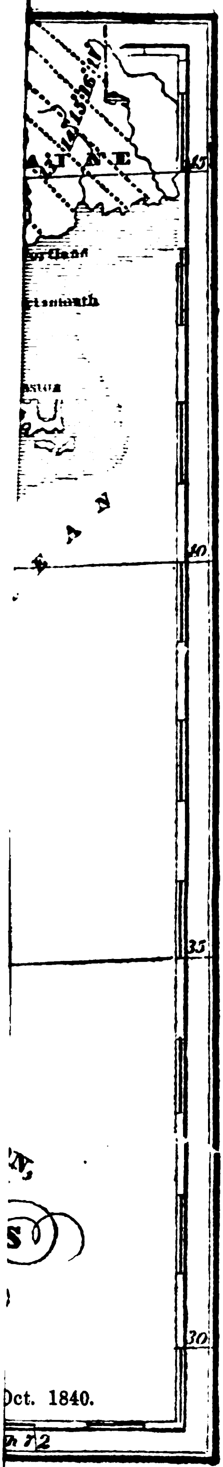
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A

TREATISE

ON

LAND-SURVEYING:

COMPRISING

THE THEORY

DEVELOPED FROM FIVE ELEMENTARY PRINCIPLES;

AND THE PRACTICE

WITH THE CHAIN ALONE, THE COMPASS, THE TRANSIT, THE THEODOLITE, THE PLANE TABLE, &c.

ILLUSTRATED BY
FOUR HUNDRED ENGRAVINGS,
AND A MAGNETIC CHART.

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PREFACE.

LAND-SURVEYING is perhaps the oldest of the mathematical arts. Indeed, Geometry itself, as its name—"Land-measuring"—implies, is said to have arisen from the efforts of the Egyptian sages to recover and to fix the land-marks annually swept away by the inundations of the Nile. The art is also one of the most important at the present day, as determining the title to land, the foundation of the whole wealth of the world. It is besides one of the most useful as a study, from its striking exemplifications of the practical bearings of abstract mathematics. But, strangely enough, Surveying has never yet been reduced to a systematic and symmetric whole. To effect this, by basing the art on a few simple principles, and tracing them out into their complicated ramifications and varied applications (which extend from the measurement of "a mowng lot" to that of the Heavens), has been the earnest endeavor of the present writer.

The work, in its inception, grew out of the author's own needs. Teaching Surveying, as preliminary to a course of Civil Engineering, he found none of the books in use (though very excellent in many respects) suited to his purpose. He was therefore compelled to teach the subject by a combination of familiar lectures on its principles and exemplifications of its practice. His notes continually swelling in bulk, gradually became systematized in nearly their present form, and in 1851 he printed a synopsis of them for the use of his classes. His system has thus been fully tested, and the present volume is the result.

A double object has been kept in view in its preparation; viz. to produce a very plain introduction to the subject, easy to be mastered by the young scholar or the practical man of little previous acquirement, the only pre-requisites being arithmetic and a little geometry; and at the same time to make the instruction of such a character as to lay a foundation broad enough and deep enough for the most complete superstructure which the professional student may subsequently wish to raise upon it.

For the convenience of those wishing to make a hasty examination of the book, a summary of some of its leading points and most peculiar features will here be given

I. *All the operations of Surveying are deduced from only five simple principles.* These principles are enunciated and illustrated in Chapter 1, of Part I. They will be at once recognized by the Geometer as familiar systems of "Co-ordinates;" but they were not here arbitrarily assumed in advance. They were arrived at most practically by analyzing all the numerous and incongruous methods and contrivances employed in Surveying, and rejecting, one after another, all extraneous and non-essential portions, thus reducing down the operations, one by one and step by step, to more and more general and comprehensive laws, till at last, by continual elimination, they were unexpectedly resolved into these few and simple principles; upon which it is here attempted to build up a symmetrical system.

II. The three operations common to *all* kinds of Land-surveying, viz. Making the Measurements, Drawing the Maps, and Calculating the Contents, are fully examined *in advance*, in Part I, Chapters 2, 3, 4, so that when the various methods of Surveying are subsequently taken up, only the few new points which are peculiar to each, require to be explained.

Each kind of Surveying, founded on one of the five fundamental principles, is then explained in its turn, in the successive Parts, and each carefully kept distinct from the rest

III. A complete system of Surveying with only a chain, a rope, or any substitute, (invaluable to farmers having no other instruments,) is very fully developed in Part II.

IV. The various Problems in Chapter 5, of Part II, will be found to constitute a course of practical Geometry on the ground. As some of their demonstrations involve the "Theory of Transversals, etc," (a beautiful supplement to the ordinary Geometry), a carefully digested summary of its principal Theorems is here given, for the first time in English. It will be found in Appendix B.

V. In Compass Surveying, Part III, the Field work, in Chapter 3, is adapted to our American practice; some new modes of platting bearings are given in Chapter 4, and in Chapter 6, the rectangular method of calculating contents is much simplified.

VI. The effects of the continual change in the Variation of the magnetic needle upon the surveys of old lines, the difficulties caused by it, and the means of remedying them, are treated of with great minuteness of practical detail. A new table has been calculated for the time of "greatest Azimuth," those in common use being the same as the one prepared by Gummere in 1814, and consequently greatly in error now from the change of place of the North Star since that date.

VII. In Part IV, in Chapter 1, the Transit and Theodolite are explained in every point; in Chapter 2, all forms of Verniers are shewn by numerous engravings; and in Chapter 3, the Adjustments are elucidated by some novel modes of illustration.

VIII. In Part VII, will be found all the best methods of overcoming obstacles to sight and to measurement in angular Surveying.

IX. Part XI contains a very complete and systematic collection of the principal problems in the Division of Land.

X. The Methods of Surveying the Public Lands of the United States, of marking lines and corners, &c., are given in Part XII, from official documents, with great minuteness; since the subject interests so many land-owners residing in the Eastern as well as in the Western States.

XI. The Tables comprise a *Traverse Table*, computed for this volume, and giving increased accuracy in one-fifteenth of the usual space; a *Table of Chords*, appearing for the first time in English, and supplying the most accurate method of platting angles; and a Table of natural *Sines and Tangents*. It was thought best not to increase the size of the volume with Logarithmic Tables, not absolutely necessary for its purposes, and to be found in all Trigonometries. The tables are printed on tinted paper, on the eye-saving principle of Babbage.

XII. The great number of engraved illustrations, most of them original, is a peculiar feature of this volume, suggested by the experience of the author that one diagram is worth a page of print in giving clearness and definiteness to the otherwise vague conceptions of a student.

XIII. The practical details, and hints to the young Surveyor, have been made exceedingly full by a thorough examination of more than fifty works on the subject, by English, French and German writers, so as to make it certain that nothing which could be useful had been overlooked. It would be impossible to credit each item (though this has been most scrupulously done in the few cases in which an American writer has been referred to), but the principal names are these: Adams, Ainslie, Baker, Begat, Belcher, Bourgeois, Bourns, Brees, Bruff, Burr, Castle, Francoeur, Frome, Galbraith, Gibson, Guy, Hogard, Jackson, Lamotte, Lefevre, Mascheroni, Narrien, Nesbitt, Pearson, Puille, Puissant, Regnault, Richard, Serret, Simms, Stevenson, Weisbach, Williams.

Should any important error, either of printer or author, be discovered (as is very possible in a work of so much detail, despite the great care used) the writer would be much obliged by its prompt communication

The present volume will be followed by another on LEVELLING AND HIGHER SURVEYING: embracing *Levelling* (with Spirit-Level, Theodolite, Barometer, etc.); its applications in *Topography* or Hill-drawing, in *Mining* Surveys, etc.; the *Sextant*, and other reflecting instruments; *Maritime* Surveying · and *Geodesy*, with its practical Astronomy.

GENERAL DIVISION OF THE SUBJECT.

[*A full Analytical Table of Contents is given at the end of the volume.*]

TO TEACHERS AND STUDENTS.

As it is desirable to obtain, at the earliest possible period, a sufficient knowledge of the general principles of Surveying to commence its practice, the Student at his first reading may omit the portions indicated below, and take them up subsequently in connection with his review of his studies. The same omissions may be made by Teachers whose classes have only a short time for this study.

In PART I, omit only Articles (46), (47), (48), (51), (72), (84), (85).

In PART II, omit, in Chapter IV, (127), (128), (129), (130); *and in Chapter V, learn at first under each Problem, only one or two of the simpler methods.*

In PART III, omit only (225), (226), 232), (233), (244), (251), (280), (297), (322).

Then pass over PART IV; and in PART V, take only (379), (380); *and* (391) *to* (395).

Then pass over PART VI; and go to PART VII, (if the student has studied Trigonometry), and omit (423); (434) *to* (438); *and all of Chapter IV, except* (439) *and* (440).

PART VIII may be passed over; and PARTS IX and X may be taken in full.

In PART XI, take all of Chapter I; and in Chapters II and III, take only the simpler constructions, not omitting, however, (517), (518) *and* (538).

In PART XII, take (560), (561), (565), (566).

Appendix C, on LEVELLING, may conclude this abridged course.

LAND-SURVEYING.

PART I.

GENERAL PRINCIPLES

AND

FUNDAMENTAL OPERATIONS.

CHAPTER I.

DEFINITIONS AND METHODS.

(1) SURVEYING is the art of making such measurements as will determine the relative positions of any points on the surface of the earth; so that a *Map* of any portion of that surface may be drawn, and its *Content* calculated.

(2) The position of a point is said to be *determined*, when it is known how far that point is from one or more given points, and in what direction there-from; or how far it is in front of them or behind them, and how far to their right or to their left, &c; so that the place of the first point, if lost, could be again found by repeating these measurements in the contrary direction.

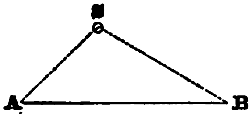
The "points" which are to be determined in Surveying, are not the mathematical points treated of in Geometry; but the corners of fences, boundary stones, trees, and the like, which are mere points in comparison with the extensive surfaces and areas which they are the means of determining. In strictness, their centres should be regarded as the points alluded to.

(3) A straight *Line* is "determined," that is, has its length and its position known and fixed, when the points at its extremities are determined; and a plane *Surface* has its form and dimensions determined, when the lines which bound it are determined. Consequently, the determination of the relative positions of *points* is all that is necessary for the principal objects of Surveying; which are to make a *map* of any surface, such as a field, farm, state, &c., and to calculate its *content* in square feet, acres, or square miles. The former is an application of Drafting, the latter of Mensuration.

(4) The position of a point may be determined by a variety of methods. Those most frequently employed in Surveying, are the following:

(5) **First Method.** *By measuring the distances from the required point to two given points.*

Fig. 1.



Thus, in Fig. 1, the point S is "determined," if it is known to be one inch from A, and half an inch from B: for, its place, if lost, could be found by describing two arcs of circles, from A and B as centres, and with the given distances as radii. The required point would be at the intersection of these arcs.

In applying this principle in surveying, S may represent any station, such as a corner of a field, an angle of a fence, a tree, a house, &c. If then one corner of a field be 100 feet from a second corner, and 50 feet from a third, the place of the first corner is known and determined with reference to the other two.

There will be two points fulfilling this condition, one on each side of the given line, but it will always be known which of them is the one desired.

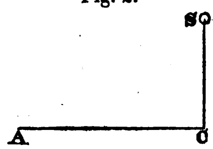
In *Geography*, this principle is employed to indicate the position of a town; as when we say that Buffalo is distant (in a straight line) 295 miles from New-York, and 390 from Cincinnati, and thus convey to a stranger acquainted with only the last two places a correct idea of the position of the first.

In *Analytical Geometry*, the lines AS and BS are known as "*Focal Co-ordinates;*" the general name "co-ordinates" being applied to the lines or angles which determine the position of a point.

(6) Second Method. *By measuring the perpendicular distance from the required point to a given line, and the distance thence along the line to a given point.*

Thus, in Fig. 2, if the perpendicular distance SC be half an inch, and CA be one inch, the point S is "determined": for, its place could be again found by measuring one inch from A to C, and half an inch from C, at right angles to AC, which would fix the point S.

Fig. 2.



The Public Lands of the United States are laid out by this method, as will be explained in Part XII.

In *Geography*, this principle is employed under the name of Latitude and Longitude.

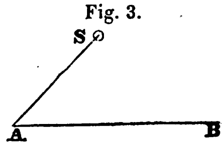
Thus, Philadelphia is one degree and fifty-two minutes of longitude east of Washington, and one degree and three minutes of latitude north of it.

In *Analytical* Geometry, the lines AC and CS are known as "*Rectangular Co-ordinates.*" The point is there regarded as determined by the intersection of two lines, drawn parallel to two fixed lines, or "*Axes,*" and at a given distance from them. These *Axes*, in the present figure, would be the line AC, and another line, perpendicular to it and passing through A, as the origin.

(7) Third Method. *By measuring the angle between a given line and a line drawn from any given point of it to the required point; and also the length of this latter line.*

Thus, in Fig. 3, if we know the angle BAS to be half a right angle, and the line AS to be one inch, the point S is determined; for, its place could be found by drawing from A, a line making the given angle with AB, and measuring on it the given distance.

Fig. 3.



In applying this principle in surveying, S, as before, may represent any station, and the line AB may be a fence, or any other real or imaginary line.

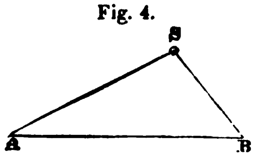
In "Compass Surveying," it is a north and south line, the direction of which is given by the magnetic needle of the compass.

In *Geography*, this principle is employed to determine the relative positions of places, by "Bearings and distances"; as when we say that San Francisco is 1750 miles nearly due west from St. Louis; the word "west" indicating the *direction*, or angle which the line joining the two places makes with a north and south line, and the number of miles giving the *length* of that line.

In *Analytical Geometry*, the line AS, and the angle BAS, are called "*Polar Co-ordinates*."

(8) **Fourth Method.** *By measuring the angles made with a given line by two other lines starting from given points upon it, and passing through the required point.*

Fig. 4.

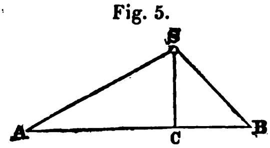


Thus, in Fig. 4, the point S is determined by being in the intersection of the two lines AS and BS, which make respectively angles of a half and of a third of a right angle with the line AB, which is one inch long; for, the place of the point could be found, if lost, by drawing from A and B lines making with AB the known angles.

In *Geography*, we might thus fix the position of St. Louis, by saying it lay nearly due north from New-Orleans, and due west from Washington.

In *Analytical Geometry*, these two angles would be called "*Angular Co-ordinates*."

Fig. 5.



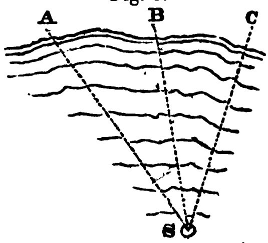
(9) In Fig. 5, are shown together all the measurements necessary for determining the same point S, by each of the four preceding methods. In the *First* Method, we measure the distances AS and BS; in the *Second* Method, the distances AC and CS, the latter at right angles to the former; in the *Third* Method, the distance

AS, and the angle SAB; and in the *Fourth* Method, the angles SAB and SBA. In all these methods the point is really determined by the intersection of two lines, either straight lines or arcs of circles. Thus, in the First Method, it is determined by the intersection of two circles; in the Second, by the intersection of two straight lines; in the Third, by the intersection of a straight line and a circle; and in the Fourth, by the intersection of two straight lines.

(10) Fifth Method. *By measuring the angles made with each other by three lines of sight passing from the required point to three points whose positions are known.*

Thus, in Fig. 6, the point S is determined by the angles ASB, and BSC made by the three lines SA, SB and SC.

Fig. 6.

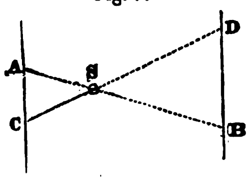


Geographically, the position of Chicago would be determined by three straight lines passing from it to Washington, Cincinnati, and Mobile, and making known angles with each other; that of the first and second lines being about one-third, and that of the second and third lines, about one-half of a right angle.

From the *three lines* employed, this may be named the Method of *Trilinear* co-ordinates.

(11) The position of a point is sometimes determined by the intersection of two lines, which are themselves determined by their extremities being given. Thus, in Fig. 7, the point S is determined by its being situated in the intersection of AB and CD. This method is sometimes employed to fix the position of a Station on a Rail-Road line, &c., when it occurs in a place where a stake cannot be driven, such as in a pond; and in a few other cases; but is not used frequently enough to require that it should be called a *sixth* principle of Surveying. It is said to be employed

Fig. 7.



by smugglers to fix the spot where they have sunk contraband goods.

