | for 100 | are 83.87 | and 54.46 |
| | for 63 | 52.84 | 34.31 |
| | for 163 | 136.71 | 88.77 |

3. Given the course 39°, and the distance 18.23.

| | | | |
|---|---|---|---|
| The departure and diff. of lat. | for 18. | are 11.33 | and 13.99 |
| | for .23 | 0.14 | 0.18 |
| | for 18.23 | 11.47 | 14.17 |

4. Given the course 41° 15′, and the departure 60.

Under 41¼°, and against the departure 60, will be found the difference of latitude 68.42 and the distance 91.

5. Given the distance 63, and the departure 56.

Opposite the distance 63, find the departure 56; in the adjoining column will be the latitude 28.85, and at the bottom, the course 62¾°.

6. Given the departure 72, and the difference of latitude 37.

Opposite these numbers in the columns of latitude and departure, will be found the distance 81, and at the foot of the columns, the course 62¾°.

51. The traverse table is useful, not only for taking out departure and difference of latitude; but for finding by inspection the sides and angles of *any right angled triangle* whatever. In plane sailing, the distance is the hypothenuse, (see Fig. 20.) the difference of latitude is the perpendicular, the departure is the base, and the course is the acute angle

at the perpendicular. If then the hypothenuse of any right-angled triangle whatever, be found in the column of distances, in the traverse table; the perpendicular will be opposite in the latitude column, and the base in the departure column; the angle at the perpendicular, being at the top or bottom of the page.

Ex. 1. Given the hypothenuse 24, and the angle at the perpendicular $54°\frac{1}{2}$; to find the base and perpendicular by inspection.

Opposite 24 in the distance column, and over $54°\frac{1}{2}$ will be found the base 19.54 in the departure column, and the perpendicular 13.94 in the latitude column.

2. Given the angle at the perpendicular $37°\frac{1}{4}$, and the base 46; to find the hypothenuse and perpendicular.

Under $37°\frac{1}{4}$, look for 46 in the departure column; and opposite this will be found the perpendicular 60.5 in the latitude column, and the hypothenuse 76 in the distance column.

3. Given the perpendicular 36, and the base 30.21; to find the hypothenuse and angles.

Look in the columns of latitude and departure, till the numbers 36 and 30.21 are found opposite each other; these will give the hypothenuse 47, and the angle at the perpendicular 40°.

## SECTION II.

### PARALLEL AND MIDDLE LATITUDE SAILING.

52. By the methods of calculation in plane sailing, a ship's course, distance, departure, and difference of latitude are found. There is one other particular which it is very important to determine, the *difference of longitude*. The departure gives the distance between two meridians in *miles*. But the situations of places on the earth, are known from their latitudes and longitudes; and these are measured in *degrees*. The lines of longitude, as they are drawn on the globe, are farthest

from each other at the equator, and gradually converge towards the poles. A ship, in making a hundred miles of departure, may change her latitude in one case 2 degrees, in another 10, and in another 20. It is important, then, to be able to convert departure into difference of longitude; that is, to determine how many degrees of longitude answer to any given number of miles, on any parallel of latitude. This is easily done by the following

### Theorem.

53. As the cosine of latitude,
To radius;
So is the departure,
To the difference of longitude.

By this is to be understood, that the cosine of the latitude is to radius; as the distance between two meridians measured on the given parallel, to the distance between the same meridians measured on the equator.

Let P (Fig. 21.) be the pole of the earth, A a point at the equator, L a place whose latitude is given, and LO a line perpendicular to PC. Then CL or CA is a semi-diameter of the earth, which may be assumed as the radius of the tables; PL is the complement of the latitude, and OL the sine of PL, that is, the *cosine* of the latitude.

If the whole be now supposed to revolve about PC as an axis, the radius CA will describe the equator, and OL the given parallel of latitude. The circumferences of these circles are as their semi-diameters OL and CA, (Sup. Euc. 8. 1.) And this is the ratio which any portion of one circumference has to a like portion of the other. Therefore OL is to CA, that is, the cosine of latitude is to radius, as the distance between two meridians measured on the given parallel, to the distance between the same meridians measured on the equator.

Cor. 1. Like portions of different parallels of latitude are to each other, as the cosines of the latitudes.

Cor. 2. A degree of longitude is commonly measured on the equator. But if it be considered as measured on a parallel of latitude, the *length* of the degree will be as the cosine of the latitude.

*The following table contains the length of a degree of longitude for each degree of latitude.*

| d.l. | miles. | d.l. | miles. | d.l. | miles. | d.l. | miles. | d.l. | miles. | d.l. | miles. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 59.99 | 16 | 57.67 | 31 | 51.43 | 46 | 41.68 | 61 | 29.09 | 76 | 14.52 |
| 2 | 59.96 | 17 | 57.38 | 32 | 50.88 | 47 | 40.92 | 62 | 28.17 | 77 | 13.50 |
| 3 | 59.92 | 18 | 57.06 | 33 | 50.32 | 48 | 40.15 | 63 | 27.24 | 78 | 12.47 |
| 4 | 59.85 | 19 | 56.73 | 34 | 49.74 | 49 | 39.36 | 64 | 26.30 | 79 | 11.45 |
| 5 | 59.77 | 20 | 56.38 | 35 | 49.15 | 50 | 38.57 | 65 | 25.36 | 80 | 10.42 |
| 6 | 59.67 | 21 | 56.01 | 36 | 48.54 | 51 | 37.76 | 66 | 24.40 | 81 | 9.39 |
| 7 | 59.55 | 22 | 55.63 | 37 | 47.92 | 52 | 36.94 | 67 | 23.44 | 82 | 8.35 |
| 8 | 59.42 | 23 | 55.23 | 38 | 47.28 | 53 | 36.11 | 68 | 22.48 | 83 | 7.31 |
| 9 | 59.26 | 24 | 54.81 | 39 | 46.63 | 54 | 35.27 | 69 | 21.50 | 84 | 6.27 |
| 10 | 59.08 | 25 | 54.38 | 40 | 45.96 | 55 | 34.41 | 70 | 20.52 | 85 | 5.23 |
| 11 | 58.89 | 26 | 53.93 | 41 | 45.28 | 56 | 33.55 | 71 | 19.53 | 86 | 4.19 |
| 12 | 58.68 | 27 | 53.46 | 42 | 44.59 | 57 | 32.68 | 72 | 18.54 | 87 | 3.14 |
| 13 | 58.46 | 28 | 52.97 | 43 | 43.88 | 58 | 31.80 | 73 | 17.54 | 88 | 2.09 |
| 14 | 58.22 | 29 | 52.47 | 44 | 43.16 | 59 | 30.90 | 74 | 16.54 | 89 | 1.05 |
| 15 | 57.95 | 30 | 51.96 | 45 | 42.43 | 60 | 30.00 | 75 | 15.53 | 90 | 0.00 |

The length of a degree of longitude in different parallels is also shown by the *Line of Longitude*, placed over or under the line of chords, on the plane scale. (See Trig. 165.)

54. The sailing of a ship on a parallel of latitude* is called *Parallel Sailing*. In this case, the departure is equal to the distance. The difference of longitude may be found by the preceding theorem; or if the difference of longitude be given, the departure may be found by inverting the terms of the proportion. (Alg. 380. 3.)

55. The *Geometrical Construction* is very simple. Make CBD (Fig. 22.) a right angle, draw BC equal to the departure in miles, lay off the angle at C equal to the latitude in degrees, and draw the hypothenuse CD for the difference of longitude. The angle C, and the sides BC and CD, of this triangle, have the same relations to each other, as the latitude, departure, and difference of longitude.

For Cos. C : BC : : R : CD (Trig. 121.)
And Cos. Lat. : Depart : : R : Diff. Lon. (Art. 53.)

* See Note C.

56. The parts of the triangle may be found by *inspection* in the traverse table. (Art. 51.) The angle opposite the departure is D the complement of the latitude, and the difference of longitude is the hypothenuse CD. If then the departure be found in the departure column under or over the given number of degrees in the co-latitude, the difference of longitude will be opposite in the distance column.

*Example* I.

A ship leaving a port in Lat. 38° N. Lon. 16° E. sails west on a parallel of latitude 117 miles in 24 hours. What is her longitude at the end of this time?

Cos. 38° : Rad. : : 117 : $148\frac{1}{2}'=2°\ 28\frac{1}{2}'$ the difference of longitude.

This subtracted from 16° leaves 13° $31\frac{1}{2}'$ the longitude required.

*Example* II.

What is the distance of two places in Lat. 46° N. if the longitude of the one is 2° 13′ W. and that of the other 1° 17′ E.?

As the two places are on opposite sides of the first meridian, the difference of longitude is $2°\ 13'+1°\ 17'=3°\ 30'$, or 210 minutes. Then

Rad : Cos. 46° : : 210 : 145.88 miles, the departure, or the distance between the two places.

*Example* III.

A ship having sailed on a parallel of latitude 138 miles, finds her difference of longitude 4° 3′ or 243 minutes. What is her latitude?

Diff. Lon. 243 : Dep. 138 : : Rad. : Cos. Lat. 55° $23\frac{3}{4}'$.

*Example* IV.

On what part of the earth are the degrees of longitude *half* as long as at the equator? Ans. In latitude 60.

## MIDDLE LATITUDE SAILING.

57. By the method just explained, is calculated the difference of longitude of a ship *sailing on a parallel of latitude.* But instances of this mode of sailing are comparatively few. It is necessary then to be able to calculate the longitude when the course is *oblique.* If a ship sail from A to C, (Fig. 18.) the departure is equal to $om+sn+tC$. But the sum of these small lines is *less* than BC, and *greater* than AD. (Art. 40.) The departure, then, is the meridian distance measured not on the parallel from which the ship sailed, nor on that upon which she has arrived, but upon one which is between the two. If the exact situation of this intermediate parallel could be determined, by a process sufficiently simple for common practice, the difference of longitude would be easily obtained. The parallel usually taken for this purpose, is an *arithmetical mean* between the two extreme latitudes. This is called the *Middle Latitude.* The meridian distance on this parallel is not exactly equal to the departure. But for small distances, the error is not material, except in high latitudes.

The middle latitude is equal to *half the sum* of the two extreme latitudes, if they are both north or both south: but to *half their difference*, if one is north and the other south.

58. In middle latitude sailing, all the calculations are made in the same way as in plane sailing, excepting the proportions in which the *difference of longitude* is one of the terms. The departure is derived from the difference of longitude, and the difference of longitude from the departure, in the same manner as in parallel sailing, (Arts. 53, 54.) only substituting in the theorem the term *middle latitude* for latitude.

### THEOREM I.

As THE COSINE OF MIDDLE LATITUDE,
To RADIUS;
So IS THE DEPARTURE,
To THE DIFFERENCE OF LONGITUDE.

59. The learner will be very much assisted in stating the proportions, by keeping the geometrical construction steadily in his mind. In Fig. 20 we have the lines and angles in plane sailing, and in Fig. 22, those in parallel sailing. By

bringing these together, as in Fig. 23, we have all the parts in middle latitude sailing. The two right angled triangles, being united at the common side BC, which is the departure, form the oblique angled triangle ACD.

60. The angle at D is the complement of the middle latitude. (Art. 55.) Then in the triangle ACD, (Trig. 143.)

Sin D : AC::Sin A : DC; that is,

THEOREM II.

AS THE COSINE OF MIDDLE LATITUDE,
TO THE DISTANCE;
SO IS THE SINE OF THE COURSE,
TO THE DIFFERENCE OF LONGITUDE.

61. The two preceding theorems, with the proportions in plane sailing, are sufficient for solving all the cases in middle latitude sailing. A third may be added, for the sake of reducing two proportions to one.

In the triangle BCD (Fig. 23.) Cos. BCD : R::BC : CD
And in the triangle ABC, AB : R::BC : Tan A.

The means being the same in these two proportions, the extremes are reciprocally proportional. (Alg. 387.) We have then

Cos BCD : AB::Tan A : CD; that is,

THEOREM III.

As the cosine of middle latitude,
To the difference of latitude;
So is the tangent of the course,
To the difference of longitude.

Among the other data in middle latitude sailing, one of the extreme latitudes must always be given.

*Example* I.

At what distance and in what direction, is Montock Point from Martha's Vineyard; the former being in Lat. 41° 04′N. Lon. 72°W., and the latter in Lat. 41° 17′ N. Lon. 70° 48′ W.?

Here are given the two latitudes and longitudes, to find the course and distance.

| | |
|---|---|
| The difference of longitude is | 72′ |
| The difference of latitude | 13′ |
| The middle latitude | 41° 10′$\frac{1}{2}$ |

Beginning with the triangle in which there are two parts given, by theorem I,

R : Diff. Lon. : : Cos. Mid. Lat. : Depart. = 54.2.

And by plane sailing, Case VI,

Diff. Lat. : Rad. : : Depart : Tan. Course = 76° 30⅔′.

Or to find the course at a single statement, by theorem III,

Diff. Lat. : Cos Mid. Lat. : : Diff. Lon. : Tan Course = 76° 30⅔′

To find the distance by plane sailing, Case III,

Cos. Course : Diff. Lat. : : Rad. : Dist. = 55.73.

*Example* II.

A ship leaving New York light-house in Lat. 40° 28′ N. and Lon. 74° 08′ W. sails SE. 67 miles in 24 hours. Required her latitude and longitude at the end of that time.

By plane sailing,

Rad. : Dist. : : Cos. Course : Diff. Lat. = 47.4′.

The latitude required, therefore, is 39° 40.6′, and the middle latitude 40° 04.3′.

Then by Theorem II,

Cos. Mid. Lat. : Dist. : : Sin. Course : Diff. Lon. = 61.9′.

Or by Theorem III,

Cos. Mid. Lat. : Diff. Lat. : : Tan. Course : Diff. Lon. = 61.9′.

The longitude required is 73° 06.1′.

*Example* III.

A ship leaving a port in Lat. 49° 57′ N. Lon. 5° 14′ W. sails S. 39° W. till her latitude is 45° 31′. Required her longitude and distance.

Ans. 10° 34.3′ W. and 342.3 miles.

*Example* IV.

A ship sailing from Lat. 49° 57′ N. and Lon. 5° 14′ W. steers west of south, till her longitude is 23° 43′, and her departure 789 miles. Required her course, distance, and latitude.

Course 51° 5′ W.
Latitude 39° 20′ N.
Distance 1014 miles.

## SECTION III.

### MERCATOR'S SAILING.*

Art. 62. The calculations in middle latitude sailing are simple, and sufficiently accurate for short distances, particularly near the equator. But they become quite erroneous, when applied to great distances, and to high latitudes. The only method in common use, which is strictly accurate, is that called *Mercator's Sailing*, or Wright's Sailing. This is founded on the construction of a *chart*, published in 1556 by Gerard Mercator. About forty years after, Mr. Edward Wright gave demonstrations of the principles of this chart, and applied them to the solution of problems in navigation.

63. In the construction of Mercator's chart, the earth is supposed to be a sphere. Yet the meridians, instead of converging towards the poles, as they do on the globe, are drawn *parallel* to each other. The distance of the meridians, therefore, is every where too great, except at the equator. To compensate this, the degrees of *latitude* are proportionally enlarged. On the artificial globe, the parallels of latitude are drawn at equal distances. But on Mercator's chart, the distances of the parallels increase from the equator to the poles, so as every where to have the same ratio to the distances of the meridians, which they have on the globe. Thus in latitude 60°, where the distance of the meridians must be *doubled*, to make it the same as at the equator, a degree of latitude is also made twice as great as at the equator. The dimensions of places are extended in the projection, in proportion as they are nearer the poles. The diameter of an island in latitude 60° would be represented twice as great as if it were on the equator, and its area four times as great.

* Robertson's Navigation, London Phil. Trans. for 1666 and 1696, Hutton's Dictionary, Introduction to Hutton's Mathematical Tables, Bowditch's Practical Navigator, Emerson's and M'Laurin's Fluxions, M'Kay's Navigation, Emerson's Prin. Navig., Barrow's Navigation.

64. *Table of Meridional Parts.*—If a meridian on a sphere be divided into degrees or minutes, the portions are all *equal.* But in Mercator's projection, they are extended more and more as they are farther from the equator. To facilitate the calculations in navigation, *tables* have been prepared, which contain the length of any number of degrees and minutes on this extended meridian, or the distance of any point of the projection from the equator. These are called tables of *Meridional Parts.* The common method of computing them is derived from the following proposition.

65. ANY MINUTE PORTION OF A PARALLEL OF LATITUDE,
IS TO A LIKE PORTION OF THE MERIDIAN;
AS RADIUS,
TO THE SECANT OF THE LATITUDE.

For, by the theorem in parallel sailing, (Art. 53.) the cosine of latitude is to radius, as the departure to the difference of longitude measured on the equator; that is, as a part of the parallel of latitude, to a like part of the equator. But on a sphere, the equator and meridian are equal.

Therefore Cos. Lat. : Rad. : : *a part of the parallel* : *a like part of the meridian.*

But Cos. Lat. : Rad. : : Rad. : Sec. Lat. (Trig. 93. 3.)

By equality of ratios then, (Alg. 384.)

*A part of the paral.* : *a like part of the merid.* : : Rad. : Sec. Lat.

By like parts of the parallel of latitude and the meridian are here meant minutes, seconds or other portions of a degree. The proposition is true when applied either to the circles on a sphere, or to the lines in Mercator's projection. For the parts of the latter have the same ratio to each other, as the parts of the former. (Art. 63.) The divisions of Mercator's meridian, however, should be made very small; for the measure of each part is supposed to be taken *at* the parallel of latitude, and not at a distance from it. In the common tables, the meridian is divided into *minutes.*

66. Suppose then that the length of each minute of a degree of Mercator's meridian is required. By the proposition in the last article,

1′ *of the parallel* : 1′ *of the meridian* : : Rad. : Sec. Lat.

But in this projection, the parallels of latitude are all *equal.*

(Art. 63.) Whatever be the latitude, then, the first term of the proportion is equal to a minute at the equator, or a geographical mile; and if this is assumed as the radius of the trigonometrical tables, (Trig. 100.) the first and third terms are equal, and therefore the second and fourth must be equal also. (Alg. 395.) That is, *the length of any one minute of Mercator's meridian is equal to the natural secant of the latitude of that part of the meridian.*

| | | |
|---|---|---|
| The *first* | minute of the meridian is equal to the secant of | *one* minute, |
| The *second* | | *two* minutes, |
| The *third* | | *three* minutes, |

&c. &c.

The table of meridional parts is formed by adding together the several minutes thus found.* Beginning from the equator, an arc of the meridian

of *two* minutes=*sec.* 1′+*sec.* 2′,
of *three* minutes=*sec.* 1′+*sec.* 2′+*sec.* 3′,
of *four*. minutes=*sec.* 1′+*sec.* 2′+*sec.* 3′+*sec.* 4′,
&c. &c.

See the table at the end of this number.

To find from the table the length of any given number of degrees and minutes, look for the degrees at the top of the page, and the minutes on the side; then against the minutes, and under the degrees, will be the length of the arc in nautical miles.

67. *Meridional Difference of Latitude.*—An arc of Mercator's meridian contained between *two* parallels of latitude, is called meridional difference of latitude. It is found by subtracting the meridional parts for the less latitude from the meridional parts for the greater, if both are north or south; or by adding them, if one latitude is north and the other south.

| | | |
|---|---|---|
| Thus the lat. of Boston is | 42° 23′ | Merid. parts 2813 |
| Baltimore is | 39° 23′ | Merid. parts 2575 |
| Proper difference of lat. | 3° | Merid. diff. of lat. 238. |

68. If one latitude and the meridional difference of latitude be *given*, the *proper* difference of latitude is found by reversing this process.

* See Note D.

When the two latitudes are on the same side of the equator, subtracting the meridional difference of latitude from the meridional parts for the greater, will give the meridional parts for the less; or adding the meridional difference to the parts for the less latitude, will give the parts for the greater. But if the two latitudes are on opposite sides of the equator, subtracting the parts for the one latitude from the meridional difference, will give the parts for the other.

Thus the meridional difference of latitude between

| | | | |
|---|---|---|---|
| New York and New Orleans is | | | 793 |
| The lat. of N. Orleans is | 29° 57′ | Merid. parts | 1885 |
| The lat. of New York | 40° 42′ | Merid. parts | 2678 |

69. *Solutions in Mercator's Sailing.*—The solutions in Mercator's sailing are founded on *the similarity of two right-angled triangles, in one of which the perpendicular sides are the proper difference of latitude and the departure; and in the other, the meridional difference of latitude and the difference of longitude.*

According to the principle of Mercator's projection, the enlargement of each minute portion of the meridian is proportioned to the enlargement of the parallel of latitude which crosses it. (Art. 63.) Any part of the meridian before it is enlarged, is *proper difference of latitude;* and after it is enlarged, is *meridional difference of latitude.* A part of the parallel, before it is enlarged, is *departure;* and after it is enlarged, is equal to the corresponding *difference of longitude;* because in this projection, the distance of the meridians is the same on any parallel, as at the equator, where longitude is reckoned.

If then we take a small portion of the distance which a ship has sailed, as A$m$, (Fig. 18.)

Prop. Diff. Lat. A$o$ : Depart. $om$ : : Merid. Diff. Lat. : Diff. Lon.

In the triangle ABC, (Fig. 24.) let the angle at A = the course $o$A$m$, (Fig. 18.) AB = the proper difference of latitude, AC=A$m$+$mn$+$n$C the distance, and BC= $om$+$sn$+$t$C the departure. Then as the triangles A$om$, $msn$, $nt$C are each similar to the triangle ABC, (Fig. 24.) the difference of latitude for any one of the small distances as A$m$, is to the corresponding departure; as the *whole difference of latitude* AB to the *whole departure* BC. Therefore,

$$\text{P. Diff. Lat. AB : Dep. BC :: Mer. Diff. Lat.} \left\{\begin{array}{l} \text{for } Am : \\ \text{for } mn : \\ \text{for } nC : \end{array}\right. \text{Diff. Lon.} \left\{\begin{array}{l} \text{for } Am, \\ \text{for } mn, \\ \text{for } nC. \end{array}\right.$$

But the whole meridional difference of latitude for the distance AC, is equal to the sum of the differences for A*m*, *mn*, and *n*C; and the whole difference of longitude is equal to the sum of the differences for A*m*, *mn*, *n*C. Therefore, (Alg. 388, Cor. 1.)

Prop. Dif. Lat. AB : Dep. BC : : Merid. Dif. Lat. : Dif. Lon.

Extend AB, (Fig. 24.) making AL equal to the meridional difference of latitude corresponding to the proper difference of latitude AB; from L draw a line parallel to BC, and extend AC to intersect this in D. Then is DL the *difference of longitude*. For it has been shown that the difference of longitude is a *fourth proportional* to the proper difference of latitude, the departure, and the meridional difference of latitude; and by similar triangles,

AB : BC : : AL : LD.

70. To solve all the cases, then, in Mercator's sailing, we have only to represent the several quantities by the parts of two similar right angled triangles, as ABC and ALD, (Fig. 24.) and to find their sides and angles. In the smaller triangle ABC the parts are the same as in plane sailing, and the calculations are made in the same manner. The sides AL and DL are added for finding the difference of longitude; or when the difference of longitude is given, to derive from it one of the other quantities. The *course* is common to both the triangles, and the complement of the course is either ACB or ADL. The hypothenuse AD is not one of the quantities which are given or required in navigation.

71. In the similar triangles ABC, ALD, (Fig. 24.)

AB : AL : : BC : LD; that is,

### THEOREM I.

AS THE PROPER DIFFERENCE OF LATITUDE,
TO THE MERIDIONAL DIFFERENCE OF LATITUDE;
SO IS THE DEPARTURE,
TO THE DIFFERENCE OF LONGITUDE.

72. In the triangle ALD, if AL be made radius,

Rad. : Tan. A : : AL : DL; that is,

## THEOREM II.

AS RADIUS,
TO THE TANGENT OF THE COURSE;
SO IS THE MERIDIONAL DIFFERENCE OF LATITUDE,
TO THE DIFFERENCE OF LONGITUDE.

By this theorem, the difference of longitude may be calculated, without previously finding the departure.

73. In Mercator's, as well as in middle latitude sailing, *one latitude* must always be given. This is requisite in converting proper difference of latitude and meridional difference of latitude into each other. (Arts. 67, 68.)

74. When the difference of latitude is *very small*, the difference of longitude will be more correctly found by middle latitude sailing, than by Mercator's sailing; unless a table is used in which the meridional parts are given to *decimals*. Mercator's sailing is strictly correct in *theory*. But the common tables are not carried to a degree of exactness, sufficient to mark very minute differences. On the other hand, the error of middle latitude sailing is *diminished*, as the difference of latitude is lessened.

*Example* I.

The latitudes of Montock and Martha's Vineyard are { 41° 4′N. / 41 17′N.

Their longitudes { 72° w. / 70°48′w.

Required the course and distance from one to the other.

| | | | |
|---|---|---|---|
| Lat. of Martha's Vin. | 41° 17′ | Merid. parts | 2724 |
| of Montock | 41 04 | Merid. parts | 2707 |
| Proper Diff. of Lat. | 13′ | Mer. Diff. Lat. | 17 (Art. 67.) |

The difference of longitude is 1° 12′=72 miles.

To find the course by theorem II, (Fig. 24.)
Merid. Diff. Lat. : Diff. Lon. : : Rad. : Tan. Course=76° 43′.

To find the distance by plane sailing,
Cos. Course : Prop. Diff. Lat. : : Rad. : Dist.=56.58.

The results by middle latitude sailing, page 31, are a little different, as that method is not perfectly accurate.

*Example* II.

A ship sailing from the Lizard in Lat. 49° 57′ N. Lon. 5° 14′W. proceeds S. 39° W. till her latitude is found by observation to be 45° 31′ N. What is then her longitude, and what distance has she run?

Here are given the difference of latitude and the course, to find the distance and the difference of longitude.

The proper difference of latitude is 4° 26′=266′
The meridional difference of latitude 396

Then by plane sailing,

Cos. Course : Prop. Diff. Lat. : : Rad. : Dist.=342.3.

And by theorem II,

Rad.: Tan. Course : : M.Diff. Lat.: Diff. Lon.=320′.7=5°20′.7.

This added to the longitude of the Lizard 5°14′ gives the longitude of the ship 10°34′.7 W.

*Example* III.

A ship sailing from Lat. 49°57′ N. and Lon. 5° 14′ W. steers west of south, till her latitude is 39° 20′ N. and her departure 789 miles. Required her course, distance, and longitude.

The proper difference of latitude is 10° 37′=637′
The meridional difference of latitude 899

Then by theorem I, (Fig. 24.)

P. Diff. Lat. : M. Diff. Lat. : : Dep. : Diff. Lon.=1113′.5= 18°33′.5.

The longitude of the ship is therefore 23° 47′½.

And by plane sailing,

Prop. Diff. Lat. : Rad. : : Depart. : Tan. Course=51°5′.
Rad. : Prop. Diff. Lat. : : Sec. Course : Distance=1014 miles.

*Example* IV.

A ship sailing from a port in Lat. 14° 45′N. Lon. 17° 33′ W. steers S. 28° 7′½ W. till her longitude is found by observation to be 29° 26′ W. Required her distance and latitude.

The difference of longitude is 11° 53′=713′.

By theorem II,

Tan. course : : Rad. : : Diff. Lon. : M.Diff.Lat.=1334 S.
Lat. of the port 14° 45′ N. Merid. parts 895 N.

of the ship 7 18 S. Merid. parts 439 S. (Art.68.)

Diff. of Lat. 22° 3′=1323′.

By plane sailing,

Cos. Course : Diff. Lat. : : Rad. : Distance=1500 miles.

*Example* V.

A ship sails 300 miles between north and west, from Lat. 37° N. to 41°N. What is her course and difference of longitude?

The course is N. 36° 52′ W., and the difference of longitude 3° 52′.

*Example* VI.

A ship sails S. 67° 30′ E. from Lat. 50° 10′ S. till her departure is 957 miles. What is her distance, difference of latitude, and difference of longitude?

| | |
|---|---|
| The distance is | 1036 miles. |
| The difference of latitude | 6°36′.4 |
| The difference of longitude | 26°53′. |

*Example* VII.

A ship sailing from Lat. 26° 13′ N. proceeds S. 27° W. 231 miles. What is her difference of latitude and difference of longitude?

*Example* VIII.

A ship sailing from Lat. 14° S. 260 miles, between south and west, makes her departure 173 miles. What is her course, difference of latitude, and difference of longitude?*

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* See Note E.

## SECTION IV.

### TRAVERSE SAILING.

Art. 75. By the methods in the preceding sections, are found the difference of latitude, departure, &c. for a *single course*. But it is not often the fact that a ship proceeds from one port to another in a direct line. Variable and contrary winds frequently render a change of direction necessary every few hours. The irregular path of the ship, sailing in this manner, is called a *traverse*.

*Resolving a traverse* is reducing the compound course to a single one. This is commonly done at sea every noon. From the several courses and distances in the log-book, the departure, difference of latitude, &c. are determined for the whole 24 hours. In the same manner, the courses of several successive *days* are reduced to one, so as to ascertain, at any time, the situation of the ship. The following methods by construction and by calculation, are sufficiently accurate for short distances, at least near the equator.

76. *Geometrical construction of a traverse.*—To construct a traverse, draw a meridian line and lay down the first course and distance; from the end of this, lay down the *second* course and distance; from the end of that, a *third* course, &c. Then draw a line connecting the extremities of the first and last of these, to show the whole distance, and the direction of the ship from the point of starting.

This will be easily understood by an example.

*Example* I.

A ship sails from a port in Lat. 32° N., and in 24 hours makes the following courses;

1. N. 25° E. 16 miles,
2. S. 54° E. 11,
3. N. 13° W. 7,
4. N. 61° E. 5,
5. N. 38° W. 18.

It is required to find the departure, difference of latitude, distance, and course, for the whole traverse.

On A as a center (Fig. 25.) describe a circle and draw the meridian NAS. Then considering the upper part as north, the right hand east, and the left hand west, draw the lines A1, A2, A3, A4, and A5, to correspond with the several courses; that is, make the angle NA1=25°, SA2=54°; NA3=13°, NA4=61°, and NA5=38°.

Make A1B=16, BC=11 and parallel to A2, CD=7 and parallel to A3, DF=5 and parallel to A4, FG=18 and parallel to A5; join AG, and draw GP perpendicular to NS.

Then if the surface of the earth be considered as a plane, G is the place of the ship at the end of 24 hours, AG the *distance* from port, PG the *departure*, AP the *difference of latitude*, and GAP the *course*. The angles may be measured by a line of chords, and the distances taken from a scale of equal parts.* (Trig. 148, 161, 2.)

| | |
|---|---|
| The distance is | 32.3 miles. |
| The departure | 7.38 |
| The difference of lat. | 31.45 |
| The course | 13° 12′. |

77. *Resolving a traverse, by Calculation or Inspection.* When a ship sails on different courses for a short time, the difference of latitude, at the end of that time, is equal to the difference between the sum of the northings and the sum of the southings, and the departure is nearly equal to the difference between the sum of the eastings and the sum of the westings. (See Arts. 78, 79.) If then the difference of latitude and the departure for each course be found by calculation or inspection, and placed in separate columns in a table; the difference of latitude for the whole time may be obtained exactly, and the departure nearly, by addition and subtraction; and the corresponding distance and course may be determined by trigonometrical calculation or inspection, as in the last case of plane sailing. (Art. 49.)

The following table contains the courses, distances, departure and difference of latitude in the preceding example. See Fig. 25.

---

* See Note F.

TRAVERSE TABLE.

| Courses. | Distances. | Diff. Lat. N. | Diff. Lat. S. | Departure. E. | Departure. W. |
|---|---|---|---|---|---|
| 1. N. 25° E. | AB 16 | 14.50 | | 6.76 | |
| 2. S. 54° E. | BC 11 | | 6.47 | 8.90 | |
| 3. N. 13° W. | CD 7 | 6.82 | | | 1.57 |
| 4. N. 61° E. | DF 5 | 2.42 | | 4.37 | |
| 5. N. 38° W. | FG 18 | 14.18 | | | 11.08 |
| | | 37.92 | 6.47 | 20.03 | 12.65 |
| | | 6.47 | | 12.65 | |
| N. 13° $12\frac{1}{3}'$ E. | AG 32.3 | 31.45 | | 7.38 | |

The sum of the northings is 37.92. Subtracting from this the southing 6.47, we have the difference of latitude AP 31.45 N.

The sum of the eastings is 20.03. Subtracting from this the sum of the westings 12.65 we have the departure GP 7.38 E. Then (Art. 49.)

Dif. Lat. : Rad. : : Depart : Tan. Course NAG=13° $12\frac{1}{3}'$
Rad. : Dif. Lat. : : Sec. Course : Distance AG=32.3.

| | |
|---|---|
| The latitude of the port is | 32° N. |
| The difference of latitude | 0° 31.45′ N. |
| The latitude of the ship | 32° 31.45′ N. |
| The meridional difference of lat. | 37.5 |

Then by Mercator's sailing,
Rad. : Tan. Course : : Merid. Dif. Lat. : Dif. Lon.=8.8′.

*Example* II.

A ship sailing from a port in Lat. 42° N. makes the following courses and distances.

1. S. 13° E. 21 miles,
2. S. 18° W. 16,
3. N. 84° E. 9,
4. S. 67° E. 12,
5. N. 78° E. 14,
6. S. 12° W. 35.

The difference of latitude, departure, &c. are required.

| | |
|---|---|
| The departure is | 26'.19 E. |
| The diff. of latitude, | 1° 10'$\frac{3}{4}$ S. |
| The diff. of longitude, | 35'.07 |
| The direct course, | S. 20° 18'$\frac{2}{3}$ E. |
| The distance, | 75$\frac{1}{2}$ miles. |

*Accurate method of resolving a traverse.*

78. The preceding method of resolving a traverse is frequently used at sea, because it is simple, and in most cases is sufficiently accurate for a run of 24 hours. But it is founded on the assumption, that when a ship sails from one place to another by *several courses*, she makes the *same departure*, as if she had proceeded by a *single course* to the same place. This is not strictly true. Suppose a vessel, instead of sailing directly from A to C, (Fig. 18.) proceeds by one course from A to H, and then by a different course from H to C. In the compound course, the whole departure, is $bd+g\text{H}+t\text{C}$; (Art. 40.) which, on account of the obliquity of the meridians, is *less* than $om+sn+t\text{C}$, the departure on the single course. If the compound course had been on the other side of the single one, nearer the equator, the departure would have been *greater*.

79. But the *difference of latitude* is the same, whether the ship proceeds from one place to the other, on a single course, or on several. The difference of latitude AB (Fig. 18.)= $\text{A}o+ms+nt=\text{A}b+dg+\text{H}t$. The *difference of longitude* is also the same, whether the course is single or compound. For the difference of longitude is the distance between the meridians of the two places measured on the equator.

*If then the difference of latitude and difference of longitude be calculated for each part of the compound course; the whole difference of latitude and difference of longitude will be found by addition and subtraction;* and from these may be determined the direct course and distance. The difference of longitude for each course may be obtained independently of the departure, by theorem II. of Mercator's sailing.

It will facilitate the calculation of the longitude, to place in the traverse table, the latitudes at the beginning and end of each of the courses, the corresponding meridional parts, and the meridional differences of latitude.

In the following example, the courses and distances are the same as in Art. 76. Ex. 1. The port from which the ship is supposed to sail, is in latitude 32° N.

TRAVERSE TABLE.

| Courses. | Dist. | Diff. Lat. | | Latitudes. | Merid. Parts. | Merid. dif. lat. | Diff. Long. | |
|---|---|---|---|---|---|---|---|---|
| | | N. | S. | 32° | 2028 | | E | W. |
| 1. N. 25° E. | 16 | 14.50 | | 32 14.50 | 2045.5 | 17.5 | 8.16 | |
| 2. S. 54° W. | 11 | | 6.47 | 32 8.03 | 2038 | 7.5 | 10.32 | |
| 3. N. 13° W. | 7 | 6.82 | | 32 14.85 | 2045.8 | 7.8 | | 1.80 |
| 4. N. 61° E. | 5 | 2.42 | | 32 17.27 | 2048.3 | 2.5 | 4.51 | |
| 5. N. 38° W. | 18 | 14.18 | | 32 31.45 | 2065.5 | 17.2 | | 13.44 |
| | | 37.92 | 6.47 | | | | 22.99 | 15.24 |
| | | 6.47 | | | | | 15.24 | |
| 11° 40′37″ | 32.12 | 31.45 | | | | | 7.75 | |

The difference of longitude is here found to be 7.75, and in Art. 77, 8′.8; the error there being 1.05.

To find the direct course and distance from the port to the place of the ship.

Merid. Dif. Lat. : Dif. Lon. : : Rad. : Tan. Course = 11° 40′ 37″
Rad. : Prop. Dif. Lat. : : Sec. Course : Distance = 32.12.

By comparing the results here with those in Art. 77, it will be seen that a small error was introduced there, both into the *course* and the *distance*, by making them dependent on the departure; which being obtained from the several courses, is not the same as for a single course. (Art. 78.)

Ex. 2. A ship sailing from a port in latitude 78° 15′ N. makes the following courses and distances,

1. N. 67° 30′ W. 154 miles.
2. S. 45 W. 96
3. N. 50 37½ W. 89
4. N. 11 15 E. 110
5. N. 36 33¾ W. 56
6. S. 19 41¼ E. 78

Required the difference of latitude, the difference of longitude, and the distance the ship must have sailed, to reach the same place on a single course.

| | |
|---|---|
| The difference of latitude is | 2° 7′ |
| The difference of longitude | 22° 29′ |
| The direct course | N. 63° 1′ W. |
| The distance | 279.9 miles. |

## SECTION V.

### MISCELLANEOUS ARTICLES.

### I. The Plane Chart.

Art. 80. The charts commonly used in navigation are either *Plane charts*, or *Mercator's charts*. The latter are generally to be preferred. But plane charts will answer for short distances, such as the extent of a harbor or small bay.

In the construction of the plane chart, that part of the surface of the globe which is represented on it, is supposed to be a *plane*. The meridians are drawn parallel; and the lines of latitude at equal distances. Islands, coasts, &c. are delineated upon it, by laying down the several parts according to their known latitudes and longitudes.

81. On a chart extending a small distance, each side of the *equator*, the meridians ought to be at the same distance from each other, as the parallels of latitude. A similar construction is frequently applied to different parts of the globe. But this renders the chart much more incorrect than is necessary. A circular island in latitude 60 would, by such a construction, be thrown into a figure whose length from east to west would be twice as great as from north to south; the comparative distance of the meridians being made twice as great as it ought to be. (Art. 63. Trig. 96. cor.)

But when the chart extends only a few degrees, if the distance of the meridians is proportioned to the distance of the parallels of latitude, *as the cosine of the mean latitude to radius;* (Art. 53.) the representation will not be materially incorrect. The meridian distance in the *middle* of the chart will be exact. On one side, it will be a little too great; and on the other, a little too small.

82. *To construct a Plane Chart*, then, on one side of the paper draw a scale of equal parts, which are to be counted as degrees or minutes of latitude, according to the proposed extent of the chart. Through the several divisions, draw the

parallels of latitude, and at right angles to these, draw the meridians in such a manner, that their distance from each other shall be to the distance of the parallels of latitude, as the cosine of the latitude of the middle of the chart, to radius.

After the lines on all the sides are graduated, the positions of the several places which are to be laid down, may be determined, by applying the edge of a rule or strip of paper, to the divisions for the given degree of longitude on each side, and another to the divisions for the degree of latitude. In the intersection of these, will be the point required.

The distance which a ship must sail, in going from one place to another, on a single course, may be nearly found, by applying the measure of the interval between the two places, to the scale of miles of latitude on the side of the chart.*

## II. Construction of Mercator's Chart.

83. In Mercator's chart, the meridians are drawn at equal distances, and the parallels of latitude at unequal distances, proportioned to the meridional differences of latitude. (Arts. 63, 67.) To construct this chart, then, make a scale of equal parts on one side of the paper, for the lowest parallel of latitude which is to be laid down, and divide it into degrees and minutes. Perpendicular to this, and through the dividing points for degrees, draw the lines of longitude. For the second proposed parallel of latitude, find from the table, (Art. 67.) the meridional difference of latitude between that and the parallel first laid down, and take this number of minutes from the scale on the chart, for the interval between the two parallels. In the same manner, find the interval between the second and third parallels, between the third and fourth, &c. till the projection is carried to a sufficient extent.

Places whose latitudes and longitudes are known, may be laid down in the same manner as on the plane chart, by the intersections of the meridians and lines of latitude passing through them.

If the chart is upon a small scale, the least divisions on the graduated lines may be *degrees* instead of minutes; and the meridians and parallels may be drawn for every fifth or every tenth degree. But in this case, it will be necessary to di-

* See Note G.

vide the meridional differences of latitude by 60, to reduce them from minutes to degrees.

84. *The Line of Meridional Parts* on *Gunter's scale* is divided in the same manner as Mercator's Meridian, and corresponds with the *table* of meridional parts; except that the numbers in the latter are *minutes*, while the divisions on the other are *degrees*. Directly beneath the line of meridional parts, is placed a line of *equal* parts. The divisions of the latter being considered as degrees of longitude, the divisions of the former will be degrees of latitude adapted to the same scale. The meridional *difference* of latitude is found, by extending the compasses from one latitude to the other.

A chart may be constructed from the scale, by using the line of equal parts for the degrees of longitude, and the line of meridional parts for the intervals between the parallels of latitude.

85. It is an important property of Mercator's chart, that all the rhumb-lines projected on it are *straight* lines. This renders it, in several respects, more useful to navigators, than even the artificial globe. By Mercator's sailing, theorem II. (Art. 72.)

Merid. Diff. Lat. : Diff. Lon. : : Rad. : Tan. Course.

So that, while the course remains the same, the ratio of the meridional difference of latitude to the difference of longitude is *constant*. If A, C, C′, and C″ (Fig. 26.) be several points in a rhumb-line, AB, AB′, and AB″, the corresponding meridional differences of latitude, and BC, B′C′, B″C″, the differences of longitude; then

$$AB : BC : : AB' : B'C' : : AB'' : B''C''.$$

Therefore ABC, AB′C′ and AB″C″, are similar triangles, and ACC′C″ is a right line. (Euc. 32. 6.)

## III. Oblique Sailing.

86. The application of oblique angled trigonometry to the solution of certain problems in navigation, is called oblique sailing. It is principally used in bays and harbors, to determine the bearings of objects on shore, with their distances from the ship and from each other. A few examples will be sufficient here, in addition to those already given under heights and distances.

One of the cases which most frequently occurs, is that in which the distance of a ship from land is to be determined, when leaving a harbor to proceed to sea. This is necessary, that her difference of latitude and departure may be reckoned from a fixed point, whose latitude and longitude are known.

The distance from land is found, by taking the bearing of an object from the ship, then running a certain distance, and taking the bearing again. The course being observed, there will then be given the angles and one side of a triangle, to find either of the remaining sides.

*Example* I.

The point of land C, (Fig. 27.) is observed to bear N 67° 30′ W. from A. The ship then sails S. 67° 30′ W. 9 miles from A to B; and the direction of the point from B is found to be N. 11° 15′ E. At what distance from land was the ship at A?

Let NS and N′S′ be meridians passing through A and B. Then subtracting CAN and BAS each $67°\frac{1}{2}$ from 180° we have the angle CAB=45°. And subtracting CBN′ $11°\frac{1}{4}$ from BAS or its equal ABN′, we have ABC=$56°\frac{1}{4}$. The angle at C is therefore 78° 45′. And

$$\text{Sin C} : \text{AB} :: \text{Sin B} : \text{AC} = 7.63 \text{ miles.}$$

*Example* II.

New-York light-house on Sandy Point is in Lat. 40° 28′ N. Lon. 74° 8′ W. A ship observes this to bear N. 76° 16′ W., and after sailing S. 35° 10′ W. 8 miles, finds the bearing to be N. 17° 13′ W. Required the latitude and longitude of the ship, at the first observation.

The latitude is 40° $26'\frac{1}{4}$
The longitude 73 $58\frac{1}{2}$

In this example, as the difference of latitude is small, the difference of longitude is best calculated by middle latitude sailing. (Art. 74.)

*Example* III.

A merchant ship sails from a certain port S. 51° E. at the rate of 8 miles an hour. A privateer leaving another port 7 miles N. E. of the first, sails at the rate of 10 miles an hour. What must be the course of the privateer, to meet the ship, without a change of direction in either?

Ans. S. 7° 43′ E.

### *Example* IV.

Two light-houses are observed from a ship sailing S. 38° W. at the rate of 5 miles an hour. The first bears N. 21° W., the other N. 47° W. At the end of two hours, the first is found to bear N. 5° E., the other N. 13° W. What is the distance of the light-houses from each other?

Ans. 6 miles and 30 rods.

## IV. Current Sailing.

87. When the measure given by the log-line is taken as the rate of the ship's progress, the *water* is supposed to be at *rest*. But if there is a tide or current, the log being thrown upon the water, and left at liberty, will move with it, in the same direction, and with the same velocity. The rate of sailing, as measured by the log, is the motion *through the water*.

If the ship is steered in the direction of the current, her whole motion is equal to the rate given by the log, *added* to the rate of the current. But if the ship is steered in opposition to the current, her absolute motion is equal to the *difference* between the current, and the rate given by the log. In all other cases, the current will not only affect the velocity of the ship, but will change its direction.

Suppose that a river runs directly south, and that a boat in crossing it is steered before the wind, from west to east. It will be carried down the stream as fast, as if it were merely floating on the water in a calm. And it will reach the opposite side as soon, as if the surface of the river were at rest. But it will arrive at a different point of the shore.

Let AB (Fig. 28.) be the direction in which the boat is steered, and AD the distance which the stream runs, while the boat is crossing. If DC be parallel to AB, and BC parallel to AD; then will C be the point at which the boat proceeding from A, will strike the opposite shore, and AC will be the distance. For it is driven across by the wind, to the side BC, in the same time that it is carried down by the current, to the line DC.