(12) These five methods of determining the positions of points, produce five corresponding systems of Surveying, which may be named as follows:

I. DIAGONAL SURVEYING.
II. PERPENDICULAR SURVEYING.
III. POLAR SURVEYING
IV. TRIANGULAR SURVEYING.
V. TRILINEAR SURVEYING.

(13) The above division of Surveying has been made in harmony with the principles involved and the methods employed.

The subject is, however, sometimes divided with reference to the *instruments* employed; as the chain, either alone or with cross-staff; the compass; the transit or theodolite; the sextant; the plane table, &c.

(14) Surveying may also be divided according to its *objects*.

In *Land* Surveying, the content, in acres, &c., of the tract surveyed, is usually the principal object of the survey. A map, showing the shape of the property, may also be required. Certain signs on it may indicate the different kinds of culture, &c. This land may also be required to be divided up in certain proportions; and the lines of division may also be required to be set out on the ground. One or all of these objects may be demanded in Land Surveying.

In *Topographical* Surveying, the measurement and graphical representation of the inequalities of the ground, or its "relief," i. e. its hills and hollows, as determined by the art of "Levelling," is the leading object.

In *Maritime* or *Hydrographical* Surveying, the positions of rocks, shoals and channels are the chief subjects of examination.

In *Mining* Surveying, the directions and dimensions of the subterranean passages of mines are to be determined.

(15) Surveying may also be divided according to the *extent* of the district surveyed, into *Plane* and *Geodesic*. Geodesy takes into account the curvature of the earth, and employs Spherical Trigonometry. *Plane* Surveying disregards this curvature, as a needless refinement except in very extensive surveys, such as those of a State, and considers the surface of the earth as plane, which may safely be done in surveys of moderate extent.

(16) *Land Surveying* is the principal subject of this volume; the surface surveyed being regarded as *plane;* and each of the five Methods being in turn employed. For the purposes of instruction, the subject will be best divided, partly with reference to the Methods employed, and partly to the Instruments used. Accordingly, the *First* and *Second* Methods (Diagonal and Perpendicular Surveying) will be treated of under the title "Chain Surveying," in Part II. The *Third* Method (Polar Surveying) will be explained under the titles "Compass Surveying," Part III, and "Transit and Theodolite Surveying," Part IV. The *Fourth* and *Fifth* Methods will be found under their own names of "Triangular Surveying," and "Trilinear Surveying," in Parts V and VI.

(17) In all the methods of Land Surveying, there are three stages of operation:

1° *Measuring* certain lines and angles, and recording them;

2° *Drawing them* on paper to some suitable scale;

3° *Calculating* the content of the surface surveyed.

The three following chapters will treat of each of these topics in their turn.

CHAPTER II.

MAKING THE MEASUREMENTS.

(18) THE *Measurements* which are required in Surveying, may be of lines or of angles, or of both; according to the Method employed. Each will be successively considered.

MEASURING STRAIGHT LINES.

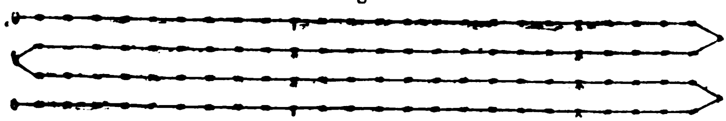
(19) The lines, or distances, which are to be measured, may be either actual or visual.

Actual lines are such as really exist on the surface of the land to be surveyed, either bounding it, or crossing it; such as fences, ditches, roads, streams, &c.

Visual lines are imaginary lines of sight, either temporarily measured on the ground, such as those joining opposite corners of a field; or simply indicated by stakes at their extremities or otherwise. If long, they are "ranged out" by methods hereafter to be described. Lines are usually measured with chains, tapes or rods, divided into yards, feet, links, or some other unit of measurement.

(20) Gunter's Chain. This is the measure most commonly used in Land surveying. It is 66 feet, or 4 rods long.* Eighty such chains make one mile.

Fig. 8.



It is composed of one hundred pieces of iron wire, or links, each bent at the end into a ring, and connected with the ring at the end of the next piece by another ring. Sometimes two or three rings are placed between the links. The chain is then less liable to

* This length was chosen (by Mr. Edward Gunter) because 10 square chains of 66 feet make one acre, (as will be shown in Chapter IV,) and the computation of areas is thus greatly facilitated. For other Surveying purposes, particularly for Rail-road work, a chain of 100 feet is preferable. On the United States Coast Survey, the unit of measurement (which at some future time will be the universal one) is the French *Metre*, equal to 3.281 feet, nearly.

twist and get entangled, or "kinked." Two or more swivels are also inserted in the chain, so that it may turn around without twisting. Every tenth link is marked by a piece of brass, having one, two, three, or four points, corresponding to the number of tens which it marks, counting from the nearest end of the chain.* The middle or fiftieth link is marked by a round piece of brass.

The hundredth part of a chain is called a link.† The great advantage of this is, that since links are decimal parts of a chain, they may be so written down, 5 chains and 43 links being 5.43 chains, and all the calculations respecting chains and links can then be performed by the common rules of decimal Arithmetic. Each link is 7.92 inches long, being $= 66 \times 12 \div 100$.

The following Table will be found convenient:

CHAINS INTO FEET.			
CHAINS.	FEET.	CHAINS.	FEET.
0.01	0.66	1.00	66.
0.02	1.32	2.	132.
0.03	1.98	3.	198.
0.04	2.64	4.	264.
0.05	3.30	5.	330.
0.06	3.96	6.	396.
0.07	4.62	7.	462.
0.08	5.28	8.	528.
0.09	5.94	9.	594.
0.10	6.60	10.	660.
0.20	13.20	20.	1320.
0.30	19.80	30.	1980.
0.40	26.40	40.	2640.
0.50	33.00	50.	3300.
0.60	39.60	60.	3960.
0.70	46.20	70.	4620.
0.80	52.80	80.	5280.
0.90	59.40	90.	5940.
1.00	66.00	100.	6600.

FEET INTO LINKS.			
FEET.	LINKS.	FEET.	LINKS.
0.10	0.15	10.	15.2
0.20	0.30	15.	22.7
0.25	0.38	20.	30.3
0.30	0.45	25.	37.9
0.40	0.60	30.	45.4
0.50	0.76	33.	50.0
0.60	0.91	35.	53.0
0.70	1.06	40.	60.6
0.75	1.13	45.	68.2
0.80	1.21	50.	75.8
0.90	1.36	55.	83.3
1.00	1.52	60.	90.9
2.	3.0	65.	98.5
3.	4.5	70.	106.1
4.	6.1	75.	113.6
5.	7.6	80.	121.2
6.	9.1	85.	128.8
7.	10.6	90.	136.4
8.	12.1	95.	143.9
9.	13.6	100.	151.5

* To prevent the very common mistake, of calling forty, sixty; or thirty, seventy; it has been suggested to make the 11th, 21st, 31st and 41st links of *brass;* which would at once show on which side of the middle of the chain was the doubtful mark. This would be particularly useful in Mining Surveying.

† This must not be confounded with the pieces of wire which have the same name, since one of them is shorter than the "link" used in calculation, by half a ring, or more, according to the way in which the chain is made.

To reduce links to feet, subtract from the number of links as many units as it contains hundreds; multiply the remainder by 2 and divide by 3.

To reduce feet to links, add to the given number half of itself, and add one for each hundred (more exactly, for each ninety-nine) in the sum.

The chain is liable to be lengthened by its rings being pulled open, and to be shortened by its links being bent. It should therefore be frequently tested by a carefully-measured length of 66 feet, set out by a standard measure, on a flat surface, such as the top of a wall, or on smooth level ground, between two stakes, their centres being marked by small nails. It may be left a little longer than the true length, since it can seldom be stretched so as to be perfectly horizontal and not hang in a curve, or be drawn out in a perfectly straight line.* Distances measured with a perfectly accurate chain will always and unavoidably be recorded as longer than they really are. To ensure the chain being always strained with the same force, a spring, like that of a spring-balance, is sometimes placed between one handle and the rest of the chain.

If a line has been measured with an incorrect chain, the true length of the line will be obtained by multiplying the number of chains and links in the measured distance by 100, and dividing by the length of the standard distance, as given by measurement of it with the incorrect chain. The proportion here employed is this: *As* the length of the standard given by the incorrect chain *Is to* the true length of the standard, *So is* the length of the line given by the measurement *To* the true length. Thus, suppose that a line has been measured with a certain chain, and found by it to be ten chains long, and that the chain is afterwards found to have been so stretched that the standard distance, measured by it, appears to be only 99 links long. The measured line is therefore longer than it had been thought to be, and its true length is obtained by multiplying ten by 100, and dividing by 99.

* The chain used by the Government Surveyors of France, which is 10 Metres, or about half a Gunter's chain in length, is made from one-fifth to two-fifths of an inch longer than the standard. An inaccuracy of one five hundredth of its length (= 1½ inches on a Gunter's chain) is the utmost allowed not to vitiate the survey.

(21) Pins. Ten iron pins or "arrows," usually accompany the chain.* They are about a foot long, and are made of stout iron wire, sharpened at one end, and bent into a ring at the other. Pieces of red and white cloth should be tied to their heads, so that they can be easily found in grass, dead leaves, &c.

They should be strung on a ring, which has a spring catch to retain them. Their usual form is shown in in Fig. 9. Fig. 10 shows another form, made very large, and therefore very heavy near the point, so that when held by the top and dropped, it may fall vertically. The uses of this will be seen presently.

Fig. 9.

Fig. 10.

(22) On irregular ground, two stout stakes about six feet long are needed to put the forward chain-man in line, and to enable whichever of the two is lowest, to raise his end of the chain in a truly vertical line, and to strain the chain straight.

A number of long and slender rods are also necessary for "ranging out" lines between distant points, in the manner to be explained hereafter; in Part II, Chapter V.

(23) How to Chain. Two men are required; a forward chain-main, and a hind chain-man; or leader and follower. The latter takes the handles of the chain in his left hand, and the chain itself in his right hand, and throws it out in the direction in which it is to be drawn. The former takes a handle of the chain and one pin in his right hand, and the other pins (and the staff, if used,) in his left hand, and draws out the chain. The follower then walks beside it, examining carefully that it is not twisted or bent. He then returns to its hinder end, which he holds at the beginning of the line to be measured, puts his eye exactly over it, and, by the words "Right," "Left," directs the leader how to put his staff, or the pin which he holds up, "in line," so that it may seem to cover and hide the flag-staff, or other object at the end of the line.

* Eleven pins are sometimes used, one being of brass. Nine of iron, with four or eight of brass, may also be employed. Their uses will be explained in the following article.

The leader all the while keeps the chain tightly stretched, and his end of it touching his staff. Every time he moves the chain, he should straighten it by an undulating shake. When the staff (or pin) is at last put "in line," the follower says "Down." The leader then puts in the single pin precisely at the end of the chain, and replies "Down." The follower then (and never before hearing this signal that the point is fixed) loosens his end of the chain, retaining it in his hand. The leader draws on the chain, making a step to one side of the pin just set, to avoid dragging it out. He should keep his eye steadily on the object ahead, or, in a hollow, should line himself approximately by looking back. The follower should count his steps, so as to know where to look for the pin in high grass, &c. As he approaches the pin, he calls "Halt." On reaching it, he holds the handle of the chain against it, pressing his knee against both to keep the pin firm. He then, with his eye over the pin, "lines" the leader as before. When the "Down" has been again called by the follower, and answered by the leader, the former pulls out the pin with the chain-hand, and carries it in his other hand, and they go on as before.* The operation is repeated till the leader has arrived at the end of the line, or has put down all his pins.

When the leader has put down his tenth pin, he draws on the chain its length farther, and after being lined, puts his foot on the handle to keep it firm, and calls "Tally." The follower then drops his end of the chain, goes up to the leader and gives him back all the pins, both counting them to make sure that none have been lost. One pin is then put down at the forward end of the chain, and they go on as before.

Some Surveyors cause the leader to call "tally" at the tenth pin, and then exchange pins; but then the follower has only the hole made by the pin, or some other indefinite mark, to measure from.

Eleven pins are sometimes preferred, the eleventh being of brass, or otherwise different from the rest, and being used to mark the

* When a chain's length would end in a ditch, pool of water, &c. and the chainmen are afraid of wetting their feet, they can measure part of a chain, to the edge of the water, then stretch the chain across it, and then measure another portion of a chain, so that with the former portion, it may make up a full chain.

end of the eleventh chain; another being substituted for it before the leader goes on.

The two chain-men may change duties at each change of pins, if they are of equal skill, but the more careful and intelligent of two laborers should generally be made "follower."

When the leader reaches the end of the line, he stops, and holds his end of the chain against it. The follower drops his end and counts the links beyond the last pin, noting carefully on which side of the "fifty" mark it comes. Each pin now held by the follower, including the one in the ground, represents 1 chain; each time "tally" has been called, and the pins exchanged, represents 10 chains, and the links just counted make up the total distance.

(24) Tallies. In chaining very long distances, there is danger of miscounting the number of "tallies," or tens. To avoid mistakes, pebbles, &c., may be changed from one pocket into another at each change of pins; or bits of leather on a cord may be slipped from one side to the other; or knots tied on a string; but the best plan is the following. Instead of ten iron pins, use nine iron pins, and four, or eight, or ten pins of brass, or very much longer than the rest. At the end of the tenth chain, the iron pins being exhausted, a brass pin is put down by the leader. The follower then comes up, and returns the nine iron pins, but retains the brass one, with the additional advantage of having this pin to measure from. At the end of the twentieth chain, the same operation is repeated; and so on. When the measurement of the line is completed, each brass pin held by the follower counts ten chains, and each iron pin one, as before.

(25) Chaining on Slopes. All the distances employed in Land-surveying must be measured horizontally, or on a level; for reasons to be given in chapter IV. When the ground slopes, it is therefore necessary to make certain allowances or corrections. If the slope be gentle, hold the up-hill end of the chain on the ground, and raise the down-hill end till the chain is level. To ensure the elevated end being exactly over the desired spot, raise it along a staff kept vertical, or drop a pin held by the point with the ring downwards, (if you have not the heavy pointed ones shown in Fig. 10), or, which is better, use a plumb-line. A person standing beside the chain, and at a little distance from it, can best tell if it be nearly level. If the hill be so steep that a whole chain cannot be held up level, use only half or qnarter of it at a time. Great care is necessary in this operation. To measure down a steep hill, stretch the whole chain in line. Hold the upper end fast on the ground. Raise up the 20 or 30 link-mark, so that that portion of the chain is level. Drop a plumb-line or pin. Then let the follower come forward and hold down that link on this spot, and the leader hold up another short portion, as before. Chaining down a slope is more accurate than chaining up it, since in the latter case the follower cannot easily place his end of the chain exactly over the pin.

Fig. 11.

(26) A more accurate, though more troublesome, method, is to measure the angle of the slope; and make the proper allowance by calculation, or by a table, previously prepared. The correction being found, the chain may be drawn forward the proper number of links, and the correct distance of the various points to be noted will thus be obtained at once, without any subsequent calculation or reduction. If the survey is made with the Theodolite, the slope of the ground can be measured directly. A "Tangent Scale," for the same purpose, may be formed on the sides of the sights of a Compass. It will be described when that instrument is explained.

In the following table, the first column contains the angle which the surface of the ground makes with the horizon; the second column contains its slope, named by the ratio of the perpendicular to the base; and the third, the correction in links for each chain measured on the slope, i. e. the difference between the hypothenuse, which is the distance measured, and the horizontal base, which is the distance desired.

TABLE FOR CHAINING ON SLOPES.

ANGLE.	SLOPE.	CORRECTION IN LINKS.	ANGLE.	SLOPE.	CORRECTION IN LINKS.
3°	1 in 19	0.14	13°	1 in $4\frac{1}{2}$	2.56
4°	1 in 14	0.24	14°	1 in 4	2.97
5°	1 in $11\frac{1}{2}$	0.38	15°	1 in 4	3.41
6°	1 in $9\frac{1}{2}$	0.55	16°	1 in $3\frac{3}{4}$	2.87
7°	1 in 8	0.75	17°	1 in $3\frac{1}{2}$	4.37
8°	1 in 7	0.97	18°	1 in $3\frac{1}{4}$	4.89
9°	1 in $6\frac{1}{2}$	1.23	19°	1 in 3	5.45
10°	1 in 6	1.53	20°	1 in $2\frac{3}{4}$	6.03
11°	1 in $5\frac{1}{4}$	1.84	25°	1 in 2	9.37
12°	1 in $4\frac{3}{4}$	2.19	30°	1 in $1\frac{3}{4}$	13.40

(27) Chaining is the fundamental operation in all kinds of Surveying. It has for this reason been very minutely detailed. The "follower" is the most responsible person, and the Surveyor will best ensure his accuracy by taking that place himself. If he has to employ inexperienced laborers, he will do well to cause them to measure the distance between any two points, and then remeasure it in the opposite direction. The difference of their two results will impress on them the necessity of great carefulness.