In the same manner, if A*m* be any *part* of AB, and *mn* be the corresponding progress of the stream, the distance sailed will be A*n*. And if the velocity of the ship and of the stream continue uniform, A*m* is to *mn*, as AB to BC, so that A*n*C is a *straight line.* (Euc. 32. 6.) The lines AB, BC, and AC, form the three sides of a triangle. Hence,

88. If the direction and rate of a ship's motion through the water, be represented by the position and length of one side of a triangle, and the direction and rate of the current, by a second side; the absolute direction and distance will be shown by the third side.

*Example* I.

If the breadth of a river running south (Fig. 28.) be 300 yards, and a boat steers S. 75° E. at the rate of 10 yards in a minute, while the progress of the stream is 24 yards in a minute; what is the actual course, and what distance must the boat go in crossing?

Cos. BAP : AP : : R : AB=310.6

And 10 : 24 : : AB : BC=745.44.

Then in the triangle ABC,

(BC + AB) : (BC − AB) : : Tan. $\frac{1}{2}$ (BAC+BCA) : Tan. $\frac{1}{2}$ (BAC—BCA)=17° 33′ 50″.

The angle BAC is 55° 3′ 50″ Then

Sin. BAC : BC : : Sin. ABC : AC=879 the distance.

And DAC=BCA=19° 56′ 10″ the course.

*Example* II.

A boat moving through the water at the rate of five miles an hour, is endeavoring to make a certain point lying S. 22½° W. while the tide is running S. 78¾° E. three miles an hour. In what direction must the boat be steered, to reach the point by a single course? Ans. S. 58° 33′ W.

89. But the most simple method of making the calculation for the effect of a current, in common cases, especially in resolving a traverse, is to consider the direction and rate of the current as *an additional separate course and distance;* and to find the corresponding departure and difference of latitude. A boat sailing from A (Fig. 28.) by the united action of the wind and current, will arrive at the same point, as if it were first carried by the wind alone from A to B, and then by the current alone from B to C.

*Example* I.

A ship sails S. 17° E. for 2 hours, at the rate of 8 miles an hour; then S. 18° W. for 4 hours, at the rate of 7 miles an hour; and during the whole time, a current sets N. 76° W. at the rate of two miles an hour. Required the direct course and distance.

| | | Dist. | N. | S. | E. | W. |
|---|---|---|---|---|---|---|
| First Course | S. 17° E. | 16 | | 15.3 | 4.68 | |
| Second do. | S. 18° W. | 28 | | 26.6 | | 8.65 |
| Current | N. 76° W. | 12 | 2.9 | | | 11.64 |
| | | | | 41.9 | | 20.29 |
| | | | | 2.9 | | 4.68 |
| | | | *D. Lat.* | 39. | *Dep.* | 15.61 |

The course is 21° 48′ 50″, and the distance 42 miles.

*Example* II.

A ship sails SE. at the rate of 10 miles an hour by the log, in a current setting E. NE. at the rate of 5 miles an hour. What is her true course? and what will be her distance at the end of two hours?

The course is 66° 13′, and the distance 25.56 miles.

## V. Hadley's Quadrant.

90. In the preceding sections, has been particularly explained the process of determining the place of a ship from her course and distance, as given by the compass and the log. But this is subject to so many sources of error, from variable winds, irregular currents, lee-way, uncertainty of the magnetic needle, &c. that it ought not to be depended on, except for short distances, and in circumstances which forbid the use of more unerring methods. The mariner who hopes to cross the ocean with safety, must place his chief reliance, for a knowledge of his true situation from time to time, on observations of the *heavenly bodies*. By these the latitude and longitude may be generally ascertained, with a sufficient degree of exactness. It belongs to astronomy to explain the methods of making the calculations. The subject will not be anticipated in this place, any farther than to give a description of the quadrant of reflection, commonly called *Hadley's Quadrant*,* by which the altitudes of the heavenly bo-

* See Note H.

dies, and their distances from each other, are usually measured at sea. The superiority of this, over most other astronomical instruments, for the purposes of navigation, is owing to the fact, that the observations which are made with it, *are not materially affected by the motion of the vessel.*

91. In explaining the construction and use of this quadrant, it will be necessary to take for granted the following simple principles of Optics.

1. The progress of light, when it is not obstructed, or turned from its natural course by the influence of some contiguous body, is in *right lines.* Hence a minute portion of light, called a ray, may be properly represented by a line.

2. Any object appears in the direction in which the light from that object *strikes the eye.* If the light is not made to deviate from a right line, the object appears in the direction in which it really is. But if the light is reflected, as by a common mirror, the object appears not in its true situation, but in the direction of the glass, from which the light comes to the eye.

3. *The angle of reflection is equal to the angle of incidence;* that is, the angle which the reflected and the incident rays make with the surface of the mirror, are equal; as are also the angles which they make with a perpendicular to the mirror.

92. From these principles is derived the following proposition; *When light is reflected by two mirrors successively, the angle which the last reflected ray makes with the incident ray, is* DOUBLE *the angle between the mirrors.*

If C and D (Fig. 29.) be the two mirrors, a ray of light coming from A to C, will be reflected so as to make the angle DCM=ACB; and will be again reflected at D, making HDM=CDE. Continue BC and ED to H, draw DG parallel to BH, and continue AC to P. Then is CPM the angle which the last reflected ray DP makes with the incident ray AC; and DHM is the angle between the mirrors.

By the preceding article, with Euc. 29. 1, and 15. 1,

$$GDC=DCM=ACB=PCM$$

And $HDM=EDC=EDG+GDC=DHM+PCM$.

But by Euc. 32. 1 and 15. 1,

$$CPM+PCM=DHM+HDM=2DHM+PCM$$

Therefore $CPM=2DHM$.

Cor. 1. If the two mirrors make an angle of a certain number of degrees, the *apparent direction* of the object will be changed twice as many degrees. The object at A, seen by the eye at P, without any mirror, would appear in the direction PA. But after reflection from the two mirrors, the light comes to the eye in the direction DP, and the apparent place of the object is changed from A to R.

Cor. 2. If the two mirrors be *parallel*, they will make no alteration in the apparent place of the object.

93. The principal parts of Hadley's quadrant are the following;

1. A *graduated arc* AB (Fig. 17.) connected with the radii AC and BC.

2. An *index* CD, one end of which is fixed at the center, C, while the other end moves over the graduated arc.

3. A plane mirror called the *index glass*, attached to the index at C. Its plane passes through the center of motion C, and is perpendicular to the plane of the instrument; that is, to the plane which passes through the graduated arc, and its center C.

4. Two other plane mirrors at E and M, called *horizon glasses*. Each of these is also perpendicular to the plane of the instrument. The one at E, called the *fore horizon glass*, is placed parallel to the index glass when the index is at 0. The other called the *back horizon glass*, is perpendicular to the first and to the index at 0. This is only used occasionally, when circumstances render it difficult to take a good observation with the other.

A *part* of each of these glasses is covered with quicksilver, so as to act as a mirror; while another part is left transparent, through which objects may be seen in their true situation.

5. Two *sight vanes* at G and L, standing perpendicular to the plane of the instrument. At one of these, the eye is placed to view the object, by looking on the opposite horizon glass. In the fore sight vane at G, there are two perforations, one directly opposite the transparent part of the fore horizon glass, the other opposite the silvered part. The back sight vane at L has only one perforation, which is opposite the center of the transparent part of the back horizon glass.

6. *Colored glasses* to prevent the eye from being injured by the dazzling light of the sun. These are placed at H, be-

tween the index mirror and the fore horizon glass. They may be taken out when necessary, and placed at X between the index mirror and the back horizon glass.

94. This instrument which is in form an *octant*, is called a *quadrant*, because the graduation extends to 90 degrees, although the arc on which these degrees are marked is only the eighth part of a circle. The light coming from the object is first reflected by the index glass C. (Fig. 17.) and thrown upon the horizon glass E, by which it is reflected to the eye at G. If the index be brought to 0, so as to make the index glass and the horizon glass *parallel*; the object will appear in its true situation. (Art. 92. Cor. 2.) But if the index glass be turned, so as to make with the horizon glass an angle of a certain number of degrees; the apparent direction of the object will be changed *twice as many degrees.*

Now the graduation is adapted to the apparent change in the situation of the object, and not to the motion of the index. If the index move over 45 degrees, it will alter the apparent place of the object 90 degrees. The arc is commonly graduated a short distance on the other side of 0 towards P. This part is called the *arc of excess.*

95. The quadrant is used at sea, to measure the angular distances of the heavenly bodies from each other, and their elevations above the horizon. One of the objects is seen in its true situation, by looking through the transparent part of the horizon glass. The other is seen by reflection, by looking on the silvered part of the same glass. By turning the index, the apparent place of the latter may be changed, till it is brought in contact with the other. The motion of the index which is necessary to produce this change, determines the distance of the two objects.*

96. *To find the distance of the moon from a star.*—Hold the quadrant so that its plane shall pass through the two objects. Look at the star through the transparent part of the horizon glass, and then turn the index till the nearest edge of the image of the moon is brought in contact with the star. This will measure the distance between the star and one *edge* of the moon. By adding the semi-diameter of the moon, we shall have the distance of its *center* from the star.

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* For the *adjustments* of the quadrant, see Vince's Practical Astronomy, Mackay's Navigation, or Bowditch's Practical Navigator.

The distance of the sun from the moon, or the distance of two stars from each other, may be measured in a similar manner.

97. *To measure the altitude of the sun above the horizon.*—Hold the instrument so that its plane shall pass through the sun, and be perpendicular to the horizon. Then move the index till the lower edge of the image of the sun is brought in contact with the horizon, as seen through the transparent part of the glass.

The altitude of any other heavenly body may be taken in the same manner.

98. To measure altitudes by the *back observation.*—When the index stands at 0, the index glass is at right angles with the back horizon glass. (Art. 93.) The apparent place of the object, as seen by reflection from this glass, must therefore be changed 180 degrees; (Art. 92. Cor. 1.) that is, it must appear in the opposite point of the heavens. In taking altitudes by the back observation, if the object is in the east, the observer faces the west; or if it be in the south, he faces the north; and moves the index, till the image formed by reflection is brought down to the horizon.

This method is resorted to, when the view of the horizon in the direction of the object is obstructed by fog, hills, &c.

99. *Dip or Depression of the Horizon.*—In taking the altitude of a heavenly body at sea, with Hadley's Quadrant, the reflected image of the object is made to coincide with the most distant visible part of the *surface of the ocean.* A plane passing through the eye of the observer, and thus touching the ocean, is called the *marine horizon* of the place of observation. If BAB′ (Fig. 13.) be the surface of the ocean, and the observation be made at T, the marine horizon is TA. But this is different from the *true horizon* at T, because the eye is elevated above the surface. Considering the earth as a sphere, of which C is the center, the true horizon is TH perpendicular to TC. The marine horizon TA falls *below* this. The angle ATH is called the *dip* or *depression* of the horizon. This varies with the height of the eye above the surface. Allowance must be made for it, in observations for determining the altitude of a heavenly body above the true horizon.

In the right angled triangle ATC, the angle ACT is equal to the angle of depression ATH; for each is the complement of ATC. The side AC is the semi-diameter of the earth, and the hypothenuse CT is equal to the same semi-diameter added to BT the height of the eye. Then

AC : R : : TC : Sec. ACT=ATH the depression.*

100. *Artificial Horizon.*—Hadley's Quadrant is particulaly adapted to measuring altitudes *at sea.* But it may be made to answer the same purpose on land, by means of what is called an artificial horizon. This is the level surface of some fluid which can be kept perfectly smooth. Water will answer, if it can be protected from the action of the wind, by a covering of thin glass or talc which will not sensibly change the direction of the rays of light. But quicksilver, Barbadoes tar, or clear molasses, will not be so liable to be disturbed by the wind. A small vessel containing one of these substances, is placed in such a situation that the object whose altitude is to be taken may be reflected from the surface. As this surface is in the plane of the horizon, and as the angles of incidence and reflection are equal, (Art. 91.) the image seen in the fluid must appear as far *below* the horizon, as the object is *above.* The distance of the two will, therefore, be *double* the altitude of the latter. This distance may be measured with the quadrant, by turning the index so as to bring the image formed by the instrument to coincide with that formed by the artificial horizon.

101. *The Sextant* is a more perfect instrument than the quadrant, though constructed upon the same principle. Its arc is the sixth part of a circle, and is graduated to 120 degrees. In the place of the sight vane, there is a small telescope for viewing the image. There is also a magnifying glass, for reading off the degrees and minutes. It is commonly made with more exactness than the quadrant, and is better fitted for nice observations, particularly for determining longitude, by the angular distances of the heavenly bodies.

A still more accurate instrument for the purpose is the *Circle of Reflection.* For a description of this, see Borda on the Circle of Reflection, Rees' Cyclopedia, and Bowditch's Practical Navigator.

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* See Note I, and Table II.

# SURVEYING.

## SECTION I.

### SURVEYING A FIELD BY MEASURING ROUND IT.

Art. 105. The most common method of surveying a field is to measure the length of each of the sides, and the angles which they make with the meridian. The lines are usually measured with a chain, and the angles with a compass.

106. *The Compass.*—The essential parts of a Surveyor's Compass are a graduated circle, a magnetic needle, and sight holes for taking the direction of any object. There are frequently added a spirit level, a small telescope, and other appendages. The instrument is called a Theodolite, Circumferentor, &c. according to the particular construction, and the uses to which it is applied.

For measuring the angles which the sides of a field make with each other, a graduated circle with sights would be sufficient. But a needle is commonly used for determining the position of the several lines with respect to the meridian. This is important in running boundaries, drawing deeds, &c. It is true, the needle does not often point directly north or south. But allowance may be made for the variation, when this has been determined by observation. See Sec. V.

107. *The Chain.*—The Surveyor's or Gunter's chain is four rods long, and is divided into 100 *links*. Sometimes a half chain is used, containing 50 links. A rod, pole, or perch, is $16\frac{1}{2}$ feet. Hence

1 Link $=7.92$ inches $=\frac{2}{3}$ of a foot nearly.
1 Rod $=$ 25 links $=16\frac{1}{2}$ feet.
1 Chain $=100$ links $=66$ feet.

108. The measuring unit for the *area* of a field is the *acre*, which contains 160 square rods. If then the contents in square rods be divided by 160, the quotient will be the number of acres. But it is commonly most convenient to make the computation for the area in square *chains or links*, which are decimals of an acre. For a square chain $=4\times4=16$ square rods, which is the tenth part of an acre. And a square link$=\frac{1}{100}\times\frac{1}{100}=\frac{1}{10000}$ of a square chain$=\frac{1}{100000}$ of an acre. Or thus,

| | | |
|---|---|---|
| 625 links, or | $272\frac{1}{4}$ feet | =1 square rod, |
| 10000 | 4356 | =1 chain or 16 rods, |
| 25000 | 10890 | =1 rood or 40 rods, |
| 100000 | 43560 | =1 acre or 160 rods. |

109. The contents, then, being calculated in chains and links; if *four* places of decimals be cut off, the remaining figures will be square *chains;* or if *five* places be cut off, the remaining figures will be *acres*. Thus the square of 16.32 chains, or 1632 links, is 2663424 square links, or 266.3424 square chains, or 26.63424 acres. If the contents be considered as square chains and *decimals*, removing the decimal point one place to the left will give the acres.

110. In surveying a piece of land, and calculating its contents, it is necessary, in all common cases, to suppose it to be reduced to a *horizontal level.* If a hill or any uneven piece of ground, is bought and sold; the quantity is computed, not from the irregular surface, but from the *level base* on which the whole may be considered as resting. In running the lines, therefore, it is necessary to reduce them to a level. Unless this is done, a correct plan of the survey can never be exhibited on paper.

If a line be measured upon an ascent which is a regular *plane*, though oblique to the horizon; the length of the corresponding level base may be found, by taking the angle of elevation.

Let AB (Fig. 30.) be parallel to the horizon, BC perpendicular to AB, and AC a line measured on the side of a hill. Then, the angle of elevation at A being taken with a quadrant, (Art. 4.)

$$R : \text{Cos. } A :: AC : AB, \text{ that is,}$$

*As radius, to the cosine of the angle of elevation;*
*So is the oblique line measured, to the corresponding horizontal base.*

If the chain, instead of being carried parallel to the surface of the ground, be kept constantly parallel to the *horizon;* the line thus measured will be the base line required. The line AB (Fig. 30.) is evidently equal to the sum of the parallel lines *ab*, *cd*, and *e*C.

## PLOTTING A SURVEY.

111. When the sides of a field are measured, and their bearings taken, it is easy to lay down a plan of it on paper. A north and south line is drawn, and with a line of chords, a protractor, or a sector, an angle is laid off, equal to the angle which the first side of the field makes with the meridian, and the length of the side is taken from a scale of equal parts. (Trig. 156—161.) Through the extremity of this, a second meridian is drawn parallel to the first, and another side is laid down; from the end of this, a third side, &c. till the plan is completed. Or the plot may be constructed in the same manner as a *traverse* in navigation. (Art. 76.) If the field is correctly surveyed and plotted, it is evident the extremity of the last side must coincide with the beginning of the first.

### *Example* I.

Draw a plan of a field, from the following courses and distances, as noted in the field book;

| | | Ch. | Links. |
|---|---|---|---|
| 1. | N. 78° E. | 2 | 46 |
| 2. | S. 16° W. | 3 | 54 |
| 3. | N. 83° W. | 2 | 72 |
| 4. | N. 12° E. | 2 | 13 |
| 5. | N. $60\frac{1}{2}$°E. | 0 | 95 |

Let A (Fig. 31.) be the first corner of the field.

Thro' A, draw the merid. NS, make BAN=78°, & AB=2.46
Thro' B, draw N'S' par. to NS, make S'BC=16°, & BC=3.54
Thro' C, draw N''S'' par. to NS, make DCN''=83°, & CD=2.72
&c. &c.

112. To avoid the inconvenience of drawing parallel lines, the sides of a field may be laid down *from the angles which they make with each other*, instead of the angles which they make with the meridian. The position of the line BC (Fig. 31.) is determined by the angle ABC, as well as by the angle S'BC. When the several courses are given, the angles

which any two contiguous sides make with each other, may be known by the following rules.

1. If one course is North and the other South, one East and the other West; *subtract the less from the greater.*

2. If one is North and the other South, but both East or West; *add them together.*

3. If both are North or South, but one East and the other West; *subtract their sum from* 180 *degrees.*

4. If both are North or South, and both East or West; *add together* 90 *degrees, the less course, and the complement of the greater.*

The reason of these rules will be evident by applying them to the preceding example. (Fig. 31.)

The first course is BAN, which is equal to ABS′. (Euc. 29. 1.) If from this the second course CBS′ be subtracted, there will remain the angle ABC.

If the second course CBS′, or its equal BCN″, be added to the third course DCN″; the sum will be the angle BCD.

The sum of the angles CDS, NDE, and CDE, is 180 degrees. (Euc. 13. 1.) If then the two first be subtracted from 180 degrees, the remainder will be the angle CDE.

Lastly, let EP be perpendicular to NS. Then the sum of the angles DES, PES, and AEP the complement of AEN, is equal to the angle DEA.

We have then the angle $ABC=62°$, $DEA=131\frac{1}{2}°$,
$BCD=99°$, $EAB=162\frac{1}{2}°$.
$CDE=85°$,

With these angles, the field may be plotted without drawing parallels, as in Trig. 173.

## FINDING THE CONTENTS OF A FIELD.

113. There are in common use two methods of finding the contents of a piece of land, one by dividing the plot into *triangles*, the other by calculating the *departure and difference of latitude* for each of the sides.

When a survey is plotted, the whole figure may be divided into triangles, by drawing diagonals from the different angles. The lengths of the diagonals, and of the perpendiculars on the bases of the triangles, may be measured on the same scale of equal parts from which the sides of the field were laid down. The area of each of the triangles is equal to half the product of its base and perpendicular; and their sum is the area of the whole figure. (Mens. 13.)

*Example* I.

Let the plan Fig. 32 be the same as Fig. 31, the sides of which with their bearings, are given in Art. 111.

Then the triangle $ABC=BC\times\frac{1}{2}AP=3.84$ sq. chains.
$ACE=AC\times\frac{1}{2}EP'=1.53$
$DCE=EC\times\frac{1}{2}DP''=2.89$

The contents of the whole=8.26.

114. This method cannot be relied on, where great accuracy is required, if the lines are measured by a scale and compasses only. But the parts of the several triangles may be found by *trigonometrical calculation*, independently of the projection; and then the area of each may be computed, either from two sides and the included angle, or from the three sides. (Mens. 9, 10.)

The sides of the field and their bearings being given by the survey, the angles of the original figure may all be known. (Art. 112.) Then in the triangle ABC (Fig. 32.) we have the side AB and BC, with the angle ABC, to find the other parts. (Trig. 153.) And in the triangle CDE, we have the sides DC and DE, with the angle CDE. Subtracting the angle BAC from BAE, we shall have CAE; and subtracting DEC from DEA, we shall have CEA. There will then be given in the triangle ACE, the side EA and the angles. (Trig. 150.)

The sides and bearings, as given in Art. 111, are

| | | | |
|---|---|---|---|
| 1. AB | N. 78° E. | 2.46 | chains. |
| 2. BC | S. 16 W. | 3.54 | |
| 3. CD | N. 83 W. | 2.72 | |
| 4. DE | N. 12 E. | 2.13 | |
| 5. EA | N. 60½E. | 0.95 | |

Then by Mensuration, Art. 9,

R : Sin. ABC : : AB×BC : 2 *area* ABC=7.69 sq. chains.
R : Sin. AEC : : AE×EC : 2 *area* AEC=3.06
R : Sin. CDE : : CD×DE : 2 *area* CDE=5.77

2)16.52

Contents of the whole field, 8.26.

Or the areas of the several triangles may be found by the rule in Mensuration, Art. 10; viz. If $a$, $b$, and $c$, be the sides of any triangle, and $h$=half their sum;

$$\textit{The area} = \sqrt{h \times (h-a) \times (h-b) \times (h-c)}.$$

*Example* II.

| Courses. | Ch. | Links. |
|---|---|---|
| 1. E. | 26 | 34 |
| 2. S. 10° 30′ E. | 32 | 26 |
| 3. N. 42 W. | 18 | 35 |
| 4. S. 58 W. | 23 | 52 |
| 5. N. | 30 | 55 |

Contents of the field, 69.735 acres.

The method which has been explained, of ascertaining the contents of a piece of land by dividing it into triangles, is of use in cases which do not require a greater degree of accuracy, than can be obtained by the scale and compasses. But if the areas of the triangles are to be found by trigonometrical calculation, the process becomes too laborious for common practice. The following method is often to be preferred.

FINDING THE AREA OF A FIELD BY DEPARTURE AND DIFFERENCE OF LATITUDE.