To "do up" the chain, take the middle of it in the left hand, and with the right hand take hold of the doubled chain just beyond the second link; double up the two links between your hands, and continue to fold up two double links at a time, laying each pair obliquely across the others, so that when it is all folded up, the handles will be on the outside, and the chain will have an hour-glass shape, easy to strap up and to carry.

(28) Tape. Though the chain is most usually employed for the principal measurements of Surveying, a *tape-line*, divided on one side into links, and on the other into feet and inches, is more convenient for some purposes. It should be tested very frequently, particularly after getting wet, and the correct length marked on it at every ten feet. A "Metallic Tape," less liable to stretch, has been recently manufactured, in which fine wires form its warp. When the tape is being wound up, it should be passed between two fingers to prevent its twisting in the box, which would make it necessary to unscrew its nut to take it out and untwist it. While in use, it should be made portable by being folded up by arm's lengths, instead of being wound up.

(29) Substitutes for a chain or a tape, may be found in leather driving lines, marked off with a carpenter's rule, or in a cord knotted at the length of every link. A well made rope, (such as a "patent wove line," woven circularly with the strands always straight in the line of the strain), when once well stretched, wetted and allowed to dry with a moderate strain, will not vary from a chain more than one foot in two thousand, if carefully used.

(30) Rods. When unusually accurate measurements are required, rods are employed. They may be of well seasoned wood, of glass, of iron, &c. They must be placed in line very carefully end to end; or made to coincide in other ways; as will be explained in Part V, under the title of "Triangular Surveying," in which the peculiarly accurate measurement of one line is required, as all the others are founded upon it.

(31) Pacing, Sound, and other approximate means, may be used for measuring the length of a line. They will be discussed, in Part IX, under the title "Surveying without Instruments."

(32) A *Perambulator*, or "Measuring Wheel," is sometimes used for measuring distances, particularly Roads. It consists of a wheel which is made to roll over the ground to be measured, and whose motion is communicated to a series of toothed wheels within the machine. These wheels are so proportioned, that the index wheel registers their revolutions, and records the whole distance passed over. If the diameter of the wheel be $31\frac{1}{2}$ inches, the circumference, and therefore each revolution, will be $8\frac{1}{4}$ feet, or half a rod. The roughnesses of the road and the slopes necessarily cause the registered distances to exceed the true measure.

MEASURING ANGLES.

(33) The angle made by any two lines, that is, the difference of their directions, is measured by various instruments, consisting essentially of a circle divided into equal parts, with plain sights, or telescopes, to indicate the directions of the two lines.

As the measurement of angles is not required for "Chain Surveying," which is the first Method to be discussed, the consideration of this kind of measurement will be postponed to Part III.

NOTING THE MEASUREMENTS.

(34) The measurements which have been made, whether of lines, or of angles, require to be very carefully noted and recorded. Clearness and brevity are the points desired. Different methods of notation are required for each of the systems of surveying which are to be explained, and will therefore be given in their appropriate places.

CHAPTER III.

DRAWING THE MAP.

(35) A Map of a survey represents the lines which bound the surface surveyed, and the objects upon it, such as fences, roads, rivers, houses, woods, hills, &c., in their true relative dimensions and positions. It is a miniature copy of the field, farm, &c., as it would be seen by an eye moving over it; or as it would appear, if from every point of its irregular surface, plumb lines were dropped to a level surface under it, forming what is called in geometrical language, its *horizontal projection*.

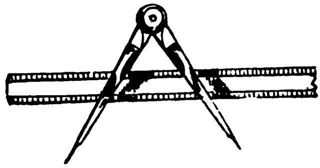
(36) Platting. A *plat* of a survey is a skeleton, or outline map. It is a figure "similar" to the original, having all its angles equal, and its sides proportional. Every inch on it represents a foot, a yard, a rod, a mile, or some other length, on the ground;

all the measured distances being diminished in exactly the same ratio.

PLATTING *is repeating on paper, to a smaller scale, the measurements which have been made on the ground.*

Its various operations may therefore be reduced, in accordance with the principles established in the first chapter, to two, viz: drawing a straight line in a given direction and of a given length; and describing an arc of a circle with a radius whose length is also given. The only instruments absolutely necessary for this, are a straight ruler, and a pair of "dividers," or "compasses." Others, however, are often convenient, and will be now briefly noticed.

Fig. 12.

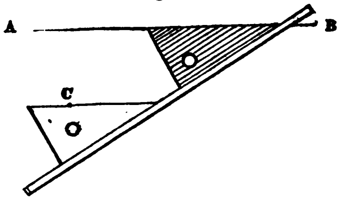


(37) Straight Lines. These are usually drawn by the aid of a straight-edged ruler. But to obtain a very long straight line upon paper, stretch a fine silk thread between any two distant points, and mark in its line various points, near enough together to be afterwards connected by a common ruler. The thread may also be blackened with burnt cork, and snapped on the paper, as a carpenter snaps his chalk line; but this is liable to inaccuracies, from not raising the line vertically.

(38) Arcs. The arcs of circles used in fixing the position of a point on paper, are usually described with compasses, one leg of which carries a pencil point. A convenient substitute is a strip of pasteboard, through one end of which a fine needle is thrust into the given centre, and through a hole in which, at the desired distance, a pencil point is passed, and can thus describe a circle about the centre, the pasteboard keeping it always at the proper distance. A string is a still readier, but less accurate, instrument.

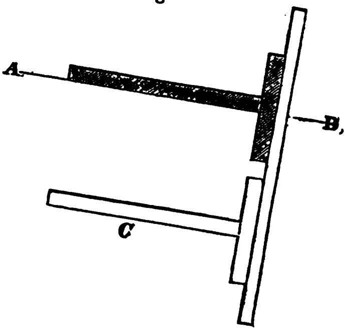
(39) Parallels. The readiest mode of drawing parallel lines is by the aid of a triangular piece of wood and a ruler. Let AB be the line to which a parallel is to be drawn, and C the point through which it must pass. Place one side of the triangle against the line, and place the ruler against another side of the triangle. Hold the ruler firm and immovable, and slide the triangle along it till the side of the triangle which had coincided with the given line, passes through the given point. This side will then be parallel to that given line, and a line drawn by it will be the line required.

Fig. 13.



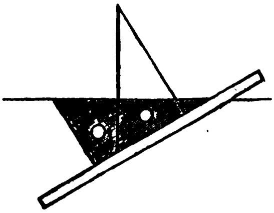
Another easy method of drawing parallels, is by means of a T square, an instrument very valuable for many other purposes. It is nothing but a ruler let into a thicker piece of wood, very truly at right angles to it. For this use of it, one side of the cross-piece must be even, or "flush," with the ruler. To use it, lay it on the paper so that one edge of the ruler coincides with the given line AB. Place another ruler against the cross-piece, hold it firm, and slide the T square along, till its edge passes through the given point C, as shown by the lower part of the figure. Then draw by this edge the desired line parallel to the given line.

Fig. 14.



(40) **Perpendiculars.** These may be drawn by the various problems given in Geometry, but more readily by a triangle which has one right angle. Place the longest side of the triangle on the given line, and place a ruler against a second side of the triangle. Hold the ruler fast, and turn the triangle so as to bring its third side against the ruler. Then will the long side be perpendicular to the

Fig. 15.



given line. By sliding the triangle along the ruler, it may be used to draw a perpendicular from any point of the line, or from any point to the line.

(41) Angles. These are most easily set out with an instrument called a Protractor, usually a semi-circle of brass. But the description of its use, and of the other and more accurate modes of laying off angles, will be postponed till they are needed in Part III, Chapter IV.

(42) Drawing to Scale. The operation of drawing on paper lines whose length shall be a half, a quarter, a tenth, or any other fraction, of the lines measured on the ground, is called "Drawing to Scale."

To set off on a line any given distance to any required scale, determine the number of chains or links which each division of the scale of equal parts shall represent. Divide the given distance by this number. The quotient will be the number of equal parts to be taken in the dividers and to be set off.

For example, suppose the scale of equal parts to be a common carpenter's rule, divided into inches and eighths. Let the given distance be twelve chains, which is to be drawn to a scale of two chains to an inch. Then six inches will be the distance to be set off. If the given distance had been twelve chains and seventy five links, the distance to be set off would have been six inches and three-eighths, since each eighth of an inch represents 25 links.

If the desired scale were three chains to an inch, each eighth of an inch would represent $37\frac{1}{2}$ links; and the distance of 1275 links would be represented by thirty-four eighths of an inch, or $4\frac{1}{4}$ inches.

A similar process will give the correct length to be set off for any distance to any scale.

If the scale used had been divided into inches and tenths, as is much the most convenient, the above distances would have become on the former scale $6\frac{37}{100}$ inches, or nearly $6\frac{4}{10}$ inches; and on the latter scale $4\frac{25}{100}$ inches, coming midway between the 2d and 3d tenth of an inch.

(43) *Conversely*, to find the real length of a line drawn on paper to any known scale, reverse the preceding operation. Take the length of the line in the dividers, apply it to the scale, and count how many equal parts it includes. Multiply their number by the number of chains or links which each represents, and the product will be the desired length of the line on the ground.

This operation and the preceding one are greatly facilitated by the use of the scales to be described in Art. (48)

(44) Scales. The choice of the scale to which a plat should be drawn, that is, how many times smaller its lines shall be than those which have been measured on the ground, is determined by several considerations. The chief one is, that it shall be just large enough to express clearly all the details which it is desirable to know. A Farm Survey would require its plat to show every field and building. A State Survey would show only the towns, rivers, and leading roads. The size of the paper at hand will also limit the scale to be adopted. If the content is to be calculated from the plat, that will forbid it to be less than 3 chains to 1 inch.

Scales are named in various ways. *They should always be expressed fractionally;* i. e. they should be so named as to indicate what fractional part of the real line measured on the ground, the representative line drawn on the paper, actually is. When custom requires a different way of naming the scale, both should be given. It would be still better, if the denominator could always be some power of 10, or at least some multiple of 2 and 5, such as $\frac{1}{500}$, $\frac{1}{1000}$, $\frac{1}{2000}$, $\frac{1}{2500}$, &c. For convenience in printing, these may be written thus: 1:500, 1:1000, 1:2000, 1:2500, &c.

Plats of *Farm Surveys* are usually named as being so many chains to an inch.

Maps of *Surveys of States* are generally named as being made to a scale of so many miles to an inch.

Maps of *Rail-road Surveys* are said to be so many feet to an inch, or so many inches to a mile.

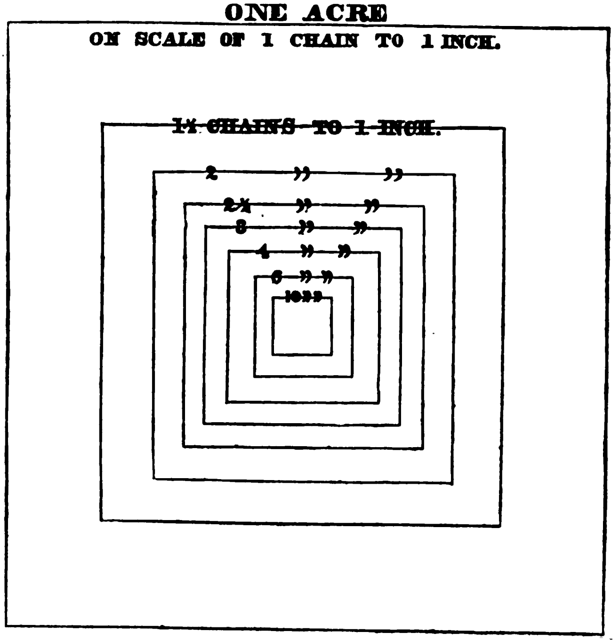
(45) Farm Surveys. If these are of small extent, two chains to one inch (which is $= \frac{1}{2 \times 66 \times 12} = \frac{1}{1584} = 1:1584$) is convenient.

A scale of one chain to one inch (1:792) is useful for plans of buildings. Three chains to one inch (1:2376) is suitable for larger farms. It is the scale prescribed by the English Tithe Commissioners for their first class maps.

In France, the *Cadastre* Surveys are lithographed on a scale about equivalent to this, being 1:2500. The original plans are drawn to a scale of 1:5000. Plans for the division of property are made on the former scale. When the district exceeds 3000 acres, the scale is 1:10,000. When it exceeds 7,500 acres, the scale is 1:20,000. A common scale in France for small surveys is 1:1000; about 1¼ chains to 1 inch.

Fig. 16.

ONE ACRE



The choice of the most suitable scale for the plat of a farm survey, may be facilitated by the Figure given above, which shows the actual space occupied by *one acre*, (the customary unit of land measure), laid out in the form of a square, on maps drawn to the various scales named in the figure.

(46) State Surveys. On these surveys, smaller scales are necessarily employed.

On the admirable *United States* Coast Survey, all the scales are expressed fractionally and decimally. "The surveys are generally platted originally on a scale of one to ten or twenty thousand, but in some instances the scale is larger or smaller.

These original surveys are reduced for engraving and publication, and when issued, are embraced in three general classes. 1°, small Harbor charts; 2°, charts of Bays, Sounds, and 3°, of the Coast General Charts.

The scales of the first class vary from 1:10,000 to 1:60,000, according to the nature of the Harbor and the different objects to be represented.

Where there are many shoals, rocks, or other objects, as in Nantucket Harbor and Hell-Gate, or where the importance of the harbor makes it necessary, a larger scale of 1:5,000, 1:10,000, and 1:20,000 is used. But where, from the size of the harbor, or its ease of access, a smaller one will point out every danger with sufficient exactness, the scales of 1:40,000 and 1:60,000 are used, as in the case of New-Bedford Harbor, Cat and Ship Island Harbor, New-Haven, &c.

The scale of the second class, in consequence of the large areas to be represented, is usually fixed at 1:80,000, as in the case of New-York Bay, Delaware Bay and River. Preliminary charts, however, are issued, of various scales from 1:80,000 to 1:200,000.

Of the third class, the scale is fixed at 1:400,000, for the General Chart of the Coast from Gay Head to Cape Henlopen, although considerations of the proximity and importance of points on the coast, may change the scales of charts of other portions of our extended coast."*

The National Survey of *Great Britain* is called, from the corps employed on it, the "Ordnance Survey."

The "Ordnance Survey" of the southern counties of England was platted on a scale of 2 inches to 1 mile, (1:31,680), and reduced for publication to that of one inch to a mile, (1:63,360). The scale of 6 inches to a mile (1:10,560) was adopted for the

* Communicated from the U. S. Coast Survey office.

northern counties of England and for the southern counties of Scotland. The same scale was employed for platting and engraving in outline the "Ordnance Survey" of Ireland. But a map on a scale of 1 inch to 1 mile (1:63,360) is about to be published, the former scale rendering the maps too unwieldy and cumbrous for consultation.

The Ordnance Survey of Scotland was at first platted on a scale of six inches to one mile, (1:10,560). That scale has since been abandoned, and it is now platted on a scale of two inches to 1 mile, (1:31,680), and the general maps are made to only half that scale.

The Ordnance Survey scale for the maps of London and other large towns, is 5 feet to 1 mile, (1:1056), or $1\frac{1}{3}$ chains to one inch.

In the "Surveys under the Public Health act" of England, the scale for the general plan is two feet to one mile, (1:2,640); and for the detailed plan, ten feet per mile, (1:528), or two-thirds of a chain per inch.

The Government Survey of *France* is platted to a scale of 1:20,000. Copies are made to 1:40,000; and the maps are engraved to a scale of 1:80,000, or about $\frac{4}{5}$ inch to 1 mile.

Cassini's famous map of France was on a scale of 1:86,400.

The French War Department employ the scales of 1:10,000; 1:20,000; 1:40,000; and 1:80,000; for the topography of France.

(47) Rail-road Surveys. For these the New-York General Rail-road Law of 1850 directs the scale of maps which are to be filed in the State Engineer's Office, to be five hundred feet to one-tenth of a foot, (= 1 : 5000.)

For the New-York Canal Maps a scale of 2 chains to 1 inch (1 : 1584) is employed.

The Parliamentary "standing orders" prescribe the plans of Rail-roads, prepared for Parliamentary purposes, to be made on a scale of not less than 4 inches to the mile, (1 : 15840): and the enlarged portions (as of gardens, court-yards, &c.) to be on a scale not smaller than 400 feet to the inch, (1 : 4800.) Accordingly the practice of English Railway Engineers is to draw the whole plan to a scale of 6 chains, or 396 feet to the inch, (1 : 4752) as being just within the Parliamentary limits.

In France, the Engineers of "Bridges and Roads" (Corps des Ponts et Chaussees) employ for the general plan of a road a scale of 1 : 5000, and for appropriations 1 : 500.

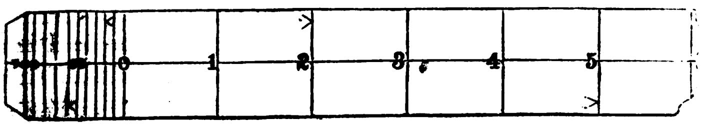
(48) In the United States Engineer service, the following scales are prescribed:

General plans of buildings, 1 inch to 10 feet, (1;120).
Maps of ground, with horizontal curves one foot apart, 1 inch to 50 feet, (1:600).
Topographical maps, one mile and a half square, 2 feet to one mile, (1:2,640).
Do. comprising three miles square, 1 foot to one mile, (1:5,280).
Do. between four and eight miles square, 6 inches to one mile, (1:10,560).
Do. comprising nine miles square, 4 inches to one mile, (1:15,840).
Maps not exceeding 24 miles square, 2 inches to one mile, (1:31,680).
Maps comprising 50 miles square, 1 inch to one mile, (1:63,360).
Maps comprising 100 miles square, ½ inch to one mile, (1:126,720.)
Surveys of Roads, Canals, &c., 1 inch to 50 feet, (1:600).

(49) The most convenient scales of equal parts are those of box-wood, or ivory, which have a *fiducial* or feather edge, along which they are divided, so that distances can be at once marked off from this edge, without requiring to be taken off with the dividers; or the length of a given line can be at once read off. Box-wood is preferable to ivory as much less liable to warp, or to vary in length with changes in the moisture in the air.

The student can, however, make for himself platting scales of drawing paper, or Bristol board. Cut a straight strip of this material, about an inch wide. Draw a line through its middle, and set

Fig. 17.



off on it a number of equal parts, each representing a chain to the desired scale. Sub-divide the left hand division into ten equal parts, each of which will therefore represent ten links to this scale. Through each point of division on the central line, draw (with the T square) perpendiculars extending to the edges, and the scale is made. It explains itself. The above figure is a scale of 2 chains to 1 inch. On it the distance 220 links would extend between the arrow-heads above the line in the figure; 560 links extends between the lower arrow-heads, &c.

A paper scale has the great advantage of varying less from a plat which has been made by it, in consequence of changes in the weather, than any other. The mean of many trials showed the difference between such a scale and drawing paper, when exposed alternately to the damp open atmosphere, and to the air of a warm dry room, to be equal to .005, while that between box-wood scales and the paper was .012, or nearly $2\frac{1}{2}$ times as much. The difference with ivory would have been even greater.

Some of the more usual plotting scales are here given in their actual dimensions.

In these five figures, different methods of drawing the scales have been given, but each method may be applied to any scale. The first and second, being the most simple, are generally the best. In the third the subdivisions are made by a diagonal line: the distances between the various pairs of arrow heads, beginning with the uppermost, are, respectively, 310, 540, and 270 links.

Fig. 18. *Scale of* 1 *chain to* 1 *inch.*

Fig. 19. *Scale of* 2 *chains to* 1 *inch.*

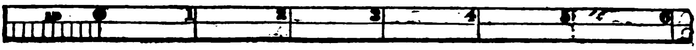
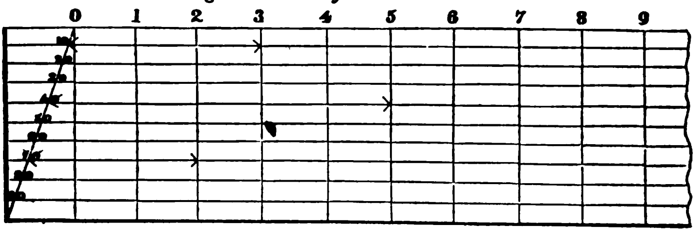
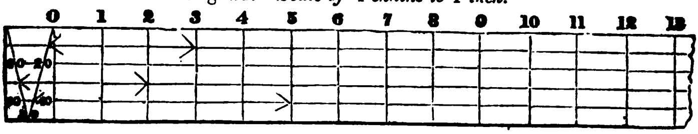


Fig. 20. *Scale of* 3 *chains to* 1 *inch.*



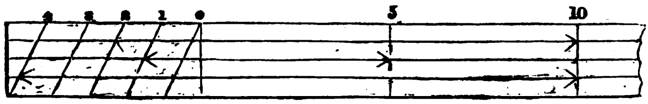
In the fourth figure the distances between the arrow heads are respectively 310, 270, and 540 links.

Fig. 21. *Scale of* 4 *chains to* 1 *inch.*



In the fifth figure the scale of 5 chains to 1 inch is subdivided diagonally to only every quarter chain, or 25 links. The distance between the upper pair of arrow-heads on it is $12\frac{1}{4}$ chains, or 12.25; between the next pair of arrow-heads, it is 6.50; and between the lower pair, 14.75.

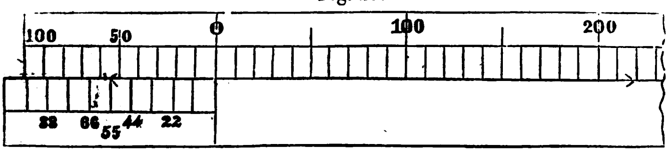
Fig. 22. *Scale of* 5 *chains to* 1 *inch.*



A diagonal scale for dividing an inch, or a half inch, into 100 equal parts, is found on the "Plain scale" in every case of instruments.

) **Vernier Scale.** This is an ingenious substitute for the diagonal scale. The one given in the following figure divides an inch into 100 equal parts, and if each inch be supposed to represent a chain, it gives single links.

Fig. 23.



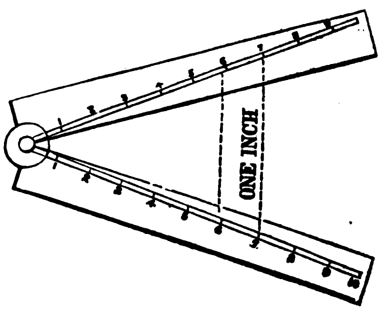
Make a scale of an inch divided into tenths, as in the upper scale of the above figure. Take in the dividers eleven of these divisions, and set off this distance from the 0 of the scale to the left of it. Divide the distance thus set off into 10 equal parts. Each of them will be one tenth of eleven tenths of one inch; i. e. eleven hundredths, or a tenth and a hundredth, and the first division on the short, or vernier scale, will overlap, or be longer than the first division on the long scale, by just *one* hundredth of an inch; the second division will overlap *two* hundredths, and so on. The principle will be more fully developed in treating of "Verniers," Part IV, Chapter II.

Now suppose we wish to take off from this scale 275 hundredths of an inch. To get the last figure, we must take five divisions on the lower scale, which will be 55 hundredths, for the reason just given. 220 will remain which are to be taken from the upper scale, and the entire number will be obtained at once by extending the dividers between the arrow-heads in the figure from 220 on the upper scale (measuring along its lower side) to 55 on the lower scale, 254 would extend from 210 on the upper scale to 44 on the lower. 318 would extend from 230 on the upper scale to 88 on the lower. Always begin then with subtracting 11 times the last figure from the given number; find the remainders on the upper scale, and the number subtracted on the lower scale.

(51) A plat is sometimes made by a nominally reduced scale in the following manner. Suppose that the scale of the plat is to be ten chains to one inch, and that a diagonal scale of inches, divided into tenths and hundredths, is the only one at hand. By dividing all the distances by ten, this scale can then be used without any further reduction. But if the content is measured from the plat to the same scale, in the manner explained in the next chapter, the result must be multiplied by 10 times 10. This is called by old Surveyors "Raising the scale," or "Restoring true measure."

(52) Sectoral Scales. The *Sector*, (called by the French "Compass of Proportion"), is an instrument sometimes convenient for obtaining a scale of equal parts. It is in two portions, turning on a hinge, like a carpenter's pocket rule. It contains a great number of scales, but the one intended for this use is lettered at its ends L in English instruments, and consists of two lines running from the centre to the ends of the scale, and each divided into ten equal parts, each of which is again subdivided into 10, so that each leg of the scale contains 100 equal parts. To illustrate its use, suppose that a scale of 7 chains to 1 inch is required. Take 1 inch in the dividers, and open the sector till this distance will just reach from the 7 on one leg to the 7 on the other. The sector is then "*set*" for this

Fig. 24.



scale, and the angle of its opening must not be again changed. Now let a distance of 580 links be required. Open the dividers till they reach from 58 to 58 on the two legs, as in the dotted line in the figure, and it is the required distance. Again, suppose that a scale of $2\frac{1}{2}$ chains to one inch is desired. Open the sector so that 1 inch shall extend from 25 to 25. Any other scale may be obtained in the same manner.

Conversely, the length of any known line to any desired scale can thus be readily determined.

(53) Whatever scale may be adopted for platting the survey, it should be drawn on the map, both for convenience of reference, and in order that the contraction and expansion, caused by changes in the quantity of moisture in the atmosphere, may affect the scale and the map alike. When the drawing paper has been wet and glued to a board, and cut off when the map is completed, its contractions have been found by many observations to average from one-fourth to one-half per cent. on a scale of 3 chains to an inch, (1:2376), which would therefore require an allowance of from one-half perch to one perch per acre.

A scale made as directed in Art. **(49)**, if used to make a plat on unstretched paper, and then kept with the plat, will answer nearly the same purpose.

Such a scale may be attached to a map, by slipping it through two or three cuts in the lower part of the sheet, and will be a very convenient substitute for a pair of dividers in measuring any distance upon it.

(54) Scale omitted. It may be required to find the unknown scale to which a given map has been drawn, its superficial content being known. Assume any convenient scale, measure the lines of the map by it, and find the content by the methods to be given in the next chapter, proceeding as if the assumed scale were the true one. Then make this proportion, founded on the geometrical principle that the areas of similar figures are as the squares of their corresponding sides: *As* the content found *Is* to the given content *So is* the square of the assumed scale *To* the square of the true scale.

CHAPTER IV.

CALCULATING THE CONTENT.

(55) The CONTENT of a piece of ground is its superficial *area*, or the number of square feet, yards, acres, or miles which it contains.

(56) Horizontal Measurement. All ground, however inclined or uneven its surface may be, should be measured horizontally, or as if brought down to a horizontal plane, so that the surface of a hill, thus measured, would give the same content as the level base on which it may be supposed to stand, or as the figure which would be formed on a level surface beneath it by dropping plumb lines from every point of it.

This method of procedure is required for both Geometrical and Social reasons.

Geometrically, it is plain that this horizontal measurement is absolutely necessary for the purpose of obtaining a correct plat. In Fig. 25, let ABCD, and BCEF, be two square lots of ground, platted horizontally. Suppose the ground to slope in all, directions from the point C, which is the summit of a hill. Then the lines BC, DC, measured on the slope, are longer than if measured on a level, and the field ABCD, of Fig. 25, platted with these long lines, would take the shape ABGD in Fig. 26; and the field BCEF, of Fig. 25, would become BHEF of Fig. 26. The two adjoining fields would thus overlap each other; and the same difficulty would occur in every case of platting any two adjoining fields by the measurements made on the slope.

Fig. 25.

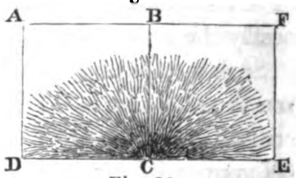


Fig. 26.

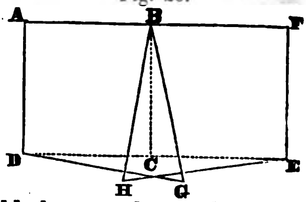


Fig. 27.

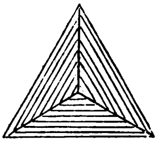
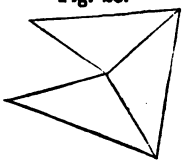


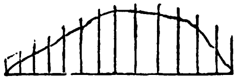
Fig. 28.



Let us suppose another case, more simple than would ever occur in practice, that of a three-sided field, of equal sides and composed of three portions each sloping down uniformly, (at the rate of one to one) from one point in the centre, as in Fig. 27. Each slope being accurately platted, the three could not come together, but would be separated as in Fig. 28.

We have here taken the most simple cases, those of uniform slopes. But with the common irregularities of uneven ground, to measure its actual surface would not only be improper, but impossible.

Fig. 29.



In the *Social* aspect of this question, the horizontal measurement is justified by the fact that no more houses can be built on a hill than could be built on its flat base; and that no more trees, corn, or other plants, which shoot up vertically, can grow on it; as is represented by the vertical lines in the Figure.* Even if a side hill should produce more of certain creeping plants, the increased difficulty in their cultivation might perhaps balance this. For this reason the surface of the soil thus measured is sometimes called *the productive base* of the ground.

Again, a piece of land containing a hill and a hollow, if measured on the surface would give a larger content than it would after the hollow had been filled up by the hill, while it would yet really be of greater value than before.

Horizontal measurement is called the "Method of Cultellation," and Superficial measurement, the "Method of Developement."†

An act of the State of New-York prescribes that "The acre, for land measure, shall be measured horizontally."

* This question is more than two thousand years old, for Polybius writes, Some even of those who are employed in the administration of states, or placed at the head of armies, imagine that unequal and hilly ground will contain more houses than a surface which is flat and level. This, however, is not the truth. For the houses being raised in a vertical line, form right angles, not with the declivity of the ground, but with the flat surface which lies below, and upon which the hills themselves also stand."

† The former from *Cultellum*, a knife, as if the hills were sliced off; the latter so named because it strips off or unfolds, as it were, the surface.

(57) Unit of Content. The *Acre* is the unit of land-measurement. It contains 4 Roods. A *Rood* contains 40 Perches. A *Perch* is a square Rod; otherwise called a Perch, or Pole. A *Rod* is $5\frac{1}{2}$ yards, or $16\frac{1}{2}$ feet.

Hence, 1 acre = 4 Roods = 160 Perches = 4,840 square yards = 43,560 square feet.

One square mile = 5280 × 5280 feet = 640 acres.

Since a chain is 66 feet long, a square chain contains 4356 square feet; and consequently *ten square chains make one acre.**

In different parts of England, the acre varies greatly. The statute acre, as in the United States, contains 160 square perches of $16\frac{1}{2}$ feet, or 43,560 square feet. The acre of Devonshire and Somersetshire, contains 160 perches of 15 feet, or 36,000 square feet. The acre of Cornwall is 160 perches of 18 feet, or 51,840 square feet. The acre of Lancashire is 160 perches of 21 feet, or 70,560 square feet. The acre of Cheshire and Staffordshire, is 160 perches of 24 feet, or 92,160 square feet. The acre of Wiltshire is 120 perches of $16\frac{1}{2}$ feet, or 32,670 square feet. The acre in Scotland consists of 10 square chains, each of 74 feet, and therefore contains 54,760 square feet. The acre in Ireland is the same as the Lancashire. The chain is 84 feet long.

The French units of land-measure are the *Are* = 100 square *Metres*, = 0.0247 acre, = one fortieth of an acre, nearly; and the *Hectare* = 100 *Ares* = 2.47 acres, or nearly two and a half. Their old land-measures were the "Arpent of Paris," containing 36,800 square feet; and the "Arpent of Waters and Woods," containing 55,000 square feet.

(58) When the content of a piece of land (obtained by any of the methods to be explained presently) is given in square links, as is customary, cut off four figures on the right, (i. e. divide by 10,000), to get it into square chains and decimal parts of a chain; cut off the right hand figure of the *square chains*, and the remaining figures will be *Acres*. Multiply the remainder by 4, and the figure, if any, outside of the new decimal point will be *Roods*.

* Let the young student beware of confounding 10 square chains with 10 chains square. The former make one acre; the latter space contains ten acres.

Multiply the remainder by 40, and the outside figures will be *Perches*. The nearest round number is usually taken for the Perches; fractions less than a half perch being disregarded.*

Thus, 86.22 square chains	= 8 Acres	2 Roods	20 Perches.
Also, 64.1818 do.	= 6 A.	1 R.	27 P.
" 43.7564 do.	= 4 A.	1 R.	20 P.
" 71.1055 do.	= 7 A.	0 R.	18 P.
" 82.50 do.	= 8 A.	1 R.	0 P.
" 8.250 do.	= 0 A.	3 R.	12 P.
" 0.8250 do.	= 0 A.	0 R.	13 P.

(59) The following Table gives by mere inspection the Roods and Perches corresponding to the Decimal parts of an Acre. It explains itself.