115. Let ABCDE (Fig. 33.) be the boundary of a field. At a given distance from A, draw the meridian line NS. Parallel to this draw L′R′, AG, BH, and DK. These may be considered as portions of meridians passing through the points A, B, D, and E. For all the meridians which cross a field of moderate dimensions, may be supposed to be *parallel*, without sensible error. At right angles to NS draw the parallels AL, BM, CO, EP, and DR. These will divide the figure LABCDR into the three trapezoids ABML, BCOM, and CDRO; and the figure LAEDR, into the two trapezoids DEPR and EALP. The area of the field is evidently equal to the difference between these two figures.

The sum of the parallel sides of a trapezoid, multiplied into their distance, is equal to twice the area. (Mens. 12.) Thus

$$(AL+BM) \times AG = 2\,\textit{area}\ ABML.$$

Now AL is a given distance, and BM=AL+BG. But BG is the *departure*, and AG the *difference of latitude*, corresponding to AB one of the sides of the field. (Arts. 39. 40.) And by Art. 44,

Rad. : Dist. AB : : { Sin. BAG : Depart. BG
Cos. BAG : Diff. Lat. AG.

Or the departure and difference of latitude may be taken from the *Traverse table*, as in Navigation. (Art. 50.)

In the same manner, from the sides BC, CD, DE, and EA, may be found the departures CH, CK, DR′, AL′, and the differences of latitude BH, DK, ER′, and EL′. We shall then have the parallel sides of each of the trapezoids, or the distances of the several corners of the field from the meridian NS. For

$$BM=AL+BG, \qquad DR=CO-CK,$$
$$CO=BM+CH, \qquad EP=DR-DR'.$$

If the field be measured in the direction ABCDE, the differences of latitude AG, BH, and DK, will be *Southings*, while R′E and EL′ will be *Northings*. The former are the breadths of the three trapezoids which form the figure LABCDR; and the latter are the breadths of the two trapezoids which form the figure LAEDR. The difference, then, between the sum of the products of the northings into the corresponding meridian distances, and the sum of the products of the southings into the corresponding meridian distances, is twice the area of the field.

It will very much facilitate the calculation, to place in a *table* the several courses, distances, northings, southings, &c. We have, then, the following

RULE.

116. *Find the northing or southing, and the easting or westing, for each side of the field, and place them in distinct columns in a table. To these add a column of Meridian Distances, for the distance of one end of each side of the field from a given meridian; a column of Multipliers, to contain the pairs of meridian distances for the two ends of each of the sides; and columns for the north and south Areas.* See Fig. 33, and the table for example 1.

*Suppose a meridian line to be drawn without the field, at any given distance from the first station; and place the assumed distance at the head of the column of Meridian Distances. To this add the first departure, if both be east or both west; but subtract, if one be east and the other west; and place the sum* *or difference in the column of Meridian Distances, against the first course. To or from the last number, add or subtract the second departure, &c. &c.*

*For the column of Multipliers, add together the first and second numbers in the column of Meridian Distances; the second and third, the third and fourth, &c. placing the sums opposite the several courses.*

*Multiply each number in the column of Multipliers into its corresponding northing or southing, and place the product in the column of north or south areas. The difference between the sum of the north areas, and the sum of the south areas, will be twice the area of the field.*

This method of finding the contents of a field, as it depends on departure and difference of latitude, which are calculated by right-angled trigonometry, is sometimes called *Rectangular Surveying.*

117. If the assumed meridian pass through the eastern or western extremity of the field, as L′ER′ (Fig. 33.) the distance EP will be reduced to nothing, and the figures AEL′ and EDR′ will be *triangles* instead of trapezoids. If the survey be made to begin at the point E, *cipher* is to be placed at the head of the column of meridian distances, and the first number in the column of multipliers will be the same, as the first in the column of meridian distances. See example II.

118. When there is a *re-entering angle* in a field, situated with respect to the meridian as CDE; (Fig. 34.) the area EDM, being included in the figure BCRA, will be *repeated* in the column of south areas. But, as it is also included in the figure DCRM, it will be contained in the column of north areas. Therefore the *difference* between the north areas and the south areas, will be twice the area of the field, in this case, as well as in others.

119. If any side is directly *east or west*, there will be no difference of latitude, and consequently no number to be placed against this course, in the columns of north and south areas. See example II, Course 1. AB. (Fig. 34.)

The number in the columns of areas will be wanting also, when any side of the field coincides with the assumed meridian. See example II, Course 5. EA. (Fig. 34.)

120. In finding the departure and difference of latitude from the traverse table, the numbers for the *links* may be looked out separately; care being taken to remove the decimal point two places to the left, because a link is the 100th part of a chain.

Thus if the course be 29°, and the distance 23.46 chains;

| | | | |
|---|---|---|---|
| The diff. of lat. & depart. | for 23 chains are | 20.12 and | 11.15 |
| | for 46 links | .40 | .22 |
| | for 23.46 | 20.52 | 11.37 |

*Example* I. See Fig. 33.

| Courses. | Dist. | Diff. Lat. N. | Diff. Lat. S. | Departure. E. | Departure. W. | M. D. A. L. 20 E. | Mult. | N. Areas. | S. Areas. |
|---|---|---|---|---|---|---|---|---|---|
| 1. BAG s. 64° E. | AB 30ch. | | AG 13.15 | GB 26.96 | | BM 46.96 | AL+BM 66.96 | | 2 ABML 880.5240 |
| 2. CBH s. 14° E. | BC 10 | | BH 9.70 | HC 2.42 | | CO 49.38 | BM+CO 96.34 | | 2 BMOC 984.4980 |
| 3. CDK s. 35° W. | CD 30 | | KD 24.57 | | CK 17.21 | DR 32.17 | CO+DR 81.55 | | 2 CDRO 2008.6835 |
| 4. KDE N. 65° W. | DE 20 | R′E 8.45 | | | DR′ 18.13 | EP 14.04 | DR+EP 46.21 | 2 DEPR 390.4745 | |
| 5. L′EA N. 8° 42′ E. | EA 39.42 | EL′ 38.97 | | L′A 5.96 | | AL 20. | EP+AL 34.04 | 2 EPLA 1326.5388 | |
| | | 47.42 | 47.42 | 35.34 | 35.34 | | | 1717.0133 | 3818.7055 |

Twice the figure ABCDRL is 3818.7055 square chains;
Twice the figure AEDRL 1717.0133

The difference 2101.6922

The contents of the field 1050.8461 sq. ch. or 105.0846 acres. (Art. 109.)

### *Example* II. See Fig. 34.

| Courses. | Dist. | Diff. Lat. N. | Diff. Lat. S. | Departure. E. | Departure. W. | M. Dist. 00 | Mult. | N. Areas | S. Areas. |
|---|---|---|---|---|---|---|---|---|---|
| 1. E. | AB 26.34 | 00 | 00 | 2[illegible] | | AB 26.34 | 00 | 00 | 00 |
| 2. S. 10½° E. | BC 32.26 | | 31.72 | 5.88 | | CR 32.22 | AB+CR 58.56 | | 2 ABCR 1857.5232 |
| 3. N. 42° W. | CD 18.35 | 13.64 | | | 12.27 | DM 19.95 | CR DM 52.17 | 2 CDMR 711.5988 | |
| 4. S. 58° W. | DE 23.52 | | 12.47 | | 19.95 | 00 | DM 19.95 | | 2 DME 248.7765 |
| 5. N. | EA 30.55 | 30.55 | | 00 | 00 | 00 | 00 | | |
| | | 44.19 | 44.19 | | | | | 711.5988 | 2106.2997 |

The conteuts of the field$=\frac{1}{2}(2106.3-711.6)=697.35$ sq. ch.
Or 69.735 acres.

In this example, the meridian distance of the first station A being nothing, *cipher* is placed at the head of the column of meridian distances. (Art. 117.) The first side AB being directly east and west, has no difference of latitude, and therefore the number in the column of areas against this course is wanting, as it is against the fifth course, which is directly north. (Art. 119.) The number against the fourth course, in the column of multipliers, is only the length of the line DM; the figure DME being a *triangle*, instead of a trapezoid.

### *Example* III.

Find the contents of a field bounded by the following lines;

| | | | | | | |
|---|---|---|---|---|---|---|
| 1. | N. | 35° | 30′ | E. | 15 ch. | 50 links. |
| 2. | N. | 72 | 45 | E. | 18 | 70 |
| 3. | S. | 70 | 45 | E. | 18 | 70 |
| 4. | S. | 53 | | W. | 12 | 45 |
| 5. | S. | 83 | 15 | E. | 24 | 10 |
| 6. | S. | 31 | 15 | W. | 15 | 20 |
| 7. | S. | 62 | 45 | W. | 22 | 60 |
| 8. | N. | 73 | 30 | W. | 27 | 30 |
| 9. | N. | 17 | 45 | W. | 14 | 56 |

The area is 145½ acres.

121. When a field is correctly surveyed, and the departures and differences of latitude accurately calculated; it is evident the sum of the northings must be equal to the sum of the southings, and the sum of the eastings equal to the sum of the westings. If upon adding up the numbers in the departure and latitude columns, the northings are not found to agree nearly with the southings, and the eastings with the westings, there must be an error, either in the survey or in the calculation, which requires that one or both should be revised. But if the difference be small, and if there be no particular reason for supposing it to be occasioned by one part of the survey rather than another; it may be apportioned among the several departures or differences of latitude, according to the different lengths of the sides of the field, by the following rule;

As the whole perimeter of the field,
To the whole error in departure or latitude;
So is the length of one of the sides,
To the correction in the corresponding departure or latitude.

This correction, if applied to the column in which the sum of the numbers is too small, is to be *added;* but if to the other column, it is to be *subtracted.** See the example on the next page.

* See the fourth Number of the Analyst, published at Philadelphia.

## *Example* IV.

| Courses. | Dist. Chains. | Diff. Lat. N. | Diff. Lat. S. | Departure. E. | Departure. W. | Cor. Lat. | Cor. Dep. | Cor. Diff. Lat. N. | Cor. Diff. Lat. S. | Cor. Dep. E. | Cor. Dep. W. | Mer. Dist. 00 | Mult. | N. Areas. | S. Areas. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1. N. 55¼° E. | 18 | 10.26 | | 14.79 | | +.06 | +.08 | 10.32 | | 14.87 | | 14.87 | 14.87 | 153.46 | |
| 2. S. 62½° E. | 14½ | | 6.70 | 12.87 | | −.05 | +.07 | | 6.65 | 12.94 | | 27.81 | 42.68 | | 283.82 |
| 3. S. 40° W. | 11 | | 8.43 | | 7.07 | −.04 | −.04 | | 8.39 | | 7.03 | 20.78 | 48.59 | | 407.67 |
| 4. S. 4¼ E. | 14 | | 13.96 | 1.04 | | −.05 | +.06 | | 13.91 | 1.10 | | 21.88 | 42.66 | | 593.40 |
| 5. N. 73¾° W. | 12½ | 3.50 | | | 12.00 | +.04 | −.05 | 3.54 | | | 11.95 | 9.93 | 31.81 | 112.61 | |
| 6. S. 52° W. | 9½ | | 5.85 | | 7.49 | −.03 | −.04 | | 5.82 | | 7.45 | 2.48 | 12.41 | | 72.23 |
| 7. N. 7° W. | 21 | 20.84 | | | 2.56 | +.07 | −.08 | 20.91 | | | 2.48 | 00 | 2.48 | 51.85 | |
| Perimeter | 100½ | 34.60 | 34.94 | 28.70 | 29.12 | | | 34.77 | 34.77 | 28.91 | 28.91 | | | 317.92 | 1357.12 |
| | | | 34.60 | | 28.70 | | | | | | | | | | 317.92 |
| | | Error .34 | | Error .42 | | | | | | | | | | Double area | 1039.20 |

In this example the whole perimeter of the field is $100\frac{1}{2}$ chains, the whole error in latitude .34, the whole error in departure .42, and the length of the first side 18. To find the corresponding errors,

$$100\tfrac{1}{2} : 18 :: \begin{cases} .34 : .06 \text{ the error in latitude,} \\ .42 : .08 \text{ the error in departure.} \end{cases}$$

The error in latitude is to be added to 10.26 making it 10.32, as in the column of corrected northings; and the error in departure is to be added to 14.79 making it 14.87, as in the column of corrected eastings. After the corrections are made for each of the courses, the remaining part of the calculation is the same as in the preceding examples.

122. If the length and direction of each of the sides of a field *except one* be given, the remaining side may be easily found by calculation. For the difference between the sum of the northings and the sum of the southings of the given sides, is evidently equal to the northing or southing of the remaining side; and the difference between the sum of the eastings and the sum of the westings of the given sides, is equal to the easting or westing of the remaining side. Having then the difference of latitude and departure for the side required, its length and direction may be found, in the same manner as in the sixth case of plane sailing. (Art. 49.)

### *Example* V.

What is the area of a field of six sides, of which five are given, viz.

1. S. 56° E. 4.18 chains.
2. N. 21 E. 4.80
3. N. 56 W. 3.06
4. S. 21 W. 0.13
5. N. $66\frac{1}{2}$ W. 1.44
6. ——— ———

The area is two acres.

### *Example* VI.

1. N. 38° W. 17.21 chains.
2. N. 13 E. 21.16
3. N. 72 E. 24.11
4. S. 41 E. 19.26
5. S. 11 W. 24.35
6. ——— ———

123. *Plotting by departure and difference of latitude.*—A survey may be easily plotted from the northings and southings, eastings and westings. For this purpose, the column of *Meridian Distances* is used. It will be convenient to add also another column, containing the distance of each station from a given *parallel of latitude*, and formed by adding the northings and subtracting the southings, or adding the southings and subtracting the northings.

Let AT (Fig. 33.) be a parallel of latitude passing through the first station of the field. Then the southing TB or LM is the distance of B, the second station, from the given parallel. To this adding the southing BH, we have LO the distance of CO from LT. Proceeding in this manner for each of the sides of the field, and copying the 7th column in the table, p. 65, we have the following differences of latitude and meridian distances.

| *Diff. Lat.* | *Merid. Dist.* |
|---|---|
| | AL 20. |
| 1. LM 13.15 | BM 46.96 |
| 2. LO 22.85 | CO 49.38 |
| 3. LR 47.42 | DR 32.17 |
| 4. LP 38.97 | EP 14.04 |

To plot the field, draw the meridian NS, and perpendicular to this, the parallel of latitude LT. From L set off the differences of latitude LM, LO, LR, and LP. Through L, M, O, R, and P, draw lines parallel to LT; and set off the meridian distances AL, BM, CO, DR, and EP. The points A, B, C, D, and E, will then be given.

124. When a field is a *regular figure*, as a parallelogram, triangle, circle, &c. the contents may be found by the rules in Mensuration, Sec. I, and II.

125. The area of a field which has been plotted, is sometimes found by *reducing the whole to a* TRIANGLE *of the same area.* This is done by changing the figure in such a manner as, at each step, to make the number of sides one less, till they are reduced to three.

Let the side AB (Fig. 35.) be extended indefinitely both ways. To reduce the two sides BC and CD to one, draw a line from D to B, and another parallel to this from C, to intersect AB continued. Draw also a line from D to the point of intersection G. Then the triangles DBC and DBG are equal. (Euc. 37. 1.) Taking from each the common part DBH, there remains BGH equal to DCH. If then the triangle DCH be thrown out of the plot, and BGH be added, we shall have the five-sided figure AGDEF equal to the six-sided figure ABCDEF.

In the same manner, the line EL may be substituted for the two sides AF and EF; and then DM, for EL and ED. This will reduce the whole to the triangle MGD, which is equal to the original figure. The area of the triangle may then be found by multiplying its base into half its height; and this will be the contents of the field.

In practice, it will not be necessary actually to draw the parallel lines BD, GC, &c. It will be sufficient to lay the edge of a rule on C, so as to be parallel to a line supposed to pass through B and D, and to mark the point of intersection G.

126. If after a field has been surveyed, and the area computed, the chain is found to be *too long* or *too short*; the true contents may be found, upon the principle that similar figures are to each other as the squares of their homologous sides. (Euc. 20. 6.) The proportion may be stated thus;

> As the square of the true chain, to the square of that by which the survey was made;
> So is the computed area of the field, to the true area.

Ex. If the area of a field measured by a chain 66.4 feet long, be computed to be 32.6036 acres; what is the area as measured by the true chain 66 feet long?

Ans. 33 acres.

127. A plot of a field may be changed to a *different scale*, that is, it may be enlarged or diminished in any given ratio, by drawing lines parallel to each of the sides of the original plan.

To enlarge the perimeter of the figure ABCDE (Fig. 36.) in the ratio of *a*G to AG; draw lines from G through each of the angular points. Then beginning at *a*, draw *ab* parallel to AB, *bc* parallel to BC, &c.

It is evident that the *angles* are the same in the enlarged figure, as in the original one. And by similar triangles,

AG : *a*G : : BG : *b*G : : CG : *c*G : : &c.

And

AG : *a*G : : AB : *ab* : : BC : *bc* : : &c.

Therefore ABCDE and *abcde* are similar figures. (Euc. Def. 1. 6.)

In the same manner, the smaller figure *a'b'c'd'e'* may be drawn, so as to have its perimeter proportioned to ABCDE as *a'*G to AG.

# SECTION II.

## METHODS OF SURVEYING IN PARTICULAR CASES.

Art. 128. Measuring round a field, in the manner explained in the preceding section, is by far the most common method of surveying. The following problems are sometimes useful. They may serve to verify or correct the surveys which are made by the usual method.

### Problem I.

*To survey a field from* TWO STATIONS.

129. Find the distance of the two stations, and their bearings from each other; then take the bearings of the several corners of the field from each of the stations.

In the field ABCDE, (Fig. 37.) let the distance of the two stations S and T be given, and their bearings from each other. By taking the bearing of A from S and T, or the angles AST and ATS, we have the direction of the lines drawn from the two stations to one of the corners of the field. The point A is determined by the *intersection* of these lines. In the same manner, the point B is determined by the intersection of SB and TB; the point C, by the intersection of SC and TC; &c. &c. The sides of the field are then laid down, by connecting the points ABCD, &c.

The *area* is obtained, by finding the areas of the several triangles into which the field is divided by lines drawn from one of the stations. Thus the area of ABCDE (Fig. 37.) is equal to

$$ABT+BCT+CDT+DET+EAT$$

or to

$$ABS+BCS+CDS+DES+EAS.$$

Now we have the base line ST given and the angles, in the triangle AST, to find AS and AT; in the triangle BST, to find BS and BT, &c. After these are found, we have two sides and the included angle in the triangles ABT, BCT, &c. from which the areas may be calculated. (Mens. 9.)

*Example.*

Let the station T (Fig. 37.) be N. 80° E. from S, the distance ST 27 chains, and the bearings of the several corners of the field from S and T as follows;

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| TA | N. 30° | W. | | SA | N. 17° | E. |
| TB | N. 15 | E. | | SB | N. 55 | E. |
| TC | S. 53 | E. | | SC | S. 73 | E. |
| TD | S. 55 | W. | | SD | S. 24 | W. |
| TE | N. 70 | W. | | SE | N. 26 | W. |

These will give the following angles;

| | | |
|---|---|---|
| ATS= 70° | AST= 63° | ATB= 45° |
| BTS=115 | BST= 25 | BTC=112 |
| CTS=133 | CST= 27 | CTD=108 |
| DTS= 25 | DST=124 | DTE= 55 |
| ETS= 30 | EST=106 | ETA= 40 |

From which, with the base line ST, are calculated the following lines and areas.

| | |
|---|---|
| AT=32.89 chains. | ABT=206.45 sq. chains. |
| BT=17.75 | BCT=294.95 |
| CT=35.84 | CDT=740.7 |
| DT=43.46 | DET=665.1 |
| ET=37.36 | EAT=395. |

Contents of the field, =230.22 acres.

The course and length of each of the *sides* of the field may be found, if necessary. After the parts mentioned above are calculated, there will be given two sides and the included angle, in the triangle ATB, to find AB, in BTC to find BC, &c.

If the base line between the two stations be *too short*, compared with the sides of the field and their distances, the survey will be liable to inaccuracy. It should not generally be less than one tenth of the longest straight line which can be drawn on the ground to be measured.

130. It is not necessary that the base line, from the extremities of which the bearings are taken, should be *within* the field. It may be one of the sides, or it may be entirely *without* the field.

Let S and T (Fig. 38.) be two stations from which all the corners of a field ABCDE may be seen. If the direction and length of the base line be measured, and the bearings of the points A, B, C, D, and E, be taken at each of the stations the areas of the several triangles may be found. The figure ABCTDE is equal to

$$DET+EAT+ABT+BCT.$$

From this subtracting DCT, we have the area of the field ABCDE.

In this manner, a piece of ground may be measured which, from natural or artificial obstructions, is *inaccessible.* Thus an island may be measured from the opposite bank, or an enemy's camp, from a neighboring eminence.

131. The method of surveying by making observations from two stations, is particularly adapted to the measurement of a *bay* or *harbor.*

The survey may be made on the water, by anchoring two vessels at a distance from each other, and observing from each the bearings of the several remarkable objects near the shore. Or the observations may be made from such elevated situations on the land, as are favorable for viewing the figure of the harbor. If all the parts of the shore cannot be seen from two stations, three or more may be taken. In this case the direction and distance of each from one of the others should be measured.

## Problem II.

*To survey a field by measuring from* one station.

132. Take the bearings of the several corners of the field, and measure the distance of each from the given station.

If the length and direction of the several lines AT, BT, CT, DT, and ET, (Fig. 37.) be ascertained; there will be given two sides and the included angle of each of the triangles ABT, BCT, CDT, DET, and EAT; from which their areas may be calculated, (Mens. 9.) and the sum of these will be the contents of the whole figure.

The station may be taken in one of the sides or angles of the field, as at C. (Fig. 32.) The lines

CD, CE, CA, CB, and the angles
DCE, ECA, ACB, being given,

the areas of the triangles may be found.

PROBLEM III.

*To survey a field by the* CHAIN ALONE.

133. MEASURE THE SIDES OF THE FIELD, AND THE DIAGONALS BY WHICH IT IS DIVIDED INTO TRIANGLES.

By measuring the sides (Fig. 32.)
AB, BC, CD, DE, EA,

and the diagonals CA and CE, we have the three sides of each of the triangles into which the whole figure is divided. They may therefore be constructed, (Trig. 172.) and their areas calculated. (Mens. 10.)

134. *To measure an* ANGLE *with the chain*, set off equal distances on the two lines which include the angle, as AB, AC, (Fig. 39.) and measure the distance from B to C. There will then be given the three sides of the isosceles triangle ABC, to find the angle at A by construction or calculation.

The chain may be thus substituted for the compass, in surveying a field by going round it, according to the method explained in the preceding section; or by measuring from one or two stations, as in problems I and II.

PROBLEM IV.

*To survey an irregular boundary by means of* OFFSETS.

135. RUN A STRAIGHT LINE IN ANY CONVENIENT DIRECTION, AND MEASURE THE PERPENDICULAR DISTANCE OF EACH ANGULAR POINT OF THE BOUNDARY FROM THIS LINE.