Decimal parts of an acre.	ROODS.				
	0	1	2	3	Perches.
	.000	.250	.500	.750	+ 0
	.006	.256	.506	.756	+ 1
	.012	.262	.512	.762	+ 2
	.019	.269	.519	.769	+ 3
	.025	.275	.525	.775	+ 4
	.031	.281	.531	.781	+ 5
	.037	.287	.537	.787	+ 6
	.044	.294	.544	.794	+ 7
	.050	.300	.550	.800	+ 8
	.056	.306	.556	.806	+ 9
	.062	.312	.562	.812	+10
	.069	.319	.569	.819	+11
	.075	.325	.575	.825	+12
	.081	.331	.581	.831	+13
	.087	.337	.587	.837	+14
	.094	.344	.594	.844	+15
	.100	.350	.600	.850	+16
	.106	.356	.606	.856	+17
	.112	.362	.612	.862	+18
	.119	.369	.619	.869	+19
	.125	.375	.625	.875	+20

Decimal parts of an acre.	ROODS.				
	0	1	2	3	Perches.
	.131	.381	.631	.881	+21
	.137	.387	.637	.887	+22
	.144	.394	.644	.894	+23
	.150	.400	.650	.900	+24
	.156	.406	.656	.906	+25
	.162	.412	.662	.912	+26
	.169	.419	.669	.919	+27
	.175	.425	.675	.925	+28
	.181	.431	.681	.931	+29
	.187	.437	.687	.937	+30
	.194	.444	.694	.944	+31
	.200	.450	.700	.950	+32
	.206	.456	.706	.956	+33
	.212	.462	.712	.962	+34
	.219	.469	.719	.969	+35
	.225	.475	.725	.975	+36
	.231	.481	.731	.981	+37
	.237	.487	.737	.987	+38
	.244	.494	.744	.994	+39
	.250	.500	.750	1.000	+40

(60) Chain Correction. When a survey has been made, and the plat has been drawn, and the content calculated; and after-

* To reduce square yards to acres, instead of dividing by 4840, it is easier, and very nearly correct, to multiply by 2, cut off four figures, and add to this product one-third of one-tenth of itself.

wards the chain is found to have been incorrect, too short or too long, the true content of the land, may be found by this proportion: *As* the square of the length of the standard given by the incorrect chain *Is to* the square of the true length of the standard *So is* the calculated content *To* the true content. Thus, suppose that the chain used had been so stretched that the standard distance measured by it appears to be only 99 links long; and that a square field had been measured by it, each side containing 10 of these long chains, and that it had been so platted. This plat, and therefore the content calculated from it, will be smaller than it should be, and the correct content will be found by the proportion $99^2 : 100^2 :: 100$ sq. chains :: 102.03 square chains. If the chain had been stretched so as to be 101 *true* links long, as found by comparing it with a correct chain, the content would be given by this proportion: $100^2 : 101^2 ::$ 100 square chains : 102.01 square chains. In the former case, the elongation of the chain was $1\frac{1}{99}$ true links; and $100^2 : (101\frac{1}{99})^2 ::$ 100 square chains : 102.03 square chains.*

(61) Boundary Lines. The lines which are to be considered as bounding the land to be surveyed, are often very uncertain, unless specified by the title deeds.

If the boundary be a brook, the middle of it is usually the boundary line. On tide-waters, the land is usually considered to extend to low water mark.

Where hedges and ditches are the boundaries of fields, as is almost universally the case in England, the dividing line is generally the top edge of the ditch farthest from the hedge, both hedge and ditch belonging to the field on the hedge side. This varies, however, with the customs of the locality. From three to six feet from the roots of the quickwood of the hedges are allowed for the ditches.

METHODS OF CALCULATION.

(62) The various methods employed in calculating the content of a piece of ground, may be reduced to four, which may be called *Arithmetical*, *Geometrical*, *Instrumental*, and *Trigonometrical*.

(63) FIRST METHOD.—ARITHMETICALLY. *From direct measurements of the necessary lines on the ground.*

The figures to be calculated by this method may be either the shapes of the fields which are measured, or those into which the fields can be divided by measuring various lines across them.

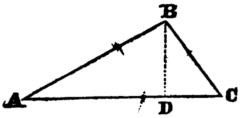
The familiar rules of mensuration for the principal figures which occur in practice, will be now briefly enunciated.

(64) Rectangles. If the piece of ground be rectangular in shape, its content is found by multiplying its length by its breadth.

(65) Triangles. When the given quantities are one side of a triangle and the perpendicular distance to it from the opposite angle; the content of the triangle is equal to half the product of the side and the perpendicular.

When the given quantities are the three sides of the triangle; add together the three sides and divide the sum by 2; from this half sum subtract each of the three sides in turn; multiply together the half sum and the three remainders; take the square root of the product; it is the content required. If the sides of the triangle be designated by a, b, c, and their sum by s, this rule will give its area $= \sqrt{[\frac{1}{2}s\ (\frac{1}{2}s - a)\ (\frac{1}{2}s - b)\ (\frac{1}{2}s - c)]}$.*

Fig. 30.



* When two sides of a triangle, and the included angle are given, its content equals half the product of its sides into the sine of the included angle. Designating the angles of the triangle by the capital letters A,B,C, and the sides opposite them by the corresponding small letters a,b,c, the area $= \frac{1}{2}\ bc$ sin. A.

When one side of a triangle and the adjacent angles are given, its content equals the square of the given side multiplied by the sines of each of the given angles, and divided by twice the sine of the sum of these angles. Using the same symbols as before, the area $= a^2 \dfrac{\text{sin. B . sin. C}}{2 \text{ sin. } (\text{B} + \text{C})}$.

When the three angles of a triangle and its altitude are given, its area, referring to the above figure, $= \frac{1}{2}\ \text{BD}^2 \cdot \dfrac{\text{sin. B}}{\text{sin. A . sin. C}}$.

(66) Parallelograms; or four-sided figures whose opposite sides are parallel. The content of a Parallelogram equals the product of one of its sides by the perpendicular distance between it and the side parallel to it.

(67) Trapezoids; or four-sided figures, two opposite sides of which are parallel. The content of a Trapezoid equals half the product of the sum of the parallel sides by the perpendicular distance between them.

If the given quantities are the four sides a, b, c, d, of which b and d are parallel; then, making $q = \frac{1}{2}(a + b + c - d)$, the area of the trapezoid will $= \frac{b+d}{b-d} \sqrt{[q(q-a)(q-c)(q-b+d).]}$*

(68) Quadrilaterals, or Trapeziums; four-sided figures, none of whose sides are parallel.

A very gross error, often committed as to this figure, is to take the average, or half sum of its opposite sides, and multiply them together for the area: thus, assuming the trapezium to be equivalent to a rectangle with these averages for sides.

In practical surveying, it is usual to measure a line across it from corner to corner, thus dividing it into two triangles, whose sides are known, and which can therefore be calculated by Art. (65).†

* When two parallel sides, b and d, and a third side, a, are given, and also the angle, C, which this third side makes with one of the parallel sides, then the content of the trapezoid $= \frac{b+d}{2} \cdot a \cdot \sin. C$.

† *When two opposite sides, and all the angles are given*, take one side and its adjacent angles, (or their supplements, when their sum exceeds 180°), consider them as belonging to a triangle, and find its area by the second formula in the note on page 43. Do the same with the other side and its adjacent angles. The difference of the two areas will be the area of the quadrilateral.

When three sides and their two included angles are given, multiply together the sine of one given angle and its adjacent sides. Do the same with the sine of the other given angle and its adjacent sides. Multiply together the two opposite sides and the sine of the supplement of the sum of the given angles. Add together the first two products, and add also the last product, if the sum of the given angles is more than 180°, or subtract it if this sum be less, and take half the result. Calling the given sides, p, q, r; and the angle between p and $q = A$; and the angle between q and $r = B$; the area of the quadrilateral

$$= \tfrac{1}{2}[p \cdot q \cdot \sin. A + q \cdot r \cdot \sin. B \pm p \cdot r \cdot \sin. (180° - A - B)].$$

When the four sides and the sum of any two opposite angles are given, proceed thus: Take half the sum of the four given sides, and from it subtract each side in turn Multiply together the four remainders, and reserve the product. Multiply together the four sides. Take half their product, and multiply it by the cosine of the given sum of the angles increased by unity. Regard the sign of

(69) *Surfaces bounded by irregularly curved lines.* The rules for these will be more appropriately given in connection with the surveys which measure the necessary lines; as explained in Part II, Chap. III.

(70) **SECOND METHOD.—GEOMETRICALLY.** *From measurements of the necessary lines upon the plat.*

(71) **Division into Triangles.** The plat of a piece of ground having been drawn from the measurements made by any of the methods which will be hereafter explained, lines may be drawn upon the plat so as to divide it into a number of triangles. Four

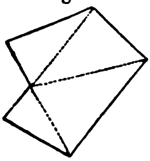
Fig. 31.


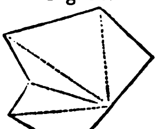
Fig. 32.


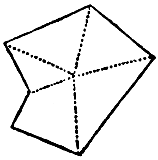
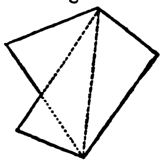
Fig. 33.


Fig. 34.


ways of doing this are shown in the figures: viz. by drawing lines from one corner to the other corners; from a point in one of the sides to the corners; from a point inside of the figure to the corners; and from various corners to other corners. The last method is usually the best. The lines ought to be drawn so as to make the triangles as nearly equilateral as possible, for the reasons given in Part V.

One side of each of these triangles, and the length of the perpendicular let fall upon it, being then measured, as directed in Art. (43,) the content of these triangles can be at once obtained by multiplying their base by their altitude, and dividing by two.

The easiest method of getting the length of the perpendicular, without actually drawing it, is, to set one point of the dividers at the angle from which a perpendicular is to be let fall, and to

the cosine. Multiply this product by the reserved product, and take the square root of the resulting product. It will be the area of the quadrilateral.

When the four sides, and the angle of intersection of the diagonals of the quadrilateral are given; square each side; add together the squares of the opposite sides; take the difference of the two sums; multiply it by the tangent of the angle of intersection, and divide by four. The quotient will be the area.

When the diagonals of the quadrilateral, and their included angle are given, multiply together the two diagonals and the sine of their included angle, and divide by two. The quotient will be the area.

open and shut their legs till an arc described by the other point will just touch the opposite side.

Otherwise; a platting scale, (described in Art. (49) may be placed so that the zero point of its edge coincides with the angle, and one of its cross lines coincides with the side to which a perpendicular is to be drawn. The length of the perpendicular can then at once be read off.

The method of dividing the plat into triangles is the one most commonly employed by surveyors for obtaining the content of a survey, because of the simplicity of the calculations required. Its correctness, however, is dependant on the accuracy of the plat, and on its scale, which should be as large as possible. Three chains to an inch is the smallest scale allowed by the English Tithe Commissioners for plats from which the content is to be determined.

In calculating in this way the content of a farm, and also of its separate fields, the sum of the latter ought to equal the former. A difference of one three-hundredth ($\frac{1}{300}$) is considered allowable.

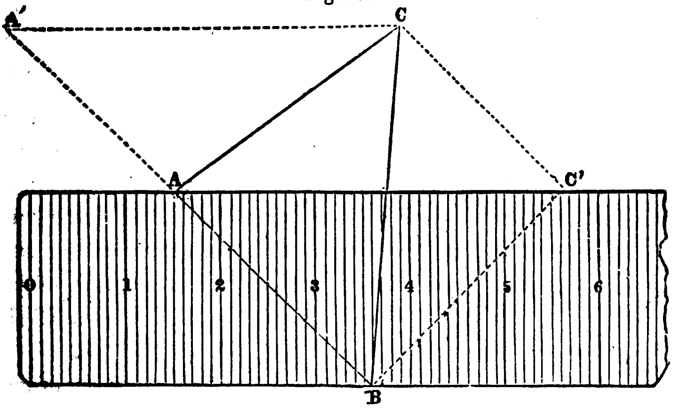
Some surveyors measure the perpendiculars of the triangles by a scale half of that to which the plat is made. Thus, if the scale of the plat be 2 chains to the inch, the perpendiculars are measured with a scale of one chain to the inch. The product of the base by the perpendicular thus measured, gives the area of the triangle at once, without its requiring to be divided by two.

Another way of attaining the same end, with less danger of mistakes, is, to construct a *new* scale of equal parts, longer than those by which the plat was made in the ratio $\sqrt{2}:1$; or $1.414:1$. When the base and perpendicular of a triangle are measured by this new scale and then multiplied together, the product will be the content of the triangle, without any division by two. In this method there is the additional advantage of the greater size and consequent greater distinctness of the scale.

When the measurement of a plat is made some time after it has been drawn, the paper will very probably have contracted or expanded so that the scale used will not exactly apply. In that case a correction is necessary. Measure very precisely the present length of some line on the plat of known length originally. Then make this proportion: *As* the square of the present length of this line *Is to* the square of its original length, *So is* the content obtained by the present measurement *To* the true content.

(72) *Graphical Multiplication.* Prepare a strip of drawing paper, of a width exactly equal to two chains on the scale of the plat; i. e. one inch wide, as in the figure, for a scale of two chains to 1 inch; two-thirds of an inch wide for a scale of 3 chains; half an inch for 4 chains; and so on. Draw perpendicular lines across the paper at distances representing one-tenth of a chain on the scale of the triangle to be measured, thus making a platting scale. Apply it to the triangle so that one edge of the scale shall pass through one corner, A, of the triangle, and the other edge through another

Fig. 35.

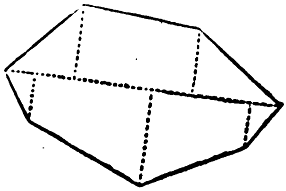


corner, B; and note very precisely what divisions of the scale are at these points. Then slide the scale in such a way that the points of the scale which had coincided with A and B, shall always remain on the line BA produced, till the edge arrives at the point C. Then will A'C, that is, the distance, or number of divisions on the scale, from the point to which the division A on the scale has arrived, to the third corner of the triangle, express the area of the triangle ABC in square chains.*

*For, from C draw a parallel to AB, meeting the edge of the scale in C'; and draw C'B. Then the given triangle ABC = ABC'. But the area of this last triangle = AC' multiplied by half the width of the scale, i. e. = AC' × 1 = AC'. But, because of the parallels, A'C = AC'. Therefore the area of the given triangle ABC = A'C; i. e. it is equal in square chains to the number of linear chains read off from the scale. This ingenious operation is due to *M. Cousinery*.

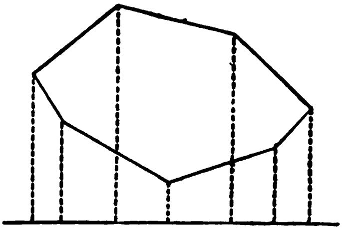
(73) Division into Trapezoids. A line may be drawn across the field, as in Fig. 36, and perpendiculars drawn to it. The field will thus be divided into trapezoids, (excepting a triangle at each end), and their content can be calculated by Art. (67).

Fig. 36.



Otherwise; a line may be drawn outside of the figure, and perpendiculars to it be drawn from each angle. In that case the difference between the trapezoids formed by lines drawn to the outer angles of the figure, and those drawn to the inner angles, will be the content.

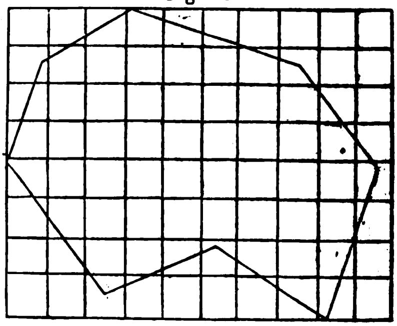
Fig. 37.



This method is very advantageously applied to surveys by the compass; as will be explained in Part III, Chap. VI.

(74) Division into Squares. Two sets of parallel lines, at right angles to each other, one chain apart (to the scale of the plat) may be drawn over the plat, so as to divide it into squares, as in the figure. The number of squares which fall within the plat represent so many square chains; and the triangles and trapezoids which fall outside of these, may then be calculated and added to the entire square chains which have been counted.

Fig. 38.



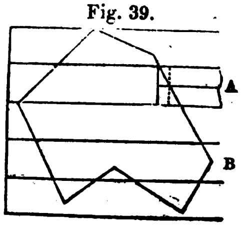
Instead of drawing the parallel lines on the plat, they may better be drawn on a piece of transparent "tracing paper," which is simply laid upon the plat, and the squares counted as before. The same paper will answer for any number of plats drawn to the same scale. This method is a valuable and easy check on the results of other calculations.

To calculate the fractional parts, prepare a piece of tracing paper, or horn, by drawing on it one square of the same size as a square of the plat, and subdividing it, by two sets of ten parallels at right angles to each other, into hundredths. This will measure the fractions remaining from the former measurement, as nearly as can be desired.

(75) Division into Parallelograms. Draw a series of parallel lines across the plat at equal distances depending on the scale. Thus, for a plat made to a scale of 2 chains to 1 inch, the distance between the parallels should be $2\frac{1}{2}$ inches; for a scale of 3 chains to 1 inch, $1\frac{1}{9}$ inch; for a scale of 4 chains to 1 inch, $\frac{5}{8}$ inch; for a scale of 5 chains to 1 inch, $\frac{4}{10}$ inch; and for any scale, make the distance between the parallels that fraction of an inch which would be expressed by 10 divided by the square of the number of chains to the inch. Then apply a common inch scale, divided on the edge into tenths, to these parallels; and every inch in length of the spaces included between each pair of them will be an acre, and every tenth of an inch will be a square chain.*

To measure the triangles at the ends of the strips between the parallels, prepare a piece of transparent horn, or stout tracing paper, of a width equal to the width between the parallels, and draw a line through its middle longitudinally. Apply it to the oblique line at the end of the space between two parallels, and it will bisect the line, and thus reduce the triangle to an equivalent rectangle, as at A in the figure. When an angle occurs between two parallels, as at B in the figure, the fractional part may be measured by any of the preceding methods.

Fig. 39.

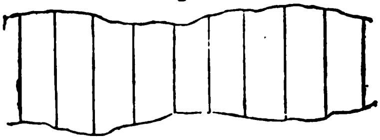


* For, calling the number of chains to the inch, $= n$, and making the width between the parallels $\frac{10}{n^2}$ inch, this width will represent $\frac{10}{n^2} \times n = \frac{10}{n}$ chains; and as the inch length represents n chains, their product, $\frac{10}{n} \times n = 10$ square chains $= 1$ acre.

A somewhat similar method is much used by some surveyors, particularly in Ireland: the plat being made on a scale of 5 chains to 1 inch, parallel lines being drawn on it, half an inch apart, and the distances along the parallels being measured by a scale, each large division of which is $\frac{8}{10}$ inch in length. Each division of this scale indicates an acre; for it represents 4 chains, and the distance between the parallels is $2\frac{1}{2}$ chains. This scale is called the "Scale of Acres."

(76) **Addition of Widths.** When the lines of the plat are very irregularly curved, as in the figure, draw across it a number of equi-distant lines as near together as the case may seem to require. Take a straight-edged piece of paper, and apply one edge of it to the middle of the first space, and mark its length from one end; apply the same edge to the middle of the next space, bringing the mark just made to one end, and making another mark at the end of the additional length; so go on, adding the length of each space to the previous ones. When all have been thus measured, the total length, multiplied by the uniform width, will give the content.

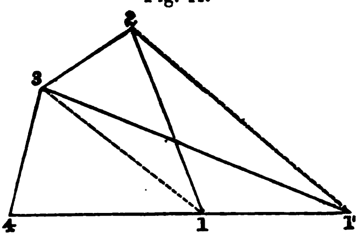
Fig. 40.



(77) **THIRD METHOD.—INSTRUMENTALLY.** *By performing certain instrumental operations on the plat.*

(78) *Reduction of a many sided figure to a single equivalent triangle.* Any plane figure bounded by straight lines may be reduced to a single triangle, which shall have the same content. This can be done by any instrument for drawing parallel lines, such as those described in Art. (39). Let the trapezium, or four sided figure, shown in Fig. 41, be required to be reduced to a single equivalent triangle. Produce one side of the figure, as 4——1. Draw a line from the first to the third angle of

Fig. 41.

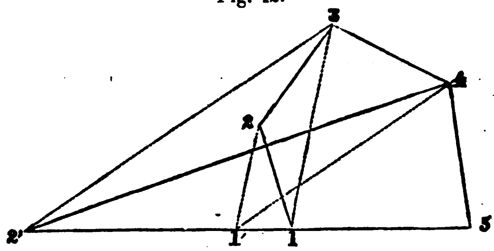


the figure. From the second angle draw a parallel to the line just drawn, cutting the produced side in a point 1′. From the point 1′ draw a line to the third angle. A triangle (1′—3—4 in the figure) will thus be formed, which will be equivalent to the original trapezium.*

The content of this final triangle can then be found by measuring its perpendicular, and taking half the product of this perpendicular by the base, as in the first paragraph of Art. (65).

(79) Let the given figure have five sides, as in Fig. 42. For brevity, the angles of the figure will be named as numbered in the engraving. Produce 5—1. Join 1—3. From 2 draw a parallel to 1—3, cutting the produced base in 1′. Join 1′—4. From 3 draw a parallel to it, cutting the base in 2′. Join 2′—4. Then will the triangle 2′—4—5 be equivalent to the five sided figure 1—2—3—4—5, for similar reasons to those of the preceding case.

Fig. 42.

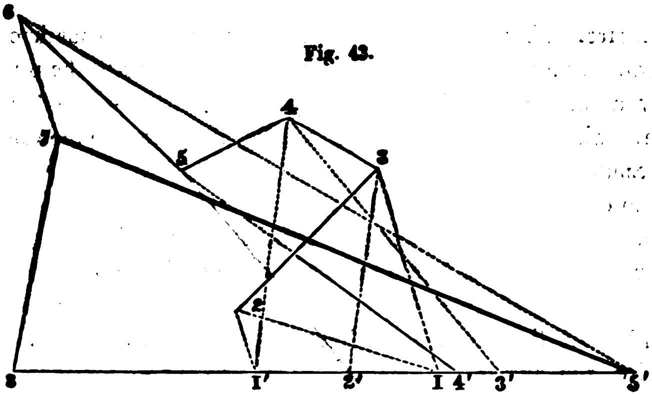


(80) Let the given figure be 1—2—3—4—5—6—7—8, as shown in Fig. 43, given at the top of the following page. All the operations are shown by dotted lines, and the finally resulting triangle 5′—7—8, is equivalent to the original figure of eight sides.

It is best, in choosing the side to be produced, to take one which has a long side adjoining it on the end not produced; so that this long side may form one side of the final triangle, the base of which will therefore be shorter, and will not be cut so acutely by the final line drawn, as to make the point of intersection too indefinite.

* For, the triangle 1—2—3 taken away from the original figure is equivalent to the triangle 1′—1—3 added to it; because both these triangles have the same base and also the same altitude, since the vertices of both lie in the same line parallel to the base.

Fig. 43.



(81) *General Rule.* When the given figure has many sides, with angles sometimes salient and sometimes re-entering, the operations of reduction are very liable to errors, if the draftsman attempts to reason out each step. All difficulties, however, will be removed by the following *General Rule:*

1. Produce one side of the figure, and call it a base. Call one of the angles at the base the first angle, and number the rest in regular succession around the figure.

2. Draw a line from the 1st angle to the 3d angle. Draw a parallel to it from the 2d angle. Call the intersections of this parallel with the base the 1st mark.

3. Draw a line from the 1st mark to the 4th angle. Draw a parallel to it from the 3d angle. Its intersection with the base is the 2d mark.

4. Draw a line from the 2d mark to the 5th angle. Draw a parallel to it from the 4th angle. Its intersection with the base is the 3d mark.

5. In general terms, which apply to every step after the first, draw a line from the last mark obtained to the angle whose number is greater by three than the number of the mark. Draw a parallel to it through the angle whose number is greater by two than that of the mark. Its intersection with the base will be a mark whose number is greater by one than that of the preceding mark.*

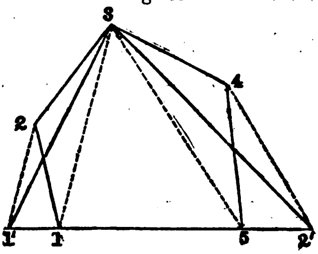
* In the concise language of Algebra, draw a line from the nth mark to the $n+3$ angle. Draw a parallel to it through the $n+2$ angle, and the intersection with the base will be the $n+1$ mark.

6. Repeat this process for each angle, till you get a mark whose number is such that the angle having a number greater by three is the last angle of the figure, i. e. the angle at the other end of the base. Then join the last mark to the angle which precedes the last angle in the figure, and the triangle thus formed will be the equivalent triangle required.

In practice it is *unnecessary* to actually draw the lines joining the successive angles and marks, but the parallel ruler is merely laid on so as to pass through them, and the points where the parallels cut the base are alone marked.

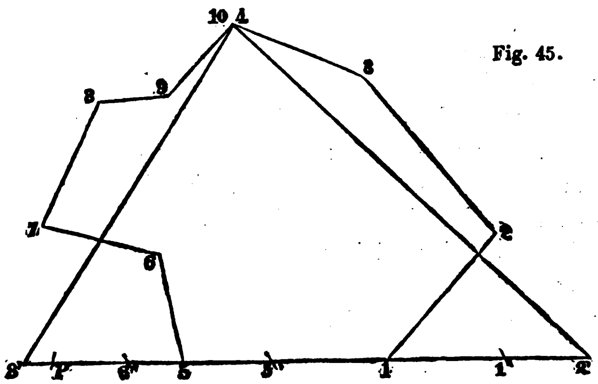
(82) It is generally more convenient, for the reasons given at the end of Art. (80), to reduce half of the figure on one side and half on the other, as is shown in Fig. 44, which represents the same field as Fig. 42. The equivalent triangle is here 1′—3—2′.

Fig. 44.



When the figure has many angles, they should not be numbered consecutively all the way around, but, after the numbers have gone around as far as the angle where it is intended to have the vertex

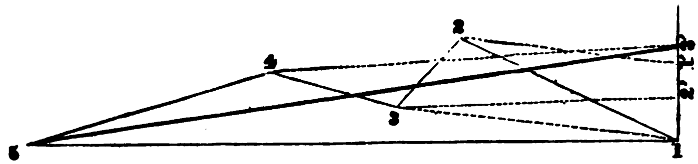
Fig. 45.



of the final triangle, the numbers should be continued from the other angle of the base, as is shown in Fig. 45. In it only the intersections are marked.*

(83) It is sometimes more convenient, not to produce one of the sides of the figure, but to draw at one end of it, as at the point 1 in Fig. 46, an indefinite line, usually a perpendicular to a line

Fig. 46.



joining two distant angles of the figure, and make this line the base of the equivalent triangle desired. The operation is shown by the dotted lines in the figure. The same General Rule applies to it, as to the previous figures.

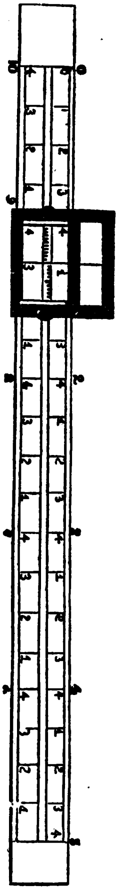
(84) **Special Instruments.** A variety of instruments have been invented for the purpose of determining areas rapidly and correctly. One of the simplest is the "*Computing Scale*," which is on the same principles as the Method of Art. (75). It is represented in Fig. 47, given on the following page. It consists of a scale divided for its whole length from the zero point into divisions, each representing $2\frac{1}{2}$ chains to the scale of the plat. The scale carries a slider, which moves along it, and has a wire drawn across its centre at right angles to the edges of the scale. On each side of this wire, a portion of the slider equal in length to one of the primary, or $2\frac{1}{2}$ chain, divisions of the scale, is laid off and divided into 40 equal parts.

This instrument is used in connection with a sheet of transparent paper, ruled with parallel lines at distances apart each equal to one chain on the scale of the plat. It is plain, that when the

* A figure with curved boundaries may be reduced to a triangle in a similar manner. Straight lines must be drawn about the figure, so as to be partly in it and partly out, giving and taking about equal quantities, so that the figure which these lines form, shall be about equivalent to the curved figure. This having been done, as will be further developed in Art. (124), the equivalent straight lined figure is reduced by the above method.

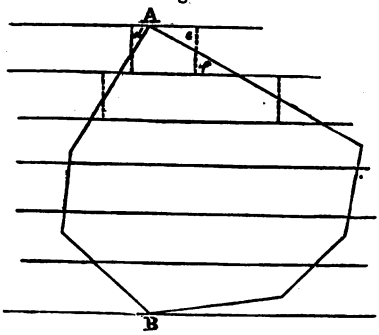
instrument is laid on this paper, with its edge on one of the parallel lines, and the slider is moved over one of the divisions of $2\frac{1}{2}$ chains, that *one rood*, or a quarter of an acre, has been measured between two of the parallel lines on the paper (since 10 square chains make one acre); and that one of the smaller divisions measures *one perch* between the same parallels. Four of the larger divisions give one acre. The scale is generally made long enough to measure at once five acres.

Fig 47.



To apply this to the plat of a field, or farm, lay the transparent paper over it in such a position that two of the ruled lines shall touch two of the exterior points of the boundaries, as at A and B. Lay the scale, with the slide set to zero, on the paper, in a direction parallel to the ruled lines, and so that the wire of the slide cuts the left hand oblique line so as to make the spaces *c* and *d* about equal. Hold the scale firm, and move the slider till the wire cuts the right hand oblique line in such a way as to equalize the spaces *e* and *f*. Without changing the slide, move the scale down the width of a space, and to the left hand end of the next space; begin there again, and proceed as before.

Fig. 48.



So go on, till the whole length of the scale is run out, (five acres having been measured), and then begin at the right hand side and work backwards to the left, reading the lower divisions, which run up to 10 acres. By continuing this process, the content of plats of any size can be obtained.

A still simpler substitute for this is a scale similarly divided, but without an attached slide. In place of it there is used a piece of horn having a line drawn across it and rivetted to the end of a short scale of box-wood, divided like the former slide. It is used like the former, except that at starting, the zero of the short scale and not the line on the horn is made to coincide with the zero of the long scale. The slide is to be held fast to the instrument when this is moved.

The *Pediometer* is another less simple instrument used for the same object. It measures any quadrilateral directly.

(85) Some very complicated instruments for the same object have been devised. One of them, Sang's *Planometer*, determines the area of any figure, by merely moving a point around the outline of the surface. This causes motion in a train of wheel work, which registers the algebraic sum of the product of ordinates to every point in that perimeter, by the increment of their abscissas, and therefore measures the included space.

Instruments of this kind have been invented in Germany by Ernst, Hansen, and Wetli.

A purely mechanical means of determining the area of any surface by means of its *weight*, may be placed here. The plat is cut out of paper and weighed by a delicate balance. The weight of a rectangular piece of the same paper containing just one acre is also found; and the "Rule of Three" gives the content. A modification of this is to paste a tracing of the plat on thin sheet lead, cut out the lead to the proper lines and weigh it.

(87) **FOURTH METHOD.—TRIGONOMETRICALLY.** *By calculating, from the observed angles of the boundaries of the piece of ground, the lengths of the lines needed for calculating the content.*

This method is employed for surveys made with angular instruments, as the compass, &c., in order to obtain the content of the land surveyed, without the necessity of previously making a plat, thus avoiding both that trouble and the inaccuracy of any calculations founded upon it. It is therefore the most accurate method; but will be more appropriately explained in Part III, Chapter VI, under the head of "Compass Surveying."

PART II.

CHAIN-SURVEYING;

By the First and Second Methods:

OR

DIAGONAL AND PERPENDICULAR SURVEYING.

(88) The chain alone is abundantly sufficient, without the aid of any other instrument, for making an accurate survey of any surface, whatever its shape or size, particularly in a district tolerably level and clear. Moreover, since a chain, or some substitute for it, formed of a rope, of leather driving reins, &c., can be obtained by any one in the most secluded place, this method of Surveying deserves more attention than has usually been given to it in this country. It will, therefore, be fully developed in the following chapters.

CHAPTER I.

SURVEYING BY DIAGONALS:

OR

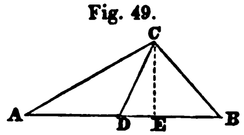
By the First Method.

(89) *Surveying by Diagonals* is an application of the *First Method* of determining the position of a point, given in Art. (**5**,) to which the student should again refer. Each corner of the field or farm which is to be surveyed is "determined" by measuring its distances from two other points. The field is then "platted" by repeating this process on paper, for each corner, in a contrary order, and the "content" is obtained by some of the methods explained in Chapter IV of Part I.

The lines which are measured in order to determine the corners of the field are usually *diagonals* of the irregular polygon which is to be surveyed. They therefore divide up this polygon into triangles; whence this mode of surveying is sometimes called "Chain Triangulation."

A few examples will make the principle and practice perfectly clear. Each will be seen to require the three operations of *measuring*, *platting*, and *calculating*.

Fig. 49.



(60) A three-sided field; as Fig. 49.

Field-work. Measure the three sides, AB, BC, and CA. Measure also, as a proof line, the distance from one of the corners, as C, to some point in the opposite side, as D, at which a mark should have been left, when measuring from A to B, at a known distance from A. A stick or twig, with a slit in its top, to receive a piece of paper with the distance from A marked on it, is the most convenient mark.