The irregular field (Fig. 40.) may be surveyed, by taking the bearing and length of each of the four lines AE, EF, FI, IA, and measuring the perpendicular distances BB′, CC′, DD′, GG′, HH′, KK′. These perpendiculars are called *offsets*. It is necessary to note in a field book the parts into which the line that is measured is divided by the offsets, as in the following example. (See Fig. 40.)

| Offsets on the left. | | Courses and Distances. | Offsets on the right. |
|---|---|---|---|
| | Chains. | AE N. 85° E. 12.74 ch. | |
| BB′ | 2.18 | AB′ 3.25 | |
| CC′ | 2.18 | B′C′ 2.13 | |
| DD′ | 1.23 | C′D′ 1.12 | |
| | | D′E 6.24 | |
| | | EF S. 24° E. 7.23 | |
| | | FI N. 87° W. 13.34 | |
| GG′ | 2.86 | FG′ 3.84 | |
| HH′ | 1.48 | G′H′ 2.22 | |
| | | H′I 7.28 | |
| | | IA N. 26° W. 5.32 | |
| | | IK′ | KK′ 2.94 |
| | | K′A | |

As the offsets are perpendicular to the lines surveyed, the little spaces ABB′, BB′CC′, CC′DD′, &c. are either right angled triangles, parallelograms, or trapezoids. To find the *contents* of the field, calculate in the first place the area between the lines surveyed, as the trapezium AEFIA, (Fig. 40.) and then add the spaces between the offsets, if they fall within the boundary line; or subtract them, if they fall without, as AIK.

When any part of a side of a field is *inaccessible*, equal offsets may be made at each end, and a line run *parallel* to the boundary.

### Problem V.

*To measure the distance between any two points on the surface of the earth, by means of a series of triangles extending from one to the other.*

136. Measure a side of one of the triangles for a BASE LINE, take the BEARING of this or some other side, and measure the ANGLES in each of the triangles.

If it be required to find the distance between the two points A and I, (Fig. 41.) so situated that the measure cannot be taken in a direct line from one to the other; let a series of triangles be arranged in such a manner between them, that one side shall be common to the first and second, as BC, to the second and third as CD, to the third and fourth, &c. Then measure the length of BC for a base line, take the bearing of the side AB, and measure the angles of each of the triangles.

These data are sufficient to determine the length and bearing of each of the sides, and the distance and bearing of I from A. For in the two first triangles ABC and BCD, the angles are given and the side BC, to find the other sides. When CD is found, there are given, in the third triangle CDE, one side and the angles, to find the other side. In the same manner, the calculation may be carried from one triangle to another, till all the sides are found.

The *bearings* of the sides, that is, the angles which they make with the meridian, may be determined from the bearing of the first side, and the angles in the several triangles. Thus if NS be parallel to AM, the angle BAP, or its equal ABN subtracted from ABD leaves NBD; and this taken from 180 degrees leaves SBD.

From the bearing and length of AB may be found the *southing* AP, and the *easting* PB. In the same manner are found the several southings PP′, P′P″, P″P‴, P‴M. The sum of the southings is the line AM. And if the distance is so small, that the several meridians may be considered parallel, the difference between the sum of the eastings and the sum of the westings, is the perpendicular IM. We have then, in the right angled triangle AMI, the sides AM and MI, to find the distance and bearing of I from A.

137. This problem is introduced here for the purpose of giving the general outlines of those important operations which have been carried on of late years, with such admirable precision, under the name of *Trigonometrical Surveying*.

Any explanation of the subject, however, which can be made in this part of the course, must be very imperfect. In the demonstration of the problem, the several triangles are supposed to be in the same plane, and the distances of the meridians so small, that they may be considered parallel. But in practice, the ground upon which the measurement is to be made is very irregular. The stations selected for the angular points of the triangles, are such elevated parts of the country as are visible to a considerable distance. They should be so situated, that a signal staff, tower, or other conspicuous object in any one of the angles, may be seen from the other two angles in the same triangle. It will rarely be the case that any two of the triangles will be in the same plane, or any one of them parallel to the horizon. Reductions will therefore be necessary to bring them to a common level. But even this level is not a plane. In the cases in which this kind of surveying is commonly practiced, the measurement is carried over an extent of country of many miles. The several points, when reduced to the same distance from the center of the earth, are to be considered as belonging to a *spherical* surface. To make the calculations then, if the line to be measured is of any considerable extent, and if nice exactness is required, a knowledge of Spherical Trigonometry is necessary.

138. The decided superiority of this method of surveying, in point of accuracy, over all others which have hitherto been tried, particularly where the extent of ground is great, is owing partly to the fact that almost all the quantities measured are *angles;* and partly to this, that for the single line which it is necessary to measure, *the ground may be chosen*, any where in the vicinity of the system of triangles. It would be next to impossible to determine the precise horizontal distance between two points, by carrying a chain over an irregular surface. But in the trigonometrical measurements which are made upon a great scale, there can generally be found somewhere in the country surveyed, a level plane, a heath, or a body of ice on a river or lake, of sufficient extent for a *base.* This is the only line which it is absolutely necessary to measure. It is usual however, to measure a second, which is called a line of *verification.* If the length of the base BC (Fig. 41.) and the angles be given, all the other lines in the figure may be found by trigonometrical calculation. But if GH be also measured, it will serve to detect any error which may have been committed, either in taking the angles, or in computing the sides, of the series of triangles between BC and GH.

139. In measuring these lines, rods of copper or platina have been used in France, and glass tubes or steel chains in England. The results have in many instances been extremely exact. A base was measured, on Hounslow Heath, by General Roy, with glass rods. Several years after, it was re-

measured by Colonel Mudge, with a steel chain of very nice construction. The difference in the two measurements was less than *three inches* in more than five miles. Two parties measured a base in Peru of 6272 toises, or more than seven miles; and the difference in their results did not exceed *two inches*.

Exact as these measurements are, the exquisite construction of the instruments which have been used for taking the *angles*, has given to that part of the process a still higher degree of perfection. The amount of the errors in the angles of each of the triangles, measured by Ramsden's Theodolite, did not exceed *three seconds*. In the great surveys in France, the angles were taken with nearly the same correctness.

140. One of the most important applications of trigonometrical surveying, is in measuring *arcs of the meridian*, or of *parallels of latitude*, particularly the former. This is necessary in determining the *figure of the earth*, a very essential problem in Geography and Astronomy. A degree of zeal has been displayed on this subject, proportioned to its practical importance. Arcs of the meridian have been measured at great expense, in England, France, Lapland, Peru, &c. Men of distinguished science have engaged in the undertaking.

A meridian line has been measured, under the direction of General Roy and Colonel Mudge, from the Isle of Wight, to Clifton in the north of England, a distance of about 200 miles. Several years were occupied in this survey. Another arc passing near Paris, has been carried quite through France, and even across a part of Spain to Barcelona. In measuring this, several distinguished mathematicians and astronomers were engaged for a number of years. These two arcs have been connected by a system of triangles running across the English Channel, the particular object of which was to determine the exact difference of longitude between the observatories of Greenwich and Paris. Besides the meridian arcs, other lines intersecting them in various directions have been measured both in England and France. With these, the most remarkable objects over the face of the country have been so connected, that the geography of the various parts of the two kingdoms is settled, with a precision which could not be expected from any other method.

141. The exactness of the surveys will be seen from a comparison of the lines of *verification* as actually measured, with the lengths of the same lines as determined by calculation. These would be affected by the amount of all the errors in measuring the base lines, in taking the angles, in computing the sides of the triangles, and in making the necessary reductions for the irregularities of surface. A base of verification measured on Romney Marsh in England, was found to differ but about *two feet* from the length of the same line as deduced from a series of triangles extending more than 60 miles. A base of verification connected with the meridian passing through France, was found not to differ *one foot* from the result of a calculation which depended on the measurement of a base 400 miles distant. A line of verification of more than 7 miles, on Salisbury Plain, differed scarcely *an inch* from the length as computed from a system of triangles extending to a base on Hounslow Heath.*

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* See Note K.

## SECTION III.

### LAYING OUT AND DIVIDING LANDS.

ART. 142. To those who are familiar with the principles of geometry, it will be unnecessary to give particular rules, for all the various methods of dividing and laying out lands. The following problems may serve as a specimen of the manner in which the business may be conducted in practice.

### PROBLEM I.

*To lay out a given number of acres in the form of a* SQUARE.

143. REDUCE THE NUMBER OF ACRES TO SQUARE RODS OR CHAINS, AND EXTRACT THE SQUARE ROOT. This will give one side of the required field. (Mens. 7.)

EX. 1. What is the side of a square piece of land containing $124\frac{1}{4}$ acres? Ans. 141 rods.

2. What is the side of a square field which contains $58\frac{3}{4}$ acres?

### PROBLEM II.

*To lay out a field in the form of a* PARALLELOGRAM, *when one side and the contents are given.*

144. DIVIDE THE NUMBER OF SQUARE RODS OR CHAINS BY THE LENGTH OF THE GIVEN SIDE. The quotient will be a side perpendicular to the given side. (Mens. 7.)

EX. What is the width of a piece of land which is 280 rods long, and which contains 77 acres?

Ans. 44 rods.

Cor. As a triangle is half a parallelogram of the same base and height, a field may be laid out in the form of a triangle whose area and base are given, by dividing twice the area by the base. The quotient will be the perpendicular from the opposite angle. (Mens. 8.)

### PROBLEM III.

*To lay out a piece of land in the form of a parallelogram, the length of which shall be to the breadth in a given* RATIO.

145. As the *length* of the parallelogram, to its breadth;
So is the *area*, to the area of a *square* of the same breadth.

The side of the square may then be found by problem I, and the length of the parallelogram by problem II.

If BCNM (Fig. 42.) is a square in the right parallelogram ABCD, or in the oblique parallelogram ABC′D′, it is evident that AB is to MB or its equal BC, as the area of the parallelogram to that of the square.

Ex. If the length of a parallelogram is to its breadth as 7 to 3, and the contents are $52\frac{1}{2}$ acres, what is the length and breadth.

### PROBLEM IV.

*The area of a parallelogram being given, to lay it out in such a form, that the length shall exceed the breadth by a given* DIFFERENCE.

146. Let $x$=BC the breadth of the parallelogram ABCD (Fig. 42.) and the side of the square BCNM.
$d$=AM the difference between the length and breadth.
$a$=the area of the parallelogram.
Then $a=(x+d)\times x=x^2+dx$. (Mens. 4.)
Reducing this equation, we have

$$\sqrt{a+\frac{1}{4}d^2}-\frac{1}{2}d=x.$$

That is, to the area of the parallelogram, add one fourth of the square of the difference between the length and the breadth, and from the square root of the sum, subtract half the difference of the sides; the remainder will be the breadth of the parallelogram.

Ex. If four acres of land be laid out in the form of a parallelogram, the difference of whose sides is 12 rods, what is the breadth?

## PROBLEM V.

*To lay out a* TRIANGLE *whose area and angles are given.*

147. CALCULATE THE AREA OF ANY SUPPOSED TRIANGLE WHICH HAS THE SAME ANGLES. THEN

AS THE AREA OF THE ASSUMED TRIANGLE,
TO THE AREA OF THAT WHICH IS REQUIRED;
SO IS THE SQUARE OF ANY SIDE OF THE FORMER,
TO THE SQUARE OF THE CORRESPONDING SIDE OF THE LATTER.

If the triangles B′CC′ and BCA (Fig. 43.) have equal angles, they are similar figures, and therefore their areas are as the squares of their like sides, for instance, as $\overline{AC}^2 : \overline{CC'}^2$. (Euc. 19. 6.) The *square* of CC′ being found, extracting the *square root* will give the line itself.

To lay out a triangle of which *one side* and the area are given, divide twice the area by the given side; the quotient will be the length of a perpendicular on this side from the opposite angle. (Mens. 8.) Thus twice the area of ABC (Fig. 45.) divided by the side AB, gives the length of the perpendicular CP.

148. This problem furnishes the means of cutting off, or laying out, a given quantity of land in various forms.

Thus, from the triangle ABC, (Fig. 43.) a smaller triangle of a given area may be cut off, by a line parallel to AB. The line CC′ being found by the problem, the point C′ will be given, from which the parallel line is to be drawn.

149. If the directions of the lines AE and BD, (Fig. 44.) and the length and direction of AB be given; and if it be required to lay off a given area, by a line parallel to AB; let the lines AE and BD be continued to C. The angles of the triangle ABC with the side AB being given, the area may be found. From this subtracting the area of the given trapezoid, the remainder will be the area of the triangle DCE; from which may be found, as before, the point E through which the parallel is to be drawn.

If the trapezoid is to be laid off on the *other side* of AB, its area must be *added* to ABC, to give the triangle D′CE′.

150. If a piece of land is to be laid off from AB, (Fig. 45.) by a line in a given direction as DE, *not parallel* to AB;

let AC parallel to DE be drawn through one end of AB. The required trapezium consists of two parts, the triangle ABC, and the trapezoid ACED. As the angles and one side of the former are given, its area may be found. Subtracting this from the given area, we have the area of the trapezoid, from which the distance AD may be found by the preceding article.

151. If a given area is to be laid off from AB, (Fig. 46.) by a line proceeding from a *given point* D; first lay off the trapezoid ABCD. If this be too small, add the triangle DCE; but if the trapezoid be too large, subtract the triangle DCE'.

### Problem VI.

*To divide the area of a triangle into parts having given ratios to each other, by lines drawn from one of the angles to the opposite base.*

152. Divide the base in the same proportion as the parts required.

If the triangle ABC (Fig. 47.) be divided by the lines CH and CD; the small triangles, having the same height, are to each other as the bases BH, DH, and AD. (Euc. 1. 6.)

### Problem VII.

*To divide an irregular piece of land into any two given parts.*

153. *Run a line at a venture, near to the true division line required, and find the area of one of the parts. If this be too large or too small, add or subtract, by the preceding articles, a triangle, a trapezoid, or a trapezium, as the case may require.*

A field may sometimes be conveniently divided by reducing it to a triangle, as in Art. 125, (Fig. 35.) and then dividing the triangle by problem VI.

## SECTION IV.

### LEVELLING.

Art. 154. It is frequently necessary to ascertain how much one spot of ground is higher than another. The practicability of supplying a town with water from a neighboring fountain, will depend on the comparative elevation of the two places above a common level. The direction of the current in a canal will be determined by the height of the several parts with respect to each other.

The art of levelling has a primary reference to the level surface of water. The surface of the ocean, a lake, or a river, is said to be level when it is at rest. If the fluid parts of the earth were perfectly *spherical*, every point in a level surface would be at the same distance from the center. The difference in the heights of two places above the ocean would be the same, as the difference in their distances from the center of the earth. It is well known that the earth, though nearly spherical, is not perfectly so. It is not necessary, however, that the difference between its true figure and that of a sphere should be brought into account, in the comparatively small distances to which the art of levelling is commonly applied. But it is important to distinguish between the *true* and the *apparent* level.

155. *The* TRUE LEVEL *is a* CURVE *which either coincides with, or is parallel to, the surface of water at rest.*

*The* APPARENT LEVEL *is a* STRAIGHT LINE *which is a* TANGENT *to the true level, at the point where the observation is made.*

Thus if ED (Fig. 48.) be the surface of the ocean, and AB a concentric curve, B is on a true level with A. But if AT be a tangent to AB, at the point A, the *apparent* level as observed at A, passes through T.

156. When levelling instruments are used, the level is determined either by a *fluid* or a *plumb-line.* The surface of the former is *parallel* to the horizon. The latter is *perpen-*

*dicular*. One of the most convenient instruments for the purpose is the *spirit level*. A glass tube is nearly filled with spirit, a small space being left for a bubble of air. The tube is so formed, that when it is horizontal, the air bubble will be in the middle between the two ends. To the glass is attached an index with sight vanes; and sometimes a small telescope, for viewing a distant object distinctly. The surveyor should also be provided with a pair of *levelling rods*, which are to be set up perpendicularly, at convenient distances, for the purpose of measuring the height from the surface of the ground to the horizontal line which passes through the spirit level.

If strict accuracy is aimed at, the spirit level should be in the *middle* between the two rods. Considering D′ED″D as the spherical surface of the earth, and B′AB″B as a concentric curve; a horizontal line passing through A is a *tangent* to this curve. If therefore AT′ and AT″ are *equal*, the points T′ and T″ are equally distant from the level of the ocean. But if the two rods are at T and T′, while the spirit level is at A, the height TD is greater than T′D′. The difference however will be trifling, if the distance of the stations T and T′ be small.

157. With these simple instruments, the spirit level and the rods, the comparative heights of any two places can be ascertained by a series of observations, without measuring their distance, and however irregular may be the ground between them. But when one of the stations is visible from the other and their distance is known; the difference of their heights may be found by a *single observation*, provided allowance be made for atmospheric refraction, and for the difference between the true and the apparent level.

## PROBLEM I.

*To find the difference in the height of two places by levelling rods.*

158. *Set up the levelling rods perpendicular to the horizon, and at equal distances from the spirit level; observe the points where the line of level strikes the rods before and behind, and measure the heights of these points above the ground; level in the same manner from the second station to the third, from the*

*third to the fourth, &c. The difference between the sum of the heights at the back stations, and at the forward stations, will be the difference between the height of the first station and the last.*

If the descent from H to H″ (Fig. 49.) be required, let the spirit level be placed at A, equally distant from the stations H and H′; observe where the line of level BF cuts the rods which are at H and H′, and measure the heights BH and FH′. The difference is evidently the descent from the first station to the second. In the same manner by placing the spirit level at A′, the descent from the second station to the third may be found. The back heights, as observed at A and A′, are BH, and B′H′,; the forward heights are FH′ and F′H″.

Now FH′−BH = the descent from H to H′,
And F′H″−B′H′ = the descent from H′ to H″;
Therefore, by addition,
(FH′+F′H″)−(BH+B′H′) = the whole descent from H to H″.

159. It is to be observed, that this method gives the *true* level, and not the apparent level. The lines BF and B′F′ are not parallel to each other; but one is parallel to a tangent to the horizon at N, the other to a tangent at N′. So that the points B and F are equally distant from the horizon, as are also the points B′ and F′. The spirit level may be placed at unequal distances from the two station rods, if a correction is made for the difference between the true and the apparent level by problem II.

160. If the stations are numerous, it will be expedient to place the back and the forward heights in separate columns in a table, as in the following example.

| | Back heights. | | Fore heights. | |
|---|---|---|---|---|
| | Feet. | In. | Feet. | In. |
| 1st. Observation | 3 | 7 | 2 | 8 |
| 2. " | 2 | 5 | 3 | 1 |
| 3. " | 6 | 3 | 5 | 7 |
| 4. " | 4 | 2 | 3 | 2 |
| 5. " | 5 | 9 | 4 | 10 |
| | 22 | 2 | 19 | 4 |
| | 19 | 4 | | |
| Difference | 2 | 10 | | |

If the sum of the forward heights is less than the sum of the back heights, it is evident that the last station must be higher than the first.

## PROBLEM II.

161. *To find the difference between the* TRUE *and the* APPARENT *level, for any given distance.*

If C (Fig. 12.) be the center of the earth considered as a sphere, AB a portion of its surface, and T a point on an apparent level with A; then BT is the difference between the true and the apparent level, for the distance AT.

Let 2BC=D, the diameter of the earth,
AT=$d$, the distance of T, in a right line from A,
BT=$h$, the height of T, or the difference between the true and the apparent level.

Then by Euc. 36. 3,

$$(2BC+BT)\times BT=\overline{AT}^2;\ \text{that is, } (D+h)\times h=d^2;$$

and reducing the equation,

$$h=\sqrt{\tfrac{1}{4}D^2+d^2}-\tfrac{1}{2}D.$$

Therefore, to find $h$ the difference between the true and the apparent level, add together one fourth of the square of the earth's diameter, and the square of the distance, extract the square root of the sum, and subtract the semi-diameter of the earth.

162. This rule is exact. But there is a more simple one, which is sufficiently near the truth for the common purposes of levelling. The height BT is so small, compared with the diameter of the earth, that D may be substituted for D+$h$, without any considerable error. The original equation above will then become

$$D\times h=d^2.\quad \text{Therefore } h=\frac{d^2}{D}.$$

That is, *the difference between the true and the apparent level, is nearly equal to the square of the distance divided by the diameter of the earth.*

Ex. 1. What is the difference between the true and the apparent level, for a distance of one English mile, supposing the earth to be 7940 miles in diameter?

Ans. 7.98 inches, or 8 inches nearly.

In the equation $h=\frac{d^2}{D}$, as D is a *constant* quantity, it is evident that $h$ varies as $d^2$. According to the last rule then, the difference between the true and the apparent level varies as the *square of the distance.* The difference for 1 mile being nearly 8 inches,

| | | In. | Feet. | In. |
|---|---|---|---|---|
| For 2 miles, it is | $8\times 2^2=$ | 32= | 2 | 8 nearly. |
| For 3 miles, | $8\times 3^2=$ | | 6 | |
| For 4 miles, | $8\times 4^2=$ | | 10 | 8 |

&c. &c. See Table IV.

Ex. 2. An observation is made to determine whether water can be brought into a town from a spring on a neighboring hill. At a particular spot in the town, the spring, which is $2\frac{1}{2}$ miles distant, is observed to be apparently on a level. What is the descent from the spring to this spot?

The descent is nearly 4 feet 2 inches for the whole distance, or 20 inches in a mile; which is more than sufficient for the water to run freely.

Ex. 3. A tangent to a certain point on the ocean, strikes the top of a mountain 23 miles distant. What is the height of the mountain? Ans. 352 feet.

163. One place may be *below* the apparent level of another, and yet *above* the true level. The difference between the true and the apparent level for 3 miles is 6 feet. If one spot, then, be only two feet below the apparent level of another 3 miles distant, it will really be 4 feet higher.

If two places are on the same true level, it is evident that each is below the apparent level of the other.

PROBLEM III.

*To find the difference in the heights of two places whose distance is known.*

164. *From the angle of elevation or depression, calculate how far one of the places is above or below the apparent level of the other; and then make allowance for the difference between the apparent and the true level.*

By taking, with a quadrant, the elevation of the object whose distance is given, we have one side and the angles of a right angled triangle, to find the perpendicular height above a *horizontal plane.* (Art. 6.) Adding this to the difference between the true and the apparent level, we have the height of the object above the true level of the place of observation. When an angle of *depression* is taken, it will be necessary to *subtract* instead of adding.

Ex. 1. The angle of elevation of a hill, as observed from the top of another $4\frac{1}{2}$ miles distant, is found to be 7 degrees. What is the difference in the heights of the two hills?

| | |
|---|---|
| Height of one above the level of the other | 2917.3 feet. |
| Difference of the levels | 13.5 |
| Difference in the height of the hills | 2930.8 |

Ex. 2. From the top of a tower, the angle of depression of a fort 4 miles distant, is found to be $3\frac{1}{4}$ degrees. What is the height of the tower above the fort?

Ans. 1189 feet.

If the operation of levelling is meant to be very exact, especially when extended to considerable distances, allowance should be made for atmospheric *refraction.**

* See Note A.