Platting. Choose a suitable scale as directed in Art. (**44**). Then, by Arts. (**42**) and (**49**), draw a line equal in length, on the chosen scale, to one of the sides; AB for example. Take in the compasses the length of another side as AC, to the same scale, and with one leg in A as a centre, describe an arc of a circle. Take the length of the third side BC, and with B as a centre, describe another arc, intersecting the first arc in a point which will be the third corner C. Draw the lines AC and BC; and ABC will be the *plat*, or miniature copy—as explained in Art. (**35**)—of the field surveyed.

Instead of describing two arcs to get the point C, two pairs of compasses may be conveniently used. Open them to the lengths, respectively, of the last two sides. Put one foot of each at the ends of the first side, and bring their other feet together, and their point of meeting will mark the desired third point of the triangle.

To "prove" the accuracy of the work, fix the point D, by setting off from A the proper distance, and measure the length of the line

DC, by Art. (43). If its length on the plat corresponds to its measurement on the ground, the work is correct.*

Calculation. The content of the field may now be found as directed in Art. (65), either from the three sides, or more easily though not so accurately, by measuring on the plat, by Art. (43), the length of the perpendicular CE, let fall from any angle to the opposite side, and taking half the product of these two lines.

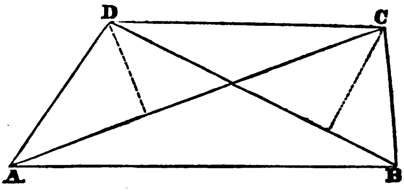
Example 1. Figure 49, is the plat, on a scale of two chains to one inch, of a field, of which the side AB is 200 links, BC is 100 links, and AC is 150 links. Its content by the rule of Art. (65), is 0.726 of a square chain, or 0A. 0R. 12P. If the perpendicular AD be accurately measured, it will be found to be 72½ links. Half the product of this perpendicular by the base will be found to give the same content.

Ex. 2. The three sides of a triangular field are respectively 89.39, 54.08, and 45.98. Required its content.

Ans. 100A. 0R. 10P.

(91) A four-sided field; as Fig. 50.

Fig. 50.



Field-work. Measure the four sides. Measure also a diagonal, as AC, thus dividing the four-sided field into two triangles. Measure also the other diagonal, or BD, for a "Proof line."

Platting. Draw a line, as AC, equal in length to the diagonal, to any scale, by Arts. (42) and (49). On each side of it, construct a triangle with the sides of the field, as directed in the preceding article.

To prove the accuracy of the work, measure on the plat the length of the "proof line," BD, by Art. (43), and if it agrees with the length of the same line measured on the ground, the field work and platting are both proved to be correct.

* It is a universal principle in all surveying operations, that the work must be tested by some means independent of the original process, and that the same result must be arrived at by two different methods. The necessary length of this proof line can also easily be calculated by the principles of Trigonometry.

Calculation. Find the content of each triangle separately, as in the preceding case, and add them together; or, more briefly, multiply either diagonal (the longer one is preferable) by the sum of the two perpendiculars, and divide the product by two.

Otherwise: reduce the four-sided figure to one triangle as in Art. (78); or, use any of the methods of the preceding chapter.

Example 3. In the field drawn in Fig. 50, on a scale of 3 chains to the inch, AB = 588 links, BC = 210, CD = 430, DA = 274, the diagonal AC = 626, and the proof diagonal BD = 500. The total content will be 1A. 0R. 17P.

Ex. 4. The sides of a four-sided field are AB = 12.41, BC = 5.86, CD = 8.25, DA = 4.24; the diagonal BD = 11.55, and the proof line AC = 11.04. Required the content.

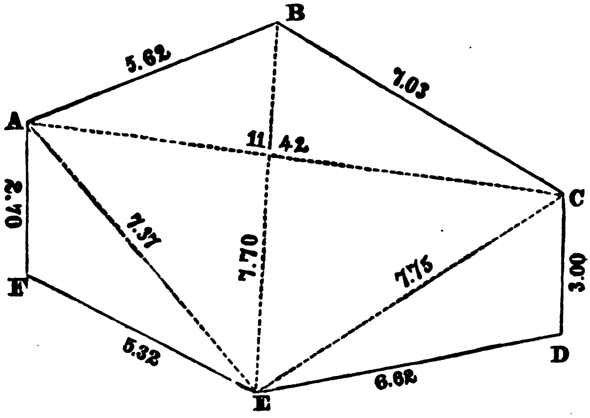
Ans. 4A. 2R. 38P.

Ex. 5. The sides of a four-sided field are as follows: AB = 8.95, BC = 5.33, CD = 10.10, DA = 6.54; the diagonal from A to C is 11.52; the proof diagonal from B to D is 10.92. Required the content. *Ans.*

Ex. 6. In a four-sided field, AB = 7.68, BC = 4.09, CD = 10.64, DA = 7.24. AC = 10.32. BD = 10.74. Required the content. *Ans.*

(92) **A many-sided field,** as Fig. 51.

Fig. 51.



Field-Work. Measure all the sides of the field. Measure also diagonals enough to divide the field into triangles; of which there will always be two less than the number of sides. Choose such diagonals as will divide the field into triangles as nearly equilateral as possible. Measure also one or more diagonals for "Proof lines." It is well for the surveyor himself to place stakes in advance at all the corners of the field, as he can then select the best mode of division.

Platting. Begin with any diagonal and plat one triangle, as in Art. (**90**). Plat a second triangle adjoining the first one, as in Art. (**91**). Plat another adjacent triangle, and so proceed, till all have been laid down in their proper places. Measure the proof lines as in the last article.

Calculation. Proceed to calculate the content of the figure, precisely as directed for the four-sided field, measuring the perpendiculars and calculating the content of each triangle in turn; or taking in pairs those on opposite sides of the same diagonal; or using some of the other methods which have been explained.

Example 7. The six-sided field, shown in Fig. 51, has the lengths of its lines, in chains and links, written upon them, and is divided into four triangles, by three diagonals. The diagonal BE is a "proof-line." The Figure is drawn to a scale of 4 chains to the inch. The content of the field is 5A. 3R. 22P.

Ex. 8. In a five-sided field, the length of the sides are as follows: AB = 2.69, BC = 1.22, CD = 2.32, DE = 3.55, EA = 3.23. The diagonals are AD = 4.81, BD = 3.33. Required its content. *Ans.*

(**93**) A field may be divided up into triangles, not only by measuring diagonals as in the last figure, but by any of the methods shown in the four figures of Art. (**71**). The one which we have been employing, corresponds to the last of those figures.

Still another mode may be used when the angles cannot be seen from one another, or from any one point within. Take three or more convenient points within the field, and measure from them to the corners, and thus form different sets of triangles.

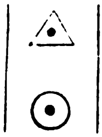
KEEPING THE FIELD NOTES.

(94) By Sketch. The most simple method is to make a sketch of the field, as nearly correct as the unassisted hand and eye can produce, and note down on it the lengths of all the lines, as in Fig. 51. But when many other points require to be noted, such as where fences, or roads, or streams are crossed in the measurement, or any other additional particulars, the sketch would become confused, and be likely to lead to mistakes in the subsequent platting from it. The following is therefore the usual method of keeping the Field-notes. A long narrow book is most convenient for it.

(95) In Columns. Draw two parallel lines about an inch apart from the bottom to the top of the page of the field-book, as in the margin. This column, or pair of lines, may be conceived to represent the measured line, *split in two*, its two halves being then separated, an inch apart, merely for convenience, so that the distances measured along the line, may be written between these halves.

Hold the book in the direction of the measurement. At the *bottom* of the page write down the name, or number, or letter, which represents the station at which the survey is to begin.

A "station" is marked with a triangle or circle, as in the margin. The latter is more easily made.



In the complicated cases, which will be hereafter explained, and in which one long base line is measured, and also many other subordinate lines, it will be well, as a help to the memory, to mark the stations on the Base line with a triangle, and the stations on the other lines with the ordinary circle.

The station from which the measurements are made is usually put on the left of the column; and the station which is measured to, is put on the right.

	⊙	*to* B
	562	
From A	⊙	

But it is more compact, and avoids interfering with the notes of "offsets" (to be explained hereafter) to write the name or number of the station *in* the column, as in the margin.

B
562
A

The measurements to different points of a line are written above one another. The numbers all refer to the beginning of the line, and are counted from it.

B
400
250
100
A

The end of a measured line is marked by a line drawn across the page above the numbers which indicate the measurements which have been made.

If the chaining does not continue along the adjoining line, but the chain-men go to some other part of the field to begin another measurement, *two* lines are drawn across the page.

When a line has been measured, the marks ┏ or ┓ are made to show whether the following line turns to the right or to the left.

A line is named, either by the names of the stations between which it is measured, as the line AB; or by its length, a line 562 links long, being called the line 562; or it is recorded as Line No. 1, Line No. 2, &c; or as Line on page 1, 2, &c., of the Field-book.

When a mark is left at any point of a line, as at D, in Fig. 49, with the intention of coming back to it again, in order to measure to some other point, the place marked is called a *False Station*, and is marked in the Field-book F. S.; or has a line drawn around it, to distinguish it; or has a station mark △ placed outside of the column, to the right or left, according to the direction in which the measurement from it is to be made. Examples of these three modes are given in the margin.

562
200 F. S.
0

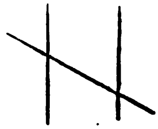
562
200
0

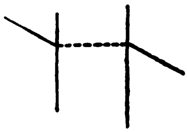
562
200 △
0

A False Station is named by its position on the line where it belongs; as thus—"200 on 562."

When a gate occurs in a measured line, the distance from the beginning of the line to the side of the gate first reached, is the one noted.

When the measured line crosses a fence, brook, road, &c., they are drawn on the field-notes in their true direction, as nearly as possible, but not in a continuous line across the column, as in the first figure in the margin, but as in the second figure, so that the two parts would form a continuous straight line, if the halves of the "split line" were brought together.





It is convenient to name the lines, in the margin, as being Sides, Diagonals, Proof lines, &c.

(96) The Field-notes of the triangular field platted in Fig. 49, are given below, according to both the methods mentioned in the preceding Article, pages 62 and 63.

In the Field-notes in the column on the right hand, it is not absolutely necessary to repeat the B and C.

PROOF LINE.		89	*to* C
	From D	F. S.	
SIDE.		150	*to* A
	From C	⊙	┐
SIDE.		100	*to* C
	From B	⊙	┐
SIDE.		200	*to* B
		80	F. S
	From A	⊙	

PROOF LINE.		C 89	
	From	80	*on* 200
SIDE.	┐	A 150 C	
SIDE.	┐	C 100 B	
SIDE.		B 200 80 A	

(97) The Field-notes of the survey platted in Fig. 51, are given below. They begin at the bottom of the left hand column.

SIDE.		F	
		532	
		300	*Gate.*
		E	
SIDE.		E	
		662	
		400	*Brook.*
		D	
SIDE.		D	
		300	
		270	*Road.*
		210	
		80	
		C	
SIDE.		C	
		703	
		150	*Gate.*
		B	
SIDE.		B	
		562	
		A	

PROOF LINE.		E	
		770	
		B	
DIAGONAL.		A	
		1142	
		C	
DIAGONAL.		C	
		775	
		480	*Road.*
		420	
		E	
DIAGONAL.		E	
		737	
		280	
		210	*Road.*
		A	
SIDE.		A	
		270	
		130	*Road.*
		80	
		F	

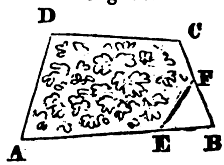
CHAPTER II.

SURVEYING BY TIE-LINES.

(98) *Surveying by Tie-lines* is a modification of the method explained in the last chapter. It frequently happens that it is impossible to measure the diagonals of a field of many sides, in consequence of obstacles to measurements, such as woods, water, houses, &c. In such cases, "*Tie-lines*," (so called because they *tie* the sides together), are employed as substitutes for diagonals.

Thus, in the four-sided field shown in the Figure, the diagonals cannot be measured because of woods intervening. As a substitute, measure off from any convenient corner of the field, as B, any distances, BE, BF, along the sides of the field. Measure also the "tie-line" EF. Measure all the sides of the field as usual.

Fig. 52.



To plat this field, construct the triangle BEF, as in Art. (90). Produce the sides BE and BF, till they become respectively equal to BA and BC, as measured on the ground. Then with A and C as centres, and with radii respectively equal to AD and CD, describe arcs, whose intersection will be D, the remaining corner of the field.

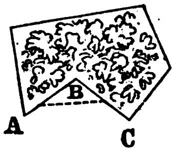
(99) It thus appears that one tie-line is sufficient to determine a four-sided field; two, a five-sided field, and so on. But, as a check on errors, it is better to measure a tie-line for each angle, and the agreement, in the plat, of all the measurements will prove the accuracy of the whole work.

Since any inaccuracy in the length of a tie-line is increased in proportion to the greater length of the sides which it fixes, the tie-lines should be measured as far from the point of meeting of these sides as possible, that is, they should be as long as possible.

The radical defect of the system is that it is "working from less to greater," (which is the exact converse of the true principle), thus magnifying inaccuracies at every step.

A tie-line may also be employed as a "proof line," in the place of a diagonal, and tested in the same manner.

Fig. 53.

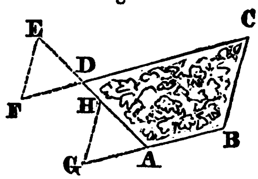


If any angle of the field is re-entering, as at B in the figure, measure a tie-line across the salient angle ABC.

(100) Chain Angles. It is convenient, though not necessary, to measure equal distances along the sides; BE, BF, in Fig. 52, and BA, BC, in Fig. 53. "Chain Angles" are thus formed.*

(101) Inaccessible Areas. The method of tie-lines can be applied to measuring fields which cannot be entered.

Fig. 54.



Thus, in the Figure, ABCD is an inaccessible wooded field, of four sides. To survey it, measure all the sides, and at any corner, as D, measure any distance DE, in the line of AD produced. Measure also another distance DF in the line of CD produced. Measure the tie-line EF, and the figure can be platted as in the case of the field of Fig. 52, the sides of the triangle being produced in the contrary direction.

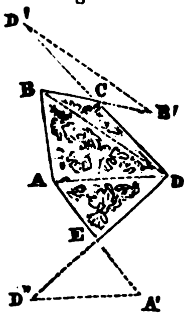
The same end would be attained by prolonging only one side, as shown at the angle A of the same figure, and measuring AG, AH, and GH. It is better in both cases to tie *all* the angles in a similar manner.

This method may be applied to a figure of any number of sides by prolonging as many of them as are necessary; all of them, if possible.

* Chain angles may be reduced to angles measured in degrees, by observing that the tie-line is the chord of the angle to a radius equal to one of the equal distances measured on the sides. Therefore, divide the length of the tie-line by the length of this distance. The quotient will be the chord of the angle to a radius of *one*. In the TABLE OF CHORDS, at the end of this volume, find this quotient, and the number of degrees and minutes corresponding to it gives the angle required. Otherwise; since the chord of any angle equals twice the sine of half the angle, we have this rule; Divide half the tie-line by the measured distance; find in a table of natural sines the angle corresponding to the quotient, and multiply this angle by two, to get the angle desired.

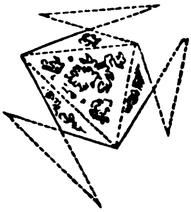
(102) If the sides CD and AD were prolonged by their full length, the content of the figure could be calculated without any plat; for the new triangle DEF would equal the triangle DAC; and the sides of the triangle ACB would then be known.

Fig. 55.



This principle may be extended still farther. For a five-sided field, as in Fig. 55, produce two pairs of sides, a distance equal to their length, forming two new triangles, as shown by the dotted lines, and measure the sides B'D', and A'D''. The three sides of each of these triangles will thus be known, and also the three sides of the triangle BAD, since AD = A'D'', and BD = B'D'.

Fig. 56.



The method of this article may be employed for a figure of six sides as shown in Fig. 56, (in which the dotted lines within the wooded field have their lengths determined by the triangles formed outside of it,) but not for figures of a greater number of sides.

CHAPTER III.

SURVEYING BY PERPENDICULARS:

OR

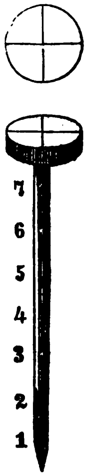
By the Second Method.

THE method of *Surveying by Perpendiculars* is founded on the *Second Method* of determining the position of a point, explained in Art. (6). It is applied in two ways, either to making a complete Survey by "*Diagonals and Perpendiculars*," or to measuring a crooked boundary by "*Off-sets*." Each will be considered in turn.

The best methods of getting perpendiculars on the ground must, however, be first explained.

TO SET OUT PERPENDICULARS.

Fig. 57.