## SECTION V.

### THE MAGNETIC NEEDLE.*

Art. 166. The direction in which a ship is steered, and the bearings of the sides of a field, are commonly determined by observing the angles which they make with the magnetic needle. This is a bar of steel to which the magnetic power has been communicated from some other artificial or natural magnet. When it is balanced on a pin, so as to turn freely in any direction, it points towards the north and south.

The *poles* of the needle are its two extremities; and the vertical plane which passes through these, is called the *magnetic meridian*. The astronomical meridian passes through the poles of the earth. These two meridians rarely coincide. The needle does not often point directly north and south.

167. *The* DECLINATION *of the needle is the angle which it makes with a north and south line;* or the angle between the magnetic and the astronomical meridians. It is said to be east or west, according as the north pole of the needle points east or west of the north pole of the earth.

The *variation* of the needle is properly the *change of its declination.* The term, however, is frequently used to signify the declination itself.

The declination of the needle is very different in different parts of the earth. In some places, it is 20 or 30 degrees: in others, little or nothing. In the variation charts given by writers on magnetism, the declination is marked, as it is found by observation on different parts of the globe. Lines are drawn connecting all the points which have the *same* declination. Thus a line is drawn through the several places in which the declination is 10 degrees; another through those

* Cavallo on Magnetism, Rees' Cyclopedia, Transactions of the Royal Society of London, the Royal Irish Academy, the American Philosophical Society, and the American Academy of Arts and Sciences.

in which it is 5 degrees, &c. These lines are very winding yet they never *cross* each other, though they extend all over the globe. One of the lines of *no declination* passes through the middle parts of the United States. The declination is towards this line, in places which are on either side of it. Thus in New England the declination is west, while on the Ohio it is east. It increases with the distance from the line of no declination.

168. The declination is not only different in different places, but different in the *same* place at different times. At London, about 230 years since, it was $11\frac{1}{4}$ degrees east. It gradually decreased till 1657, when the needle pointed directly north. From that time it deviated more and more to the west, till in 1800 the declination became about 24 degrees. At present, it appears to be nearly stationary, both at London and Paris.

In New England, the declination has been generally decreasing, for many years. At Boston, it was about 9 degrees in 1708, 8 degrees in 1742, 7 degrees in 1782, and $6\frac{1}{4}$ degrees in 1810; the rate of variation being about $1\frac{1}{2}'$ in a year, or a degree in 40 years.*

The variation in the declination is by no means uniform. If the needle moves two minutes from the meridian in one year, it may move a greater or less distance the next year. Its progress is different in different places. In some it is moving east, and in others west; in some it is coming nearer to the meridian, in others going farther from it.

169. There is also a *diurnal* variation, which appears to be owing to a change of temperature. During the fore part of the day, the north end of the needle frequently moves a few minutes of a degree to the west. In the evening, it returns nearly to the same point from which it started. This diurnal variation is found to be the greatest in the summer months, when the action of the sun is most powerful.

In addition to these various changes, there are local perturbations of the needle, occasioned probably by the attraction of ferruginous substances beneath the surface of the ground.

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* See the observations of Dr. Bowditch, in the Memoirs of the American Academy of Arts and Sciences, Vol. III. Part II. p, 337, and Prof. Olmsted's paper in the Am. Jour. of Arts and Sciences, Vol. XVI, p. 60.

170. So many irregularities must render the magnetic compass an inaccurate instrument, unless the state of the declination is ascertained by frequent observations. This is particularly necessary at sea, where the declination may be changed by a few hours' sail.

The astronomical meridian is determined by the positions of the heavenly bodies. The situation of the sun at rising or setting being known, its distance from the magnetic meridian may be observed with an *azimuth compass*, which is a mariner's compass with the addition of sight vanes for taking the direction of any object.

171. On land, a true meridian line may be drawn by observations on the *pole star*. If this were exactly in the pole, it would be always on the meridian. But the star revolves round the pole, at a short distance in a little less than 24 hours. In about 6 hours from its passing the meridian above the pole, it is at its greatest distance west; in about 12 hours, it is on the meridian beneath the pole, and in about 18 hours, at its greatest distance east. If the direction of the star can be taken, at the instant when it is on the meridian, either above or beneath the pole; a true north and south line may be found. This method, however, requires that the exact time of passing the meridian be known, and that the observations be made expeditiously.

172. But as the star comes very gradually to its greatest distance east or west, it is easy to observe these limits; and as the revolution is made in a circle round the axis of the earth, it is evident that the pole must be in the middle between the two extreme distances. To draw a true meridian line, then, *take the direction of the pole star when it is farthest west, and also when it is farthest east; and bisect the angle made by these two directions.*

When a meridian is once drawn, it may be rendered permanent, by fixing proper marks; and the declination of the needle may then be ascertained at any time, by the surveyor's compass, or more accurately by the *variation compass*, which has a long needle, and a graduated arc of so large a radius as to admit of very accurate divisions.*

* See Note L.

# NOTES.

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## Note A. Page 10 and 91.

A ray of light, in coming from a distant object to the eye, through the air, is turned from a straight line into a curve which is concave towards the earth. The effect is to *elevate* the apparent place of the object, as each point appears in the direction in which the light from that point enters the eye. The change in the apparent situation is called *astronomical refraction*, when the heavenly bodies are concerned; and *terrestrial refraction*, when the objects are on the earth. The measure of the latter is the angle at the eye, between a straight line drawn to the object, and a tangent to the curvilinear ray, as TAT′, (Fig 50.) T being the place of the object, and T′ its apparent place as seen from A.

The refraction is very much affected by the state of the atmosphere; changing with the temperature, as well as with the density indicated by the barometer. In the delicate observations made in the trigonometrical surveys in England and France, the terrestrial refraction was found to vary from $\frac{1}{4}$ to $\frac{1}{24}$ of the angle at the center of the earth subtended by the distance of the object. The mean is $\frac{1}{14}$: thus if an object at T (Fig. 50.) as seen from A in the mean state of the atmosphere, appears to be raised to T′; the angle TAT′ is about $\frac{1}{14}$ of the angle ACT subtended by the distance AT. This angle is easily found from the arc AB, which is nearly equal to AT; the whole circumference of the earth being to the arc, as 360 degrees to the angle required. The mean terrestrial refraction, as thus calculated, is 3.7″ for a *mile*, and increases as the distance nearly; the elevation of the object being supposed to be small in comparison with its distance. In measuring altitudes, the terrestrial refraction is to be *subtracted* from the observed angle of elevation.

The alteration in the *height* of the object, by the mean refraction is equal to $\frac{1}{7}$ of the curvature of the earth for the given distance, or of the difference between the true and apparent levels, as calculated by the rule in Art. 162. If the angle of refraction were equal to *half* the angle at the center of the earth subtended by the distance, the change in the height of the object would be just equal to the correction for the curvature. If an object at B (Fig. 12.) were raised by refraction, so as to be seen from A in the direction of the tangent AT; the change in the altitude would be equal to BT, which is the difference between the true and the apparent level of A. In this case the angle BAT would be half ACB, (Euc. 32. 3 and 20. 3.) But as the angle of refraction is in fact only $\frac{1}{14}$ of the angle at the center, the change in the altitude is only $\frac{1}{7}$ of the correction for curvature. The latter is about $\frac{2}{3}$ of a foot for a mile, and varies as the square of the distance. If then $d$ be the distance in miles; the correction for the curvature will be $\frac{2}{3} d^2$, and the correction for refraction $\frac{2}{21}d^2$. See Table IV.

The greatest distance at which an object can be seen on the surface of the earth, as calculated by the rule in Art. 23, depends on the *apparent* altitude. This being to the real altitude as 7 to 6, and the distance being nearly as the square root of the altitude; the distance at which an object can be seen by the mean refraction, is to the distance at which it could be seen without refraction as $\sqrt{7} : \sqrt{6}$, or as 14 : 13 nearly. See Playfair's Astronomy, Sec. II. Vince's Astronomy, Chap. VII. and the accounts of the Trigonometrical Surveys in England and France.

## Note B. p. 20.

The method of calculation in plane sailing is sometimes spoken of as inaccurate, as only approximating to the truth, in proportion to the smallness of the difference between a plane and that part of the ocean to which the calculation is applied. This view of the subject appears not to be strictly correct. It is true, that plane sailing is *incomplete*, as it does not ascertain the *longitude*. This belongs to middle latitude or Mercator's sailing. It is also true, that if a ship sails on *several courses*, the sum of the departure is not equal to the departure for the same distance on a single course, as would be the case on a plane. (Art. 78.) It is farther to be observed, that the departure, as calculated by plane sailing, is neither the meridian distance measured on the parallel of latitude from which the ship sails, nor that measured upon the parallel upon which she arrives. But the departure for a single course, as defined in Art. 40, and the difference of latitude, are as accurately calculated by plane sailing, as if the surface of the ocean were a plane. Let the whole distance be divided into portions so small, that one of the arcs shall differ less from its tangent than by any given quantity. Each of these portions is to the corresponding departure, as radius to the sine of the course; and to the difference of latitude, as radius to the cosine of the course. Therefore the whole distance is to the whole departure, as radius to the sine of the course; and to the whole difference of latitude, as radius to the cosine of the course. These proportions are exact, even for a spheroid, a cylinder, or any solid of revolution.

If there were any incorrectness in plane sailing, it would extend to Mercator's sailing also; for one is founded on the other. In Mercator's sailing, the proper difference of latitude is to the meridional difference of latitude, as the *departure* to the difference of longitude. Now the departure is calculated by plane sailing; and any error in this must produce an error in the longitude. Or if the longitude be found by the theorem in Art. 72, without previously calculating the departure; yet the *table* of meridional parts which must be used, is founded on the ratio between the departure and the difference of longitude. Art. 65.

## Note C. p. 27.

It is here supposed that the direction of the ship is at right angles with every meridian which she crosses. A number of curious questions have been started respecting sailing on a sphere; such as whether a due east and west line coincides with a parallel of latitude, &c. Most of these points are easily settled by proper definitions. But this is not the place to consider them, as they belong to spherical geometry.

## NOTE D. p. 34.

As the length of a minute of Mercator's meridian, is equal to the secant of the latitude, it will be a little more exact to take the latitude of the *middle* of the arc, rather than that of one extremity. On the extended meridian, the first minute will then be made equal to the secant of $\frac{1}{2}'$, the second to the secant of $1\frac{1}{2}'$, the third to the secant of $2\frac{1}{2}'$, &c.

The method of calculating by natural secants, though useful in forming a *table* of meridional parts, is subject to this inconvenience, that to obtain the meridional parts for any number of degrees of latitude, it is necessary to find separately the parts for each of the minutes contained in the given arc, and then to add them together. There is a different method, by which the meridional parts for an arc of any extent, may be calculated independently of any other arc. A portion of Mercator's meridian, extending from the equator to a given latitude, the semi-diameter of the earth being 1, is equal to *the hyperbolic logarithm of the co-tangent of half the complement of the latitude.* See the London Philosophical Transactions, Vol. xix. No. 219, Vince's Fluxions, Art. 190, and the Introduction to Hutton's Mathematical Tables.

## NOTE E. p. 39.

The distance which a ship sails, in going from one place to another on a rhumb line, is not the *nearest* distance; for this would be an arc of a great circle. To sail on a great circle, except on a meridian or the equator, she must be continually altering her course. If it were practicable to steer a vessel in this manner, the departure, difference of latitude, &c. might be calculated by spherical trigonometry.

## NOTE F. p. 41.

A traverse may also be constructed like the plot of a field in surveying, either by drawing parallel lines, as in Art. 111, or from the angles given by the rules in Art. 112, or more simply, as in Art. 123, by departure and difference of latitude, when these have been found by calculation or inspection.

## Note G. p. 46.

Plane sailing is sometimes represented as a method of calculation founded on the principles of the plane chart. But in the construction of this chart, a principle is assumed which is known to be erroneous. That part of the surface of the earth which is represented on it, is supposed to be a plane. This renders the construction more or less inaccurate. But in plane sailing, the calculations are strictly correct. The principle assumed here is not that the surface of the earth is a plane; but that, from the peculiar nature of the rhumb line, the distance, departure, and difference of latitude, where the course is given, have the *same ratio* to each other which they would have upon a plane.

## Note H. p. 51.

The Quadrant of reflection has received the name of *Hadley's Quadrant*, as the description of it which was first made public, was given by John Hadley, Esq. But he has not an undisputed claim to the first invention. His description of the instrument was communicated to the Royal Society of London, in May, 1731. It appears that the principle on which it is constructed had been suggested by Dr. Hooke, several years before. But the form which he proposed was not calculated to answer the purpose, as it admitted of only one reflection. Sir Isaac Newton, however, who died in 1727, left among his papers a description of a quadrant with two reflections, which is substantially the same as Hadley's. This was published in the Philosophical Transactions for 1742.

It is also stated that a quadrant similar to Hadley's had been contrived by Mr. Thomas Godfrey, of Philadelphia, before Hadley's description was communicated to the Royal Society.

Hooke's Posthumous Works, Hutton's Dictionary, Transactions of the Royal Society of London for 1731, 1734 and 1742, American Magazine for Aug. and Sept. 1758, Miller's Retrospect, i. p. 468, and Analectic Magazine, ix, 281.

## Note I. p. 56.

The proportion in Art. 99, on account of the smallness of the height BT compared with the semi-diameter of the earth, is not very suitable for calculating the depression with exactness. The following rule, which includes the effect of refraction, is better adapted to the purpose. The depression is found by multiplying 59″ into the square root of the height in feet. See Vince's Astronomy, Art. 197, Rees' Cyclopedia, and Table II.

In taking the altitude of a heavenly body with Hadley's Quadrant, when the view of the ocean is unobstructed, the reflected image is made to coincide with the most remote visible point of the water. But when there is land in the direction in which the observation is to be made, the image is brought to the water's edge; and the dip is *increased*, in proportion as the distance of the land is diminished. See table III.

## Note K. p. 81.

This is not the place for a detailed account of the various trigonometrical operations which have been undertaken, for the purpose of determining the length of a degree of latitude, in different parts of the earth. The subject belongs rather to astronomy, than to common surveying. It may not be amiss, however, to give a concise statement of the measurements which have been made in the present and the preceding century.

About the year 1700, Picard measured a degree between Paris and Amiens; and the arc was extended by Cassini to Perpignan, about 6 degrees south of Paris, and afterwards to the northward as far as Dunkirk. These measurements were made in the middle latitudes. To compare a degree here, with the length of one near the pole, and another on the equator, two expeditions were fitted out from France about the same time, one for Lapland, and the other for South America. The latter sailed in May, 1735, for Peru, and after a series of the most formidable embarrassments, they succeeded, at the end of 8 years, in accomplishing their object. They measured an arc of the meridian, crossing the equator from 3° 7′ north latitude to about 3½° south. The other party in 1736, under the direction of Maupertuis, proceeded to the head of the gulf of Bothnia, and measured an arc of the meridian extending along the river Tornea, and crossing the polar circle. The difficulties which they experienced, in this frozen and desolate region, were scarcely inferior to those with which the other adventurers were at the same time contending in South-America.

The determination of these three arcs, one in France, one in Peru, and one in Lapland, were sufficient to satisfy astronomers, that a degree of latitude near the poles is greater than one on the equator, and consequently that the equatorial diameter of the earth is longer than the polar. But it does not necessarily follow from this, that there is a regular increase in the length of a degree, from the lower to the higher latitudes. On the contrary, according to the survey which had been made by Picard and others, a degree was found to be *greater* in the south of France, than in the north. The zeal of astronomers was therefore excited to take farther measures to determine what is the exact length of a degree, in various parts of the earth, and to ascertain whether in the influence of gravitation, there are local inequalities, which affect the astronomical observations.

La Caille, about this time, measured an arc of the meridian at the Cape of Good Hope. He was not, however, provided with such instruments as would insure a great degree of precision. Boscovich, a distinguished philosopher, measured an arc of two degrees in Italy, from Rimini to Rome. Between the years 1764 and 1768, Messrs. Mason and Dixon, under the direction of the Royal Society of London, measured an arc of the meridian of about one degree and an half, crossing the line between Pennsylvania and Maryland. As the country here is very level, the whole distance was measured, not by a combination of triangles, but in the first place with a chain, and afterwards with rods of fir. A degree was also measured in Piedmont, another in Austria, and a third in Hungary; the first by Beccaria, and the two latter by Liesganig.

But the most perfect of all the trigonometrical surveys upon a great scale, are those which have been made within a few years in England and France. The instruments used for taking the angles, particularly the theodolite of Ramsden, and the repeating circle of Borda, have been brought to a surprising degree of exactness. The re-measurement of the line from Dunkirk to Barcelona, a part of which had been several times surveyed before, was commenced in 1792, under the direction of the Academy of Sciences in Paris; for the purpose of obtaining a standard of measure of lengths, weights, capacity, &c. derived from a portion of the meridian. The northern part of the arc was measured by Delambre, and the southern part by Méchain, who lost his life in 1805, in attempting to extend the line beyond Barcelona, to the Balearic Islands in the Mediterranean. This line was afterwards continued by Biot to Formentera, the southernmost of these Islands, which is in Lat. 38° 38′ 56″. The latitude of Dunkirk is 51° 2′ 9″. The whole arc is, therefore, more than 12 degrees, and is nearly bisected by the 45th parallel of latitude. This is connected with the line carried through England to Clifton in Lat. 53° 27′ 31″ : making the whole extent nearly 15 degrees.

The arc which had been surveyed by Maupertuis, on the polar circle, was re-measured by Swanberg and others, in 1802. There is a difference of 230 toises in the length of a degree, as calculated from these two measurements. In India, an arc of the meridian was measured, on the coast of Coromandel, in 1803, by Major Lambton.

According to these various measurements, we have the following lengths of a degree of latitude in different parts of the earth.

| | *Latitude.* | *Toises.* | *Fathoms.* |
|---|---|---|---|
| 1. In Peru, by Bouguer, | 0 | 56,750 | 60,480 |
| 2. In India, by Lambton, | 12° 30′ | 56,756 | 60,487 |
| 3. At the Cape of Good Hope, by La Caille, | 33 18 | 57,037 | 60,780 |
| 4. In Pennsylvania, by Mason and Dixon, | 39 12 | 56,890 | 60,630 |
| 5. In Italy, by Boscovich, | 43 | 56,980 | 60,725 |
| 6. In Piedmont, by Beccaria, | 44 44 | 57,070 | 60,820 |
| 7. In France, by Delambre and Méchain, | 45 | 57,011 | 60,760 |
| 8. In Austria, by Liesganig, | 48 43 | 57,086 | 60,835 |
| 9. In England, by Roy and Mudge, | 52 2 | 57,074 | 60,827 |
| 10. In Lapland, by Maupertuis, | 66 20 | 57,422 | 61,184 |
| Do. by Swanberg, | | 57,192 | 60,952 |

On a comparison of all the the measurements which have been made, it is found that a degree of latitude is greater near the poles, than in the middle latitudes; and greater in the middle latitudes, than near the equator. The earth is therefore compressed at the poles, and extended at the equator. But it does not appear that it is an exact spheroid, or a solid of revolution of any kind. If arcs of the meridian which are near to each other and of moderate length be compared, they will not be found to increase regularly from a lower to a higher latitude. On the southern part of the line which was measured in France, the degrees increase very slowly; towards the middle, very rapidly; and near the northern extremity, very slowly again. Similar irregularities are found in that part of the meridian which passes through England. These irregularities are too great to be ascribed to errors in the surveys. It is concluded, therefore, that the direction of the plumb line, which is used in determining the latitude, is affected by local inequalities in the action of gravitation, owing probably to the different densities of the substances of which the earth is composed. These inequalities must also have an influence upon the figure of the fluid parts of the globe, so that the surface ought not to be considered as exactly spheroidical.

See Col. Mudge's account of the Trigonometrical Survey in England. Gregory's Dissertations, &c. on the Trigonometrical Survey. Rees' Cyclopedia, Art. Degree. Playfair's Astronomy. Philosophical Transactions of London for 1768, 1785, 1787, 1790, 1791, 1795, 1797, 1800. Asiatic Researches, vol. viii. Puissant. "Traité de Géodésie." Maupertuis. "Dégré du Méridien entre Paris et Amiens." Do. "La Figure de la Terre." Cassini. "Exposé des Operations, &c." Delambre. "Bases du système métrique." Swanberg. "Exposition des Operations faites en Laponie." Laplace. "Traité de Mécanique Céleste."

## Note L. p. 94.

One of the most simple methods of determining when the pole star is on the meridian, is from the situations of two other stars, Alioth and $\gamma$ Cassiopeiæ, both which come on to the meridian a few minutes before the pole star, the one above and the other below the pole. Alioth, which is the star marked $\varepsilon$ in the Great Bear, is on the same side of the pole with the pole star, and about 30 degrees distant. The star $\gamma$ in the constellation Cassiopeia, is nearly as far on the opposite side of the pole. The right ascension of the latter in 1810 was 0h. 45m. 24s., increasing about 3¼ seconds annually. The right ascension of Alioth was 12h. 45m. 36s., increasing about 2½ seconds annually. These two stars, therefore, come on to the meridian nearly at the same time. This time may be known by observing when the same vertical line passes through them both. The right ascension of the pole star in January 1810, was 0h. 54m. 36s., and increases 13 or 14 seconds in a year. So that this star comes to the meridian about 9 or 10 minutes after $\gamma$ Cassiopeiæ. In very nice observations, it will be necessary to make allowance for nutation, aberration, and the annual variation in right ascension.

About 10 minutes after a line drawn from Alioth to $\gamma$ Cassiopeiæ is parallel to the horizon, the pole star is at its greatest distance from the meridian. As this is the case only once in 12 hours, the two limits on the east side, and on the west side, cannot both be observed the same night, except at certain seasons of the year. But on any clear night, one observation may be made; and this is sufficient for finding a meridian line, if the distance of the star from the pole, and the latitude of the place be given. The angle between the meridian and a vertical plane passing through a star, or an arc of the horizon contained between these two planes, is called the *azimuth* of the star. And by spherical trigonometry, when the star is at its greatest elongation east or west,

As the cosine of the latitude,
To radius;
So is the sine of the polar distance,
To the sine of the azimuth.

The distance of the pole star from the pole in 1810, was 1° 42′ 19.6″, and decreases 19¼ seconds annually.

To observe the direction of the pole star when its azimuth is the greatest, suspend a plumb line 15 or 20 feet long from a fixed point, with the weight swinging in a vessel of water, to protect it from the action of the wind. At the distance of 12 or 15 feet south, fix a board horizontal on the top of a firm post. On the board, place a sight vane in such a manner that it can slide a short distance to the east or west. A little before the time when the star is at its great elongation, let an assistant hold a lighted candle so as to illuminate the plumb line. Then move the sight vane, till the star seen through it is in the direction of the line. Continue to follow the motion of the star, till it appears to be stationary at its greatest elongation. Then fasten the sight vane, and fix a candle or some other object in the direction of the plumb line, at some distance beyond it.

As the declination of the needle is continually varying, the courses given by the compass in old surveys, are not found to agree with the bearings of the same lines at the present time. To prevent the disputes which arise from this source, the declination should always be ascertained, and the courses stated according to the angles which the lines make with the astronomical meridian.

It must be admitted, after all, that the magnetic compass is but an imperfect instrument. It is not used in the accurate surveys in England. In the wild lands in the United States, the lines can be run with more expedition by the compass, than in any other way. And in most of the common surveys, it answers the purpose tolerably well. But in proportion as the value of land is increased, it becomes important that the boundaries should be settled with precision, and that all the lines should be referred to a permanent meridian. The angles of a field may be accurately taken with a graduated circle furnished with two indexes. The bearings of the sides will then be given, if a true meridian line be drawn through any point of the perimeter.

## EXPLANATION OF THE TABLES.

Table I contains the parts of Mercator's meridian, to every other minute. The parts for any odd minute may be found with sufficient exactness, by taking the arithmetical mean between the next greater and the next less. For the uses of this table, see Navigation, Sec. III.

Table II gives the depression or dip of the horizon at sea for different heights. Thus if the eye of the observer is 20 feet above the level of the ocean, the angle of depression is 4′ 24″. See Art. 99. This table is calculated according to the rule in note I, which gives the depression 59″ for one foot in altitude; allowance being made for the mean terrestrial refraction.

In Table III, is contained the depression for different heights and *different distances*, when the view of the ocean is more or less obstructed by land. Thus if the height of the eye is 30 feet, and the distance of the land $2\frac{1}{2}$ miles, the depression is 8′. See Note I.

Table IV contains the curvature of the earth, or the difference between the true and the apparent level, for different distances, according to the rule in Art. 161. Thus for a distance of 17 English miles, the curvature is 192 feet.

Table V contains the distances at which objects of different heights may be seen from the surface of the ocean, in the mean state of the atmosphere. This is calculated by first finding the distance at which a given object might be seen, if there were no refraction, and then increasing this distance in the ratio of $\sqrt{7} : \sqrt{6}$. See Note A.

Table VI contains the polar distance and the right ascension in time, of the pole star, from 1800 to 1820. From this it will be seen, that the right ascension is increasing at the rate of about 14 seconds a year, and that the north polar distance is decreasing at the rate of 19½ seconds a year. From the latitude of the place, and the polar distance of the star, its azimuth may be calculated, when it is at its greatest distance from the meridian. The time when it passes the meridian may be ascertained by finding the difference between the right ascension of the star and that of the sun. See Note L.

# TABLE I.

## MERIDIONAL PARTS.

| M. | 0° | 1° | 2° | 3° | 4° | 5° | 6° | 7° | 8° | 9° | 10° | 11° | 12° | M. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 60 | 120 | 180 | 240 | 300 | 361 | 421 | 482 | 542 | 603 | 664 | 725 | 0 |
| 2 | 2 | 62 | 122 | 182 | 242 | 302 | 363 | 423 | 484 | 544 | 605 | 666 | 727 | 2 |
| 4 | 4 | 64 | 124 | 184 | 244 | 304 | 365 | 425 | 486 | 546 | 607 | 668 | 729 | 4 |
| 6 | 6 | 66 | 126 | 186 | 246 | 306 | 367 | 427 | 488 | 548 | 609 | 670 | 731 | 6 |
| 8 | 8 | 68 | 128 | 188 | 248 | 308 | 369 | 429 | 490 | 550 | 611 | 672 | 734 | 8 |
| 10 | 10 | 70 | 130 | 190 | 250 | 310 | 371 | 431 | 492 | 552 | 613 | 674 | 736 | 10 |
| 12 | 12 | 72 | 132 | 192 | 252 | 312 | 373 | 433 | 494 | 554 | 615 | 676 | 738 | 12 |
| 14 | 14 | 74 | 134 | 194 | 254 | 314 | 375 | 435 | 496 | 556 | 617 | 678 | 740 | 14 |
| 16 | 16 | 76 | 136 | 196 | 256 | 316 | 377 | 437 | 498 | 558 | 619 | 680 | 742 | 16 |
| 18 | 18 | 78 | 138 | 198 | 258 | 318 | 379 | 439 | 500 | 560 | 621 | 682 | 744 | 18 |
| 20 | 20 | 80 | 140 | 200 | 260 | 320 | 381 | 441 | 502 | 562 | 623 | 684 | 746 | 20 |
| 22 | 22 | 82 | 142 | 202 | 262 | 322 | 383 | 443 | 504 | 565 | 625 | 687 | 748 | 22 |
| 24 | 24 | 84 | 144 | 204 | 264 | 324 | 385 | 445 | 506 | 567 | 627 | 689 | 750 | 24 |
| 26 | 26 | 86 | 146 | 206 | 266 | 326 | 387 | 447 | 508 | 569 | 629 | 691 | 752 | 26 |
| 28 | 28 | 88 | 148 | 208 | 268 | 328 | 389 | 449 | 510 | 571 | 632 | 693 | 754 | 28 |
| 30 | 30 | 90 | 150 | 210 | 270 | 331 | 391 | 451 | 512 | 573 | 634 | 695 | 756 | 30 |
| 32 | 32 | 92 | 152 | 212 | 272 | 333 | 393 | 453 | 514 | 575 | 636 | 697 | 758 | 32 |
| 34 | 34 | 94 | 154 | 214 | 274 | 335 | 395 | 455 | 516 | 577 | 638 | 699 | 760 | 34 |
| 36 | 36 | 96 | 156 | 216 | 276 | 337 | 397 | 457 | 518 | 579 | 640 | 701 | 762 | 36 |
| 38 | 38 | 98 | 158 | 218 | 278 | 339 | 399 | 459 | 520 | 581 | 642 | 703 | 764 | 38 |
| 40 | 40 | 100 | 160 | 220 | 280 | 341 | 401 | 461 | 522 | 583 | 644 | 705 | 766 | 40 |
| 42 | 42 | 102 | 162 | 222 | 282 | 343 | 403 | 463 | 524 | 585 | 646 | 707 | 768 | 42 |
| 44 | 44 | 104 | 164 | 224 | 284 | 345 | 405 | 465 | 526 | 587 | 648 | 709 | 770 | 44 |
| 46 | 46 | 106 | 166 | 226 | 286 | 347 | 407 | 467 | 528 | 589 | 650 | 711 | 772 | 46 |
| 48 | 48 | 108 | 168 | 228 | 288 | 349 | 409 | 469 | 530 | 591 | 652 | 713 | 774 | 48 |
| 50 | 50 | 110 | 170 | 230 | 290 | 351 | 411 | 471 | 532 | 593 | 654 | 715 | 777 | 50 |
| 52 | 52 | 112 | 172 | 232 | 292 | 353 | 413 | 473 | 534 | 595 | 656 | 717 | 779 | 52 |
| 54 | 54 | 114 | 174 | 234 | 294 | 355 | 415 | 476 | 536 | 597 | 658 | 719 | 781 | 54 |
| 56 | 56 | 116 | 176 | 236 | 296 | 357 | 417 | 478 | 538 | 599 | 660 | 721 | 783 | 56 |
| 58 | 58 | 118 | 178 | 238 | 298 | 359 | 419 | 480 | 540 | 601 | 662 | 723 | 785 | 58 |
| M. | 0° | 1° | 2° | 3° | 4° | 5° | 6° | 7° | 8° | 9° | 10° | 11° | 12° | M. |

# TABLE I.

## MERIDIONAL PARTS.

| M. | 13° | 14° | 15° | 16° | 17° | 18° | 19° | 20° | 21° | 22° | M. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 787 | 848 | 910 | 973 | 1035 | 1098 | 1161 | 1225 | 1289 | 1354 | 0 |
| 2 | 789 | 851 | 913 | 975 | 1037 | 1100 | 1164 | 1227 | 1291 | 1356 | 2 |
| 4 | 791 | 853 | 915 | 977 | 1039 | 1102 | 1166 | 1229 | 1293 | 1358 | 4 |
| 6 | 793 | 855 | 917 | 979 | 1042 | 1105 | 1168 | 1232 | 1296 | 1360 | 6 |
| 8 | 795 | 857 | 919 | 981 | 1044 | 1107 | 1170 | 1234 | 1298 | 1362 | 8 |
| 10 | 797 | 859 | 921 | 983 | 1046 | 1109 | 1172 | 1236 | 1300 | 1364 | 10 |
| 12 | 799 | 861 | 923 | 985 | 1048 | 1111 | 1174 | 1238 | 1302 | 1367 | 12 |
| 14 | 801 | 863 | 925 | 987 | 1050 | 1113 | 1176 | 1240 | 1304 | 1369 | 14 |
| 16 | 803 | 865 | 927 | 989 | 1052 | 1115 | 1178 | 1242 | 1306 | 1371 | 16 |
| 18 | 805 | 867 | 929 | 991 | 1054 | 1117 | 1181 | 1245 | 1308 | 1373 | 18 |
| 20 | 807 | 869 | 931 | 994 | 1056 | 1119 | 1183 | 1246 | 1311 | 1375 | 20 |
| 22 | 809 | 871 | 933 | 996 | 1058 | 1121 | 1185 | 1249 | 1313 | 1377 | 22 |
| 24 | 811 | 873 | 935 | 998 | 1060 | 1123 | 1187 | 1251 | 1315 | 1380 | 24 |
| 26 | 813 | 875 | 937 | 1000 | 1063 | 1126 | 1189 | 1253 | 1317 | 1382 | 26 |
| 28 | 816 | 877 | 939 | 1002 | 1065 | 1128 | 1191 | 1255 | 1319 | 1384 | 28 |
| 30 | 818 | 879 | 942 | 1004 | 1067 | 1130 | 1193 | 1257 | 1321 | 1386 | 30 |
| 32 | 820 | 882 | 944 | 1006 | 1069 | 1132 | 1195 | 1259 | 1324 | 1388 | 32 |
| 34 | 822 | 884 | 946 | 1008 | 1071 | 1134 | 1198 | 1261 | 1326 | 1390 | 34 |
| 36 | 824 | 886 | 948 | 1010 | 1073 | 1136 | 1200 | 1264 | 1328 | 1393 | 36 |
| 38 | 826 | 888 | 950 | 1012 | 1075 | 1138 | 1202 | 1266 | 1330 | 1395 | 38 |
| 40 | 828 | 890 | 952 | 1014 | 1077 | 1140 | 1204 | 1268 | 1332 | 1397 | 40 |
| 42 | 830 | 892 | 954 | 1016 | 1079 | 1142 | 1206 | 1270 | 1334 | 1399 | 42 |
| 44 | 832 | 894 | 956 | 1019 | 1081 | 1145 | 1208 | 1272 | 1336 | 1401 | 44 |
| 46 | 834 | 896 | 958 | 1021 | 1084 | 1147 | 1210 | 1274 | 1339 | 1403 | 46 |
| 48 | 836 | 898 | 960 | 1023 | 1086 | 1149 | 1212 | 1276 | 1341 | 1406 | 48 |
| 50 | 838 | 900 | 962 | 1025 | 1088 | 1151 | 1215 | 1278 | 1343 | 1408 | 50 |
| 52 | 840 | 902 | 964 | 1027 | 1090 | 1153 | 1217 | 1281 | 1345 | 1410 | 52 |
| 54 | 842 | 904 | 966 | 1029 | 1092 | 1155 | 1219 | 1283 | 1347 | 1412 | 54 |
| 56 | 844 | 906 | 969 | 1031 | 1094 | 1157 | 1221 | 1285 | 1349 | 1414 | 56 |
| 58 | 846 | 908 | 971 | 1033 | 1096 | 1159 | 1223 | 1287 | 1352 | 1416 | 58 |
| M. | 13° | 14° | 15° | 16° | 17° | 18° | 19° | 20° | 21° | 22° | M. |

# TABLE I.

## MERIDIONAL PARTS.

| M. | 23° | 24° | 25° | 26° | 27° | 28° | 29° | 30° | 31° | 32° | M. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1419 | 1484 | 1550 | 1616 | 1684 | 1751 | 1819 | 1888 | 1958 | 2028 | 0 |
| 2 | 1421 | 1486 | 1552 | 1619 | 1686 | 1753 | 1822 | 1891 | 1960 | 2031 | 2 |
| 4 | 1423 | 1488 | 1554 | 1621 | 1688 | 1756 | 1824 | 1893 | 1963 | 2033 | 4 |
| 6 | 1425 | 1491 | 1557 | 1623 | 1690 | 1758 | 1826 | 1895 | 1965 | 2035 | 6 |
| 8 | 1427 | 1493 | 1559 | 1625 | 1693 | 1760 | 1829 | 1898 | 1967 | 2038 | 8 |
| 10 | 1430 | 1495 | 1561 | 1628 | 1695 | 1762 | 1831 | 1900 | 1970 | 2040 | 10 |
| 12 | 1432 | 1497 | 1563 | 1630 | 1697 | 1765 | 1833 | 1902 | 1972 | 2043 | 12 |
| 14 | 1434 | 1499 | 1565 | 1632 | 1699 | 1767 | 1835 | 1905 | 1974 | 2045 | 14 |
| 16 | 1436 | 1502 | 1568 | 1634 | 1701 | 1769 | 1838 | 1907 | 1977 | 2047 | 16 |
| 18 | 1438 | 1504 | 1570 | 1637 | 1704 | 1772 | 1840 | 1909 | 1979 | 2050 | 18 |
| 20 | 1440 | 1506 | 1572 | 1639 | 1706 | 1774 | 1842 | 1912 | 1981 | 2052 | 20 |
| 22 | 1443 | 1508 | 1574 | 1641 | 1708 | 1776 | 1845 | 1914 | 1984 | 2054 | 22 |
| 24 | 1445 | 1510 | 1577 | 1643 | 1711 | 1778 | 1847 | 1916 | 1986 | 2057 | 24 |
| 26 | 1447 | 1513 | 1579 | 1645 | 1713 | 1781 | 1849 | 1918 | 1988 | 2059 | 26 |
| 28 | 1449 | 1515 | 1581 | 1648 | 1715 | 1783 | 1852 | 1921 | 1991 | 2061 | 28 |
| 30 | 1451 | 1517 | 1583 | 1650 | 1717 | 1785 | 1854 | 1923 | 1993 | 2064 | 30 |
| 32 | 1453 | 1519 | 1585 | 1652 | 1720 | 1787 | 1856 | 1925 | 1995 | 2066 | 32 |
| 34 | 1456 | 1521 | 1588 | 1654 | 1722 | 1790 | 1858 | 1928 | 1998 | 2069 | 34 |
| 36 | 1458 | 1524 | 1590 | 1657 | 1724 | 1792 | 1861 | 1930 | 2000 | 2071 | 36 |
| 38 | 1460 | 1526 | 1592 | 1659 | 1726 | 1794 | 1863 | 1932 | 2002 | 2073 | 38 |
| 40 | 1462 | 1528 | 1594 | 1661 | 1729 | 1797 | 1865 | 1935 | 2005 | 2076 | 40 |
| 42 | 1464 | 1530 | 1596 | 1663 | 1731 | 1799 | 1868 | 1937 | 2007 | 2078 | 42 |
| 44 | 1467 | 1532 | 1599 | 1666 | 1733 | 1801 | 1870 | 1939 | 2010 | 2080 | 44 |
| 46 | 1469 | 1535 | 1601 | 1668 | 1735 | 1803 | 1872 | 1942 | 2012 | 2083 | 46 |
| 48 | 1471 | 1537 | 1603 | 1670 | 1738 | 1806 | 1875 | 1944 | 2014 | 2085 | 48 |
| 50 | 1473 | 1539 | 1605 | 1672 | 1740 | 1808 | 1877 | 1946 | 2017 | 2088 | 50 |
| 52 | 1475 | 1541 | 1608 | 1675 | 1742 | 1810 | 1879 | 1949 | 2019 | 2090 | 52 |
| 54 | 1477 | 1543 | 1610 | 1677 | 1744 | 1813 | 1881 | 1951 | 2021 | 2092 | 54 |
| 56 | 1480 | 1546 | 1612 | 1679 | 1747 | 1815 | 1884 | 1953 | 2024 | 2095 | 56 |
| 58 | 1482 | 1548 | 1614 | 1681 | 1749 | 1817 | 1886 | 1956 | 2026 | 2097 | 58 |
| M. | 23° | 24° | 25° | 26° | 27° | 28° | 29° | 30° | 31° | 32° | M. |

# TABLE I.

## MERIDIONAL PARTS.

| M. | 33° | 34° | 35° | 36° | 37° | 38° | 39° | 40° | 41° | 42° | M. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 2100 | 2171 | 2244 | 2318 | 2393 | 2468 | 2545 | 2623 | 2702 | 2782 | 0 |
| 2 | 2102 | 2174 | 2247 | 2320 | 2395 | 2471 | 2548 | 2625 | 2704 | 2784 | 2 |
| 4 | 2104 | 2176 | 2249 | 2323 | 2398 | 2473 | 2550 | 2628 | 2707 | 2787 | 4 |
| 6 | 2107 | 2179 | 2252 | 2325 | 2400 | 2476 | 2553 | 2631 | 2710 | 2790 | 6 |
| 8 | 2109 | 2181 | 2254 | 2328 | 2403 | 2478 | 2555 | 2633 | 2712 | 2792 | 8 |
| 10 | 2111 | 2184 | 2257 | 2330 | 2405 | 2481 | 2558 | 2636 | 2715 | 2795 | 10 |
| 12 | 2114 | 2186 | 2259 | 2333 | 2408 | 2484 | 2560 | 2638 | 2718 | 2798 | 12 |
| 14 | 2116 | 2188 | 2261 | 2335 | 2410 | 2486 | 2563 | 2641 | 2720 | 2801 | 14 |
| 16 | 2119 | 2191 | 2264 | 2338 | 2413 | 2489 | 2566 | 2644 | 2723 | 2803 | 16 |
| 18 | 2121 | 2193 | 2266 | 2340 | 2415 | 2491 | 2568 | 2646 | 2726 | 2806 | 18 |
| 20 | 2123 | 2196 | 2269 | 2343 | 2418 | 2494 | 2571 | 2649 | 2728 | 2809 | 20 |
| 22 | 2126 | 2198 | 2271 | 2345 | 2420 | 2496 | 2573 | 2651 | 2731 | 2811 | 22 |
| 24 | 2128 | 2200 | 2274 | 2348 | 2423 | 2499 | 2576 | 2654 | 2733 | 2814 | 24 |
| 26 | 2131 | 2203 | 2276 | 2350 | 2425 | 2501 | 2578 | 2657 | 2736 | 2817 | 26 |
| 28 | 2133 | 2205 | 2279 | 2353 | 2428 | 2504 | 2581 | 2659 | 2739 | 2820 | 28 |
| 30 | 2135 | 2208 | 2281 | 2355 | 2430 | 2506 | 2584 | 2662 | 2742 | 2822 | 30 |
| 32 | 2138 | 2210 | 2283 | 2358 | 2433 | 2509 | 2586 | 2665 | 2744 | 2825 | 32 |
| 34 | 2140 | 2213 | 2286 | 2360 | 2435 | 2512 | 2589 | 2667 | 2747 | 2828 | 34 |
| 36 | 2143 | 2215 | 2288 | 2363 | 2438 | 2514 | 2591 | 2670 | 2750 | 2830 | 36 |
| 38 | 2145 | 2217 | 2291 | 2365 | 2440 | 2517 | 2594 | 2673 | 2752 | 2833 | 38 |
| 40 | 2147 | 2220 | 2293 | 2368 | 2443 | 2519 | 2597 | 2675 | 2755 | 2836 | 40 |
| 42 | 2150 | 2222 | 2296 | 2370 | 2445 | 2522 | 2599 | 2678 | 2758 | 2839 | 42 |
| 44 | 2152 | 2225 | 2298 | 2373 | 2448 | 2524 | 2602 | 2680 | 2760 | 2841 | 44 |
| 46 | 2155 | 2227 | 2301 | 2375 | 2451 | 2527 | 2604 | 2683 | 2763 | 2844 | 46 |
| 48 | 2157 | 2230 | 2303 | 2378 | 2453 | 2530 | 2607 | 2686 | 2766 | 2847 | 48 |
| 50 | 2159 | 2232 | 2306 | 2380 | 2456 | 2532 | 2610 | 2688 | 2768 | 2849 | 50 |
| 52 | 2162 | 2235 | 2308 | 2383 | 2458 | 2535 | 2612 | 2691 | 2771 | 2852 | 52 |
| 54 | 2164 | 2237 | 2311 | 2385 | 2461 | 2537 | 2615 | 2694 | 2774 | 2855 | 54 |
| 56 | 2167 | 2239 | 2313 | 2388 | 2463 | 2540 | 2617 | 2696 | 2776 | 2858 | 56 |
| 58 | 2169 | 2242 | 2316 | 2390 | 2466 | 2542 | 2620 | 2699 | 2779 | 2860 | 58 |
| M. | 33° | 34° | 35° | 36° | 37° | 38° | 39° | 40° | 41° | 42° | M. |

# TABLE I.

## MERIDIONAL PARTS.

| M. | 43° | 44° | 45° | 46° | 47° | 48° | 49° | 50° | 51° | 52° | M. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 2863 | 2946 | 3030 | 3116 | 3203 | 3292 | 3382 | 3474 | 3569 | 3665 | 0 |
| 2 | 2866 | 2949 | 3033 | 3118 | 3206 | 3295 | 3385 | 3478 | 3572 | 3668 | 2 |
| 4 | 2869 | 2951 | 3036 | 3121 | 3209 | 3298 | 3388 | 3481 | 3575 | 3672 | 4 |
| 6 | 2871 | 2954 | 3038 | 3124 | 3212 | 3301 | 3391 | 3484 | 3578 | 3675 | 6 |
| 8 | 2874 | 2957 | 3041 | 3127 | 3214 | 3303 | 3394 | 3487 | 3582 | 3678 | 8 |
| 10 | 2877 | 2960 | 3044 | 3130 | 3217 | 3306 | 3397 | 3490 | 3585 | 3681 | 10 |
| 12 | 2880 | 2963 | 3047 | 3133 | 3220 | 3309 | 3400 | 3493 | 3588 | 3685 | 12 |
| 14 | 2882 | 2965 | 3050 | 3136 | 3223 | 3312 | 3403 | 3496 | 3591 | 3688 | 14 |
| 16 | 2885 | 2968 | 3053 | 3139 | 3226 | 3316 | 3407 | 3499 | 3594 | 3691 | 16 |
| 18 | 2888 | 2971 | 3055 | 3142 | 3229 | 3319 | 3410 | 3503 | 3598 | 3695 | 18 |
| 20 | 2891 | 2974 | 3058 | 3144 | 3232 | 3322 | 3413 | 3506 | 3601 | 3698 | 20 |
| 22 | 2893 | 2976 | 3061 | 3147 | 3235 | 3325 | 3416 | 3509 | 3604 | 3701 | 22 |
| 24 | 2896 | 2979 | 3064 | 3150 | 3238 | 3328 | 3419 | 3512 | 3607 | 3704 | 24 |
| 26 | 2899 | 2982 | 3067 | 3153 | 3241 | 3331 | 3422 | 3515 | 3610 | 3708 | 26 |
| 28 | 2902 | 2985 | 3070 | 3156 | 3244 | 3334 | 3425 | 3518 | 3614 | 3711 | 28 |
| 30 | 2904 | 2988 | 3073 | 3159 | 3247 | 3337 | 3428 | 3521 | 3617 | 3714 | 30 |
| 32 | 2907 | 2991 | 3075 | 3162 | 3250 | 3340 | 3431 | 3525 | 3620 | 3717 | 32 |
| 34 | 2910 | 2993 | 3078 | 3165 | 3253 | 3343 | 3434 | 3528 | 3623 | 3721 | 34 |
| 36 | 2913 | 2996 | 3081 | 3168 | 3256 | 3346 | 3437 | 3531 | 3626 | 3724 | 36 |
| 38 | 2915 | 2999 | 3084 | 3171 | 3259 | 3349 | 3440 | 3534 | 3630 | 3727 | 38 |
| 40 | 2918 | 3002 | 3087 | 3173 | 3262 | 3352 | 3443 | 3537 | 3633 | 3731 | 40 |
| 42 | 2921 | 3005 | 3090 | 3176 | 3265 | 3355 | 3447 | 3540 | 3636 | 3734 | 42 |
| 44 | 2924 | 3007 | 3093 | 3179 | 3268 | 3358 | 3450 | 3543 | 3639 | 3737 | 44 |
| 46 | 2926 | 3010 | 3095 | 3182 | 3271 | 3361 | 3453 | 3547 | 3643 | 3741 | 46 |
| 48 | 2929 | 3013 | 3098 | 3185 | 3274 | 3364 | 3456 | 3550 | 3646 | 3744 | 48 |
| 50 | 2932 | 3016 | 3101 | 3188 | 3277 | 3367 | 3459 | 3553 | 3649 | 3747 | 50 |
| 52 | 2935 | 3019 | 3104 | 3191 | 3280 | 3370 | 3462 | 3556 | 3652 | 3750 | 52 |
| 54 | 2937 | 3021 | 3107 | 3194 | 3283 | 3373 | 3465 | 3559 | 3655 | 3754 | 54 |
| 56 | 2940 | 3024 | 3110 | 3197 | 3286 | 3376 | 3468 | 3562 | 3659 | 3757 | 56 |
| 58 | 2943 | 3027 | 3113 | 3200 | 3289 | 3379 | 3471 | 3566 | 3662 | 3760 | 58 |
| M. | 43° | 44° | 45° | 46° | 47° | 48° | 49° | 50° | 51° | 52° | M. |

# TABLE I.

## MERIDIONAL PARTS.

| M. | 53° | 54° | 55° | 56° | 57° | 58° | 59° | 60° | 61° | 62° | M. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 3764 | 3865 | 3968 | 4074 | 4183 | 4294 | 4409 | 4527 | 4649 | 4775 | 0 |
| 2 | 3767 | 3868 | 3971 | 4077 | 4186 | 4298 | 4413 | 4531 | 4653 | 4779 | 2 |
| 4 | 3770 | 3871 | 3975 | 4081 | 4190 | 4302 | 4417 | 4535 | 4657 | 4784 | 4 |
| 6 | 3774 | 3875 | 3978 | 4085 | 4194 | 4306 | 4421 | 4539 | 4662 | 4788 | 6 |
| 8 | 3777 | 3878 | 3982 | 4088 | 4197 | 4309 | 4425 | 4543 | 4666 | 4792 | 8 |
| 10 | 3780 | 3882 | 3985 | 4092 | 4201 | 4313 | 4429 | 4547 | 4670 | 4796 | 10 |
| 12 | 3784 | 3885 | 3989 | 4095 | 4205 | 4317 | 4433 | 4551 | 4674 | 4801 | 12 |
| 14 | 3787 | 3889 | 3992 | 4099 | 4208 | 4321 | 4436 | 4555 | 4678 | 4805 | 14 |
| 16 | 3790 | 3892 | 3996 | 4103 | 4212 | 4325 | 4440 | 4559 | 4682 | 4809 | 16 |
| 18 | 3794 | 3895 | 3999 | 4106 | 4216 | 4328 | 4444 | 4564 | 4687 | 4814 | 18 |
| 20 | 3797 | 3899 | 4003 | 4110 | 4220 | 4332 | 4448 | 4568 | 4691 | 4818 | 20 |
| 22 | 3800 | 3902 | 4006 | 4113 | 4223 | 4336 | 4452 | 4572 | 4695 | 4822 | 22 |
| 24 | 3804 | 3906 | 4010 | 4117 | 4227 | 4340 | 4456 | 4576 | 4699 | 4826 | 24 |
| 26 | 3807 | 3909 | 4014 | 4121 | 4231 | 4344 | 4460 | 4580 | 4703 | 4831 | 26 |
| 28 | 3811 | 3913 | 4017 | 4124 | 4234 | 4347 | 4464 | 4584 | 4707 | 4835 | 28 |
| 30 | 3814 | 3916 | 4021 | 4128 | 4238 | 4351 | 4468 | 4588 | 4712 | 4839 | 30 |
| 32 | 3817 | 3919 | 4024 | 4132 | 4242 | 4355 | 4472 | 4592 | 4716 | 4844 | 32 |
| 34 | 3821 | 3923 | 4028 | 4135 | 4246 | 4359 | 4476 | 4596 | 4720 | 4848 | 34 |
| 36 | 3824 | 3926 | 4031 | 4139 | 4249 | 4363 | 4480 | 4600 | 4724 | 4852 | 36 |
| 38 | 3827 | 3930 | 4035 | 4142 | 4253 | 4367 | 4484 | 4604 | 4728 | 4857 | 38 |
| 40 | 3831 | 3933 | 4038 | 4146 | 4257 | 4370 | 4488 | 4608 | 4733 | 4861 | 40 |
| 42 | 3834 | 3937 | 4042 | 4150 | 4260 | 4374 | 4492 | 4612 | 4737 | 4865 | 42 |
| 44 | 3838 | 3940 | 4045 | 4153 | 4264 | 4378 | 4495 | 4616 | 4741 | 4870 | 44 |
| 46 | 3841 | 3944 | 4049 | 4157 | 4268 | 4382 | 4499 | 4620 | 4745 | 4874 | 46 |
| 48 | 3844 | 3947 | 4052 | 4161 | 4272 | 4386 | 4503 | 4625 | 4750 | 4879 | 48 |
| 50 | 3848 | 3951 | 4056 | 4164 | 4275 | 4390 | 4507 | 4629 | 4754 | 4883 | 50 |
| 52 | 3851 | 3954 | 4060 | 4168 | 4279 | 4394 | 4511 | 4633 | 4758 | 4887 | 52 |
| 54 | 3854 | 3958 | 4063 | 4172 | 4283 | 4398 | 4515 | 4637 | 4762 | 4892 | 54 |
| 56 | 3858 | 3961 | 4067 | 4175 | 4287 | 4401 | 4519 | 4641 | 4766 | 4896 | 56 |
| 58 | 3861 | 3964 | 4070 | 4179 | 4291 | 4405 | 4523 | 4645 | 4771 | 4901 | 58 |
| M. | 53° | 54° | 55° | 56° | 57° | 58° | 59° | 60° | 61° | 62° | M. |

# TABLE I.

## MERIDIONAL PARTS.

| M. | 63° | 64° | 65° | 66° | 67° | 68° | 69° | 70° | 71° | 72° | M. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 4905 | 5039 | 5179 | 5324 | 5474 | 5631 | 5795 | 5966 | 6146 | 6335 | 0 |
| 2 | 4909 | 5044 | 5184 | 5328 | 5479 | 5636 | 5800 | 5972 | 6152 | 6341 | 2 |
| 4 | 4914 | 5049 | 5188 | 5333 | 5484 | 5642 | 5806 | 5978 | 6158 | 6348 | 4 |
| 6 | 4918 | 5053 | 5193 | 5338 | 5489 | 5647 | 5811 | 5984 | 6164 | 6354 | 6 |
| 8 | 4923 | 5058 | 5198 | 5343 | 5495 | 5652 | 5817 | 5989 | 6170 | 6361 | 8 |
| 10 | 4927 | 5062 | 5203 | 5348 | 5500 | 5658 | 5823 | 5995 | 6177 | 6367 | 10 |
| 12 | 4931 | 5067 | 5207 | 5353 | 5505 | 5663 | 5828 | 6001 | 6183 | 6374 | 12 |
| 14 | 4936 | 5071 | 5212 | 5358 | 5510 | 5668 | 5834 | 6007 | 6189 | 6380 | 14 |
| 16 | 4940 | 5076 | 5217 | 5363 | 5515 | 5674 | 5839 | 6013 | 6195 | 6387 | 16 |
| 18 | 4945 | 5081 | 5222 | 5368 | 5520 | 5679 | 5845 | 6019 | 6201 | 6394 | 18 |
| 20 | 4949 | 5085 | 5226 | 5373 | 5526 | 5685 | 5851 | 6025 | 6208 | 6400 | 20 |
| 22 | 4954 | 5090 | 5231 | 5378 | 5531 | 5690 | 5856 | 6031 | 6214 | 6407 | 22 |
| 24 | 4958 | 5095 | 5236 | 5383 | 5536 | 5695 | 5862 | 6037 | 6220 | 6413 | 24 |
| 26 | 4963 | 5099 | 5241 | 5388 | 5541 | 5701 | 5868 | 6043 | 6226 | 6420 | 26 |
| 28 | 4967 | 5104 | 5246 | 5393 | 5546 | 5706 | 5874 | 6049 | 6233 | 6427 | 28 |
| 30 | 4972 | 5108 | 5250 | 5398 | 5552 | 5712 | 5879 | 6055 | 6239 | 6433 | 30 |
| 32 | 4976 | 5113 | 5255 | 5403 | 5557 | 5717 | 5885 | 6061 | 6245 | 6440 | 32 |
| 34 | 4981 | 5118 | 5260 | 5408 | 5562 | 5723 | 5891 | 6067 | 6252 | 6447 | 34 |
| 36 | 4985 | 5122 | 5265 | 5413 | 5567 | 5728 | 5896 | 6073 | 6258 | 6453 | 36 |
| 38 | 4990 | 5127 | 5270 | 5418 | 5573 | 5734 | 5902 | 6079 | 6264 | 6460 | 38 |
| 40 | 4994 | 5132 | 5275 | 5423 | 5578 | 5739 | 5908 | 6085 | 6271 | 6467 | 40 |
| 42 | 4999 | 5136 | 5280 | 5428 | 5583 | 5745 | 5914 | 6091 | 6277 | 6473 | 42 |
| 44 | 5003 | 5141 | 5284 | 5433 | 5588 | 5750 | 5919 | 6097 | 6283 | 6480 | 44 |
| 46 | 5008 | 5146 | 5289 | 5438 | 5594 | 5756 | 5925 | 6103 | 6290 | 6487 | 46 |
| 48 | 5012 | 5151 | 5294 | 5443 | 5599 | 5761 | 5931 | 6109 | 6296 | 6494 | 48 |
| 50 | 5017 | 5155 | 5299 | 5448 | 5604 | 5767 | 5937 | 6115 | 6303 | 6500 | 50 |
| 52 | 5021 | 5160 | 5304 | 5454 | 5610 | 5772 | 5943 | 6121 | 6309 | 6507 | 52 |
| 54 | 5026 | 5165 | 5309 | 5459 | 5615 | 5778 | 5948 | 6127 | 6315 | 6514 | 54 |
| 56 | 5030 | 5169 | 5314 | 5464 | 5620 | 5783 | 5954 | 6133 | 6322 | 6521 | 56 |
| 58 | 5035 | 5174 | 5319 | 5469 | 5625 | 5789 | 5960 | 6140 | 6328 | 6528 | 58 |
| M. | 63° | 64° | 65° | 66° | 67° | 68° | 69° | 70° | 71° | 72° | M. |

# TABLE I.

## MERIDIONAL PARTS.

| M. | 73° | 74° | 75° | 76° | 77° | 78° | 79° | 80° | 81° | 82° | M. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 6534 | 6746 | 6970 | 7210 | 7467 | 7745 | 8046 | 8375 | 8739 | 9145 | 0 |
| 2 | 6541 | 6753 | 6978 | 7218 | 7476 | 7754 | 8056 | 8387 | 8752 | 9160 | 2 |
| 4 | 6548 | 6760 | 6986 | 7227 | 7485 | 7764 | 8067 | 8398 | 8765 | 9174 | 4 |
| 6 | 6555 | 6768 | 6994 | 7235 | 7494 | 7774 | 8077 | 8410 | 8778 | 9189 | 6 |
| 8 | 6562 | 6775 | 7001 | 7243 | 7503 | 7783 | 8088 | 8422 | 8791 | 9203 | 8 |
| 10 | 6569 | 6782 | 7009 | 7252 | 7512 | 7793 | 8099 | 8433 | 8804 | 9218 | 10 |
| 12 | 6576 | 6790 | 7017 | 7260 | 7521 | 7803 | 8109 | 8445 | 8817 | 9233 | 12 |
| 14 | 6583 | 6797 | 7025 | 7268 | 7530 | 7813 | 8120 | 8457 | 8830 | 9248 | 14 |
| 16 | 6590 | 6804 | 7033 | 7277 | 7539 | 7822 | 8131 | 8469 | 8843 | 9262 | 16 |
| 18 | 6597 | 6812 | 7041 | 7285 | 7548 | 7832 | 8141 | 8480 | 8856 | 9277 | 18 |
| 20 | 6603 | 6819 | 7048 | 7294 | 7557 | 7842 | 8152 | 8492 | 8869 | 9292 | 20 |
| 22 | 6610 | 6826 | 7056 | 7302 | 7566 | 7852 | 8163 | 8504 | 8883 | 9307 | 22 |
| 24 | 6617 | 6834 | 7064 | 7311 | 7576 | 7862 | 8174 | 8516 | 8896 | 9322 | 24 |
| 26 | 6624 | 6841 | 7072 | 7319 | 7585 | 7872 | 8185 | 8528 | 8909 | 9337 | 26 |
| 28 | 6631 | 6849 | 7080 | 7328 | 7594 | 7882 | 8196 | 8540 | 8923 | 9353 | 28 |
| 30 | 6639 | 6856 | 7088 | 7336 | 7603 | 7892 | 8207 | 8552 | 8936 | 9368 | 30 |
| 32 | 6646 | 6864 | 7096 | 7345 | 7612 | 7902 | 8218 | 8565 | 8950 | 9383 | 32 |
| 34 | 6653 | 6871 | 7104 | 7353 | 7622 | 7912 | 8229 | 8577 | 8963 | 9399 | 34 |
| 36 | 6660 | 6879 | 7112 | 7362 | 7631 | 7922 | 8240 | 8589 | 8977 | 9414 | 36 |
| 38 | 6667 | 6886 | 7120 | 7371 | 7640 | 7932 | 8251 | 8601 | 8991 | 9430 | 38 |
| 40 | 6674 | 6894 | 7128 | 7379 | 7650 | 7942 | 8262 | 8614 | 9005 | 9445 | 40 |
| 42 | 6681 | 6901 | 7136 | 7388 | 7659 | 7953 | 8273 | 8626 | 9018 | 9461 | 42 |
| 44 | 6688 | 6909 | 7145 | 7397 | 7668 | 7963 | 8284 | 8638 | 9032 | 9477 | 44 |
| 46 | 6695 | 6917 | 7153 | 7406 | 7678 | 7973 | 8295 | 8651 | 9046 | 9493 | 46 |
| 48 | 6702 | 6924 | 7161 | 7414 | 7687 | 7983 | 8307 | 8663 | 9060 | 9509 | 48 |
| 50 | 6710 | 6932 | 7169 | 7423 | 7697 | 7994 | 8318 | 8676 | 9074 | 9525 | 50 |
| 52 | 6717 | 6939 | 7177 | 7432 | 7706 | 8004 | 8329 | 8688 | 9088 | 9541 | 52 |
| 54 | 6724 | 6947 | 7185 | 7441 | 7716 | 8014 | 8341 | 8701 | 9103 | 9557 | 54 |
| 56 | 6731 | 6955 | 7194 | 7449 | 7725 | 8025 | 8352 | 8714 | 9117 | 9573 | 56 |
| 58 | 6738 | 6963 | 7202 | 7458 | 7735 | 8035 | 8364 | 8726 | 9131 | 9589 | 58 |
| M. | 73° | 74° | 75° | 76° | 77° | 78° | 79° | 80° | 81° | 82° | M. |

## TABLE II.

*Depression of the Horizon of the Sea.*

| Height of the Eye in feet. | Depression. |
|---|---|
| 1 | 0′ 59″ |
| 2 | 1 24 |
| 3 | 1 42 |
| 4 | 1 58 |
| 5 | 2 12 |
| 6 | 2 25 |
| 7 | 2 36 |
| 8 | 2 47 |
| 9 | 2 57 |
| 10 | 3 7 |
| 11 | 3 16 |
| 12 | 3 25 |
| 13 | 3 33 |
| 14 | 3 41 |
| 15 | 3 48 |
| 16 | 3 56 |
| 17 | 4 3 |
| 18 | 4 10 |
| 19 | 4 17 |
| 20 | 4 24 |
| 22 | 4 37 |
| 24 | 4 49 |
| 26 | 5 1 |
| 28 | 5 13 |
| 30 | 5 23 |
| 35 | 5 49 |
| 40 | 6 14 |
| 45 | 6 36 |
| 50 | 6 57 |
| 60 | 7 37 |
| 70 | 8 14 |
| 80 | 8 48 |
| 90 | 9 20 |
| 100 | 9 50 |
| 120 | 10 47 |
| 140 | 11 39 |
| 160 | 12 27 |
| 180 | 13 12 |
| 200 | 13 55 |

## TABLE III.

*Dip of the sea at different Distances from the Observer.*

| Dist. of land in sea miles. | Height of the Eye above the Sea, in feet. | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 |
| 0 ¼ | 11′ | 22′ | 34 | 45′ | 56′ | 68 | 79′ | 90 |
| 0 ½ | 6 | 11 | 17 | 22 | 28 | 34 | 39 | 45 |
| 0 ¾ | 4 | 8 | 12 | 15 | 19 | 23 | 27 | 30 |
| 1 0 | 4 | 6 | 9 | 12 | 15 | 17 | 20 | 23 |
| 1 ¼ | 3 | 5 | 7 | 9 | 12 | 14 | 16 | 19 |
| 1 ½ | 3 | 4 | 6 | 8 | 10 | 11 | 14 | 15 |
| 2 0 | 2 | 3 | 5 | 6 | 8 | 10 | 11 | 12 |
| 2 ½ | 2 | 3 | 5 | 6 | 7 | 8 | 9 | 10 |
| 3 0 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 8 |
| 3 ½ | 2 | 3 | 4 | 5 | 6 | 6 | 7 | 7 |
| 4 0 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 7 |
| 5 0 | 2 | 3 | 4 | 4 | 5 | 5 | 6 | 6 |
| 6 0 | 2 | 3 | 4 | 4 | 5 | 5 | 6 | 6 |

## TABLE IV.

*Curvature of the Earth.*

| Dist. in miles. | Height. Inches. | Dist. in miles. | Height. Feet. |
|---|---|---|---|
| ¼ | ½ | 15 | 149 |
| ½ | 2 | 16 | 170 |
| 1 | 8 | 17 | 192 |
| | Feet. | 18 | 215 |
| 2 | 2.6 | 19 | 240 |
| 3 | 6. | 20 | 266 |
| 4 | 10.6 | 25 | 415 |
| 5 | 16.6 | 30 | 599 |
| 6 | 23.9 | 35 | 814 |
| 7 | 32.5 | 40 | 1064 |
| 8 | 42.5 | 45 | 1346 |
| 9 | 53.8 | 50 | 1662 |
| 10 | 66.4 | 60 | 2394 |
| 11 | 80.2 | 70 | 3258 |
| 12 | 95.4 | 80 | 4255 |
| 13 | 112. | 90 | 5386 |
| 14 | 130. | 100 | 6649 |

# TABLE V.

*Distances at which Objects can be seen at Sea.*

| Height in feet. | Distance in Eng. miles. | Height in feet. | Distance in Eng. miles. |
|---|---|---|---|
| 1 | 1.3 | 60 | 10.2 |
| 2 | 1.9 | 70 | 11.1 |
| 3 | 2.3 | 80 | 11.8 |
| 4 | 2.6 | 90 | 12.5 |
| 5 | 2.9 | 100 | 13.2 |
| 6 | 3.2 | 200 | 18.7 |
| 7 | 3.5 | 300 | 22.9 |
| 8 | 3.7 | 400 | 26.5 |
| 9 | 4. | 500 | 29.6 |
| 10 | 4.2 | 600 | 32.4 |
| 12 | 4.6 | 700 | 35. |
| 14 | 4.9 | 800 | 37.4 |
| 16 | 5.3 | 900 | 39.7 |
| 18 | 5.6 | 1000 | 41.8 |
| 20 | 5.9 | 2000 | 59.2 |
| 25 | 6.6 | 3000 | 72.5 |
| 30 | 7.3 | 4000 | 83.7 |
| 35 | 7.8 | 5000 | 93.5 |
| 40 | 8.4 | 10000 | 133. |
| 45 | 8.9 | 15000 | 163. |
| 50 | 9.4 | 20000 | 188. |

## TABLE VI.

*The Polar Distance and Right Ascension of the Pole Star.*

| | Polar Distance. | | | Ann. Var. | Right Ascension. | | | Ann. Var. |
|---|---|---|---|---|---|---|---|---|
| | | | | | h. | m. | s. | s. |
| 1800 | 1° | 45 | 35″ | —19″.5 | 0 | 52 | 24 | +12.9 |
| 1801 | 1 | 45 | 15 | | | 52 | 37 | |
| 1802 | 1 | 44 | 56 | | | 52 | 50 | |
| 1803 | 1 | 44 | 36 | | | 53 | 3 | |
| 1804 | 1 | 44 | 17 | | | 53 | 16 | |
| 1805 | 1 | 43 | 57 | | | 53 | 29 | |
| 1806 | 1 | 43 | 38 | | | 53 | 42 | |
| 1807 | 1 | 43 | 18 | | | 53 | 55 | |
| 1808 | 1 | 42 | 58 | | | 54 | 9 | |
| 1809 | 1 | 42 | 39 | | | 54 | 22 | s. |
| 1810 | 1 | 42 | 19 | | | 54 | 36 | +13.6 |
| 1811 | 1 | 42 | | | | 54 | 50 | |
| 1812 | 1 | 41 | 40 | | | 55 | 04 | |
| 1813 | 1 | 41 | 21 | | | 55 | 18 | |
| 1814 | 1 | 41 | 1 | | | 55 | 33 | |
| 1815 | 1 | 40 | 42 | | | 55 | 47 | |
| 1816 | 1 | 40 | 23 | | | 56 | 02 | |
| 1817 | 1 | 40 | 04 | | | 56 | 17 | |
| 1818 | 1 | 39 | 45 | | | 56 | 32 | |
| 1819 | 1 | 39 | 25 | | | 56 | 46 | s. |
| 1820 | 1 | 39 | 05 | —19″.4 | | 57 | 01 | +14.3 |

[illegible]