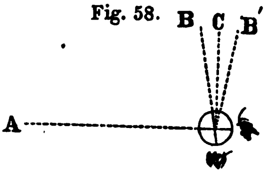


(104) Surveyor's Cross. The simplest instrument for this purpose is the *Surveyor's Cross*, or *Cross-Staff*, shown in the figure. It consists of a block of wood, of any shape, having in it two saw-cuts, made very precisely at right angles to each other, about half an inch deep, and with centre-bit holes made at the bottom of the cuts to assist in finding the objects. This block is fixed on a pointed staff, on which it can turn freely, and which should be precisely 8 links ($63\frac{1}{3}$ inches) long, for the convenience of short measurements.

To use the Cross-staff to erect a perpendicular, set it at the point of the line at which a perpendicular is wanted. Turn its head till, on looking through one saw-cut, you see the ends of the line. Then will the other saw-cut point out the direction of the perpendicular, and thus guide the measurement desired.

To find where a perpendicular to the line, from some object, as a corner of a field, a tree, &c., would meet the line, set up the cross-staff at a point of the line which seems to the eye to be about the spot. Note about how far from the object the perpendicular at this point strikes, and move the cross-staff that distance; and repeat the operation till the correct spot is found.

Fig. 58.

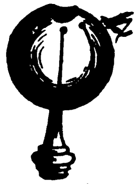


(105) To test the accuracy of the instrument, sight through one slit to some point A, and place a stake B in the line of sight of the other slit. Then turn its head a quarter of the way around, so that the second slit looked through, points to A. Then see if the other slit covers B again, as it will if correct. If it does not do so, but sights to some other point, as B′, the apparent error is double the real one, for it now points as far to the right of the true point, C as it did before to its left.

This test is the first example we have had of the invaluable principle of *Reversion*, which is used in almost every test of the accuracy of Surveying and Astronomical instruments, its peculiar merit being that it doubles the real error, and thus makes it twice as easy to perceive and correct it.

(106) The instrument, in its most finished form, is made of a hollow brass cylinder, which has two pairs of slits exactly opposite to each other, one of each pair being narrow and the other wide, with a horse-hair stretched from the top to the bottom of the latter. It is also, sometimes, made with eight faces, and two more pairs of slits added, so as to set off half a right angle.

Fig. 59.

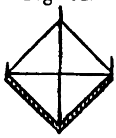


Another form is a hollow brass sphere, as in the figure. This enables the surveyor to set off perpenpendiculars on very steep slopes.

Fig. 60.

Another form of the surveyor's cross consists of two pairs of plain "Sights," each shaped as in the figure, placed at the ends of two bars at right angles to each other. The slit, and the opening with a hair stretched from its top to its bottom, are respectively at the top of one sight and at the bottom of the opposite sight.* This is used in the same manner as the preceding form, but is less portable and more liable to get out of order.

Fig. 61.

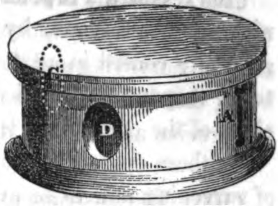


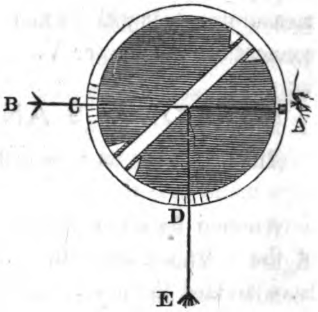
A temporary substitute for these instruments may be made by sticking four pins into the corners of a square piece of board; and sighting across them, in the direction of the line and at right angles to it.

(107) **Optical Square.** The most convenient and accurate instrument is, however, the Optical Square. The figures give a perspective view of it, and also a plan with the lid removed. It is a small circular box, containing a strip of looking-glass, from the upper half of which the silvering is removed. This glass is placed

* The French call the narrow opening *œilleton*, and the wide one *croisée*.

Fig. 62.





so as to make precisely half a right angle with the line of sight, which passes through a slit on one side of the box, and a vertical hair stretched across the opening on the other side, or a mark on the glass. The box is held in the hand over the spot where the perpendicular is desired, (a plumb line in the hand will give perfect accuracy) and the observer applies his eye to the slit A, looking through the upper or unsilvered part of the glass, and turns the box till he sees the other end of the line B, through the opening C. The assistant, with a rod, moves along in the direction where the perpendicular is desired, being seen in the silvered parts of the glass, by reflection through the opening D, till his rod, at E, is seen to coincide with, or to be exactly under, the object B. Then is the line DE at right angles to the line AB, by the optical principle of the equality of the angles of incidence and reflection.

To find where a perpendicular from a distant object would strike the line, walk along the line, with the instrument to the eye, till the image of the object is seen, in the silvered part of the glass, to coincide with the direction of the line seen through the unsilvered part.

The instrument may be tested by sighting along the perpendicular, and fixing a point in the original line; on the principle of "Reversion."

The surveyor can make it for himself, fastening the glass in the box by four angular pieces of cork, and adjusting it by cutting away the cork on one side, and introducing wedges on the other side. The box should be blackened inside.

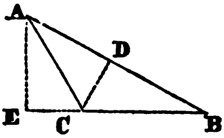
(108) Chain Perpendiculars. Perpendiculars may be set out with the chain alone, by a variety of methods. These methods generally consist in performing on the ground, the operations executed on paper in practical geometry, the chain being used, in the place of the compasses, to describe the necessary arcs.

As these operations, however, are less often used for the method of surveying now to be explained, than for overcoming obstacles to measurement, it will be more convenient to consider them in that connection, in Chapter V.

DIAGONALS AND PERPENDICULARS.

(109) In Chapter I, of this Part, we have seen that plats of surveys made with the chain alone, have their contents most easily determined by measuring, on the plat, the perpendiculars of each of the triangles, into which the diagonals measured on the ground have divided the field. In the *Method of Surveying by Diagonals and Perpendiculars*, now to be explained, the perpendiculars are measured *on the ground.* The content of the field can, therefore, be found at once, (by adding together the half products of each perpendicular by the diagonal on which it is let fall,) without the necessity of previously making a plat, or of measuring the sides of the field. This is, therefore, the most rapid and easy method of surveying when the content alone is required, and is particularly applicable to the measurement of the ground occupied by crops, for the purpose of determining the number of bushels grown to the acre, the amount to be paid for mowing by the acre, &c.

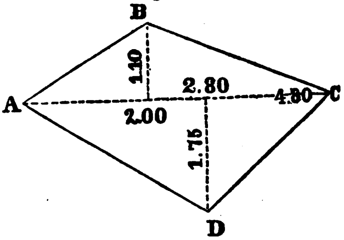
Fig. 63.



(110) A three-sided field. Measure the longest side, as AB, and the perpendicular, CD, let fall on it from the opposite angle C. Then the content is equal to half the product of the side by the perpendicular. If obstacles prevent this, find the point, where a perpendicular let fall from an angle, as A, to the opposite side produced, as BC, would meet it, as at E in the figure. Then half the product of AE by CB is the content of the triangle.

(111) A four-sided field. Measure the diagonal AC. Leave marks at the points on this diagonal at which perpendiculars from B and from D would meet it; finding these points by trial, as previously directed in Arts. (**104**) and (**107**). The best marks at these "False Stations," have been described in Art. (**90**). Return to these false stations and measure the perpendiculars. When these perpendiculars are measured before finishing the measurement of the diagonal, great care is necessary to avoid making mistakes in the length of the diagonal, when the chainmen return to continue its measurement. One check is to leave at the mark as many pins as have been taken up by the hind-chainman in coming to that point from the beginning of the line.

Fig. 64.



Example 9. Required the content of the field of Fig. 64.

Ans. 0A. 2R. 29P.

The field may be platted from these measurements, if desired, but with more liability to inaccuracy than in the first method, in which the sides are measured. The plat of the figure is 3 chains to 1 inch.

The field-notes may be taken by writing the measurements on a sketch, as in the figure; or in more complicated cases, by the column method, as below. A new symbol may be employed, this mark, ⊢, or ⊣, to show the False Station, from which a perpendicular is to be measured.

PERP.		110	*to* B
	From 200 *on* 480	F. S.	⊣
PERP.		175	*to* D
	From 280 *on* 480	F. S.	⊢
DIAGONAL.		480	*to* C
		280	⊢
	⊣	200	
	From A	⊙	

Example 10. *Calculation.*

sq. lks.

$ABC = \frac{1}{2} \times 480 \times 110 = 26400$

$ADC = \frac{1}{2} \times 480 \times 175 = 42000$

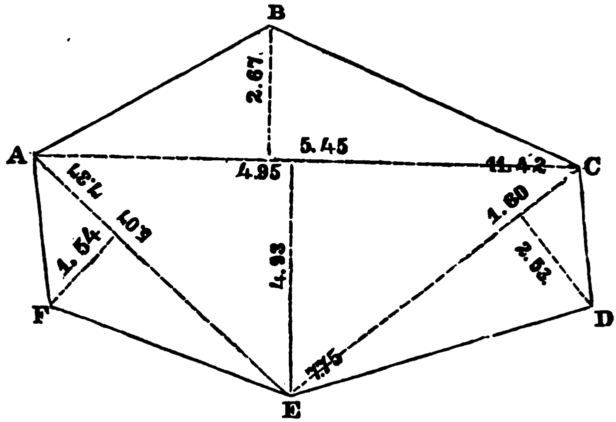
sq. chains 6.8400

Acres 0.684

It is still easier to take the two triangles together; multiplying the diagonal by the sum of the perpendiculars and dividing by two.

(112) A many-sided field. Fig. 65, and the accompanying field-notes represent the field which was surveyed by the First Method and platted in Fig. 51.

Fig. 65.



	1.54	*to* F
From 5.07 *on* 7.37	F. S.	
	2.53	*to* D
From 1.60 *on* 7.75	F. S.	
	4.93	*to* E
From 5.45 *on* 11.42	F. S.	
	2.67	*to* B
From 4.95 *on* 11.42	F. S.	
	7.37	*to* A
⊣	5.07	
From E	⊙	┏
	7.75	*to* E
⊣	1.60	
From C	⊙	┏
	11.42	*to* C
	5.45	⊢
⊣	4.95	
From A	⊙	

Example 11. *Calculation.* The content of the triangles may be expressed thus:

sq. lks.

ABC = $\frac{1}{2}\times1142\times267=152457$
AEC = $\frac{1}{2}\times1142\times493=281503$
CDE = $\frac{1}{2}\times\ 775\times253=\ 98037$
AEF = $\frac{1}{2}\times\ 737\times154=\ 56749$

sq. chains 58.8746
Acres 5.88746

or, 5A. 3R. 22P.

The first two triangles might have been taken together, as in the previous field.

Content calculated from the perpendiculars will generally vary slightly from that obtained by measuring on the plat.

(113) A small field which has many sides, may sometimes be conveniently surveyed by taking one diagonal and measuring the perpendiculars let fall on it from each angle of the field, and thus dividing the whole area into triangles and trapezoids; as in Fig. 36, page 48.

The line on which the perpendiculars are to be let fall, may also be outside of the field, as in Fig. 37, page 48.

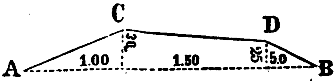
Such a survey can be platted very readily, but the length of the perpendiculars renders it less accurate.

This procedure supplies a transition to the method of "Offsets," which is to be explained in the next article.

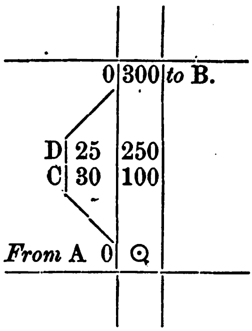
OFFSETS.

(114) Offsets are short perpendiculars, measured from a straight line, to the angles of a crooked or zigzag line, near which the straight line runs. Thus, in the figure, let ABCD be a crooked fence, bounding one side of a field. Chain along the straight line AB, which runs from one end of the fence to the other, and, when opposite each corner, note the distance from the beginning, or the point A, and also measure and note the perpendicular distance of each corner C and D from the line. These corners will then be "determined" by the *Second Method*, *Art.* (6).

Fig. 66.



The Field-notes, corresponding to Fig. 66, are as in the margin. The measurements along the line are written in the column, as before, counting from the beginning of the line, and the offsets are written beside it, on the right or left, opposite the distance at which they are taken. A sketch of the crooked line is also usually made in the Field-notes, though not absolutely necessary in so simple a case as this. The letters C and D would not be used in practice, but are here inserted to show the connection between the Field-notes and the plat.

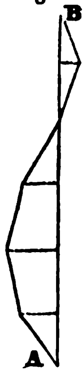


In taking the Field-Notes, the widths of the offsets should not be drawn proportionally to the distances between them, but the breadths should be greatly exaggerated in proportion to the lengths.

(115) A more extended example, with a little different notation, is given below. In the figure, which is on a scale of 8 chains to one inch for the distances along the line, the breadths of the offsets are exaggerated to four times their true proportional dimensions.

	B	
	1500	0
	1250	20
0	1000	0
30	750	
50	500	
40	250	
	0	
	A	

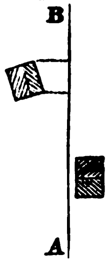
Fig. 67.



(116) The plat and Field-notes of the position of two houses, determined by offsets, are given below, on a scale of 2 chains to 1 inch.

	250	*To* B
20 30	185	
30 20	150	
	90	10 40
	50	10 30
From A.	⊙	

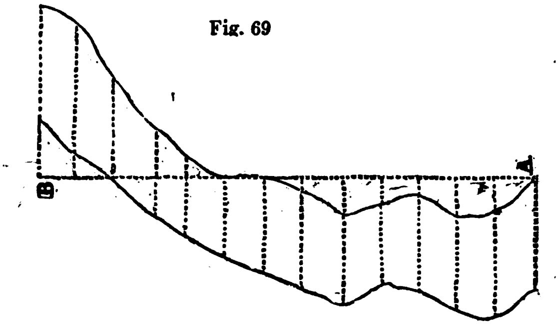
Fig. 68.



(117) Double offsets are sometimes convenient; and sometimes triple and quadruple ones. Below are given the notes and the plat, 1 chain to 1 inch, of a road of varying width, both sides of which are determined by double offsets. It will be seen that the line AB crosses one side of the road at 160 links from A, and the other side of it at 220.

Two methods of keeping the Field-notes are given. In the first form, the offsets to each side of the road are given separately and connected by the sign +. In the second form, the total distance of the second offset is given, and the two measurements connected by the word "*to*." This is easier both for measuring and platting.

Fig. 69



	B			B	
	260	30+60		260	30 *to* 90
	240	10+70		240	10 *to* 80
0	220	50	0	220	50
20	200	30	20	200	30
40	180	10	40	180	10
45	160	0	45	160	0
50+ 0	140		50 *to* 0	140	
55+ 5	120		60 *to* 5	120	
50+20	100		70 *to* 20	100	
45+15	80		60 *to* 15	80	
50+10	60		60 *to* 10	60	
50+20	40		70 *to* 20	40	
55+20	20		75 *to* 20	20	
60+ 0	A		60 *to* 0	A	

(118) These offsets may generally be taken with sufficient accuracy by measuring them as nearly at right angles to the base line as the eye can estimate. The surveyor should stand by the chain, facing the fence, at the place which he thinks opposite to the corner to which he wishes to take an offset, and measure "square" to it by the eye, which a little practice will enable him to do with much correctness.

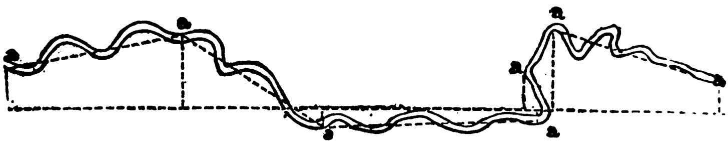
The offsets may be measured, if short, with an *Offset-staff*, a light stick, 10 or 15 links in length, and divided accordingly; or if they are long, with a tape. They are generally but a few links in length. A chain's length should be the extreme limit, as laid down by the English "Tithe Commissioners," and that should be employed only in exceptional cases. When the "Cross-staff" is in use, its divided length of 8 links, renders the offset-staff needless.

When offsets are to be taken, the method of chaining to the end of a line, described in Art. (23), page 21, is somewhat modified. After the leader arrives at the end of the line, he should draw on the chain till the follower, with the back end of the chain, reaches the last pin set. This facilitates the counting of the links to the places at which the offsets are taken.

The offsets are to be taken to every angle of the fence or other crooked line; that is, to every point where it changes its direction. These angles or prominent bends can be best found by one of the party walking along the crooked fence and directing another at the chain what points to measure opposite to. If the line which is to be thus determined is *curved*, the offsets should be taken to points so near each other, that the portions of the curved line lying between them may, without much error, be regarded as straight. It will be most convenient, for the subsequent calculations, to take the offsets at equal distances apart along the straight line from which they are measured.

In the case of a crooked brook, such as is shown in the figure given below, offsets should be taken to the most prominent angles, such as are marked *a a a* in the figure, and the intermediate bends may be merely sketched by eye.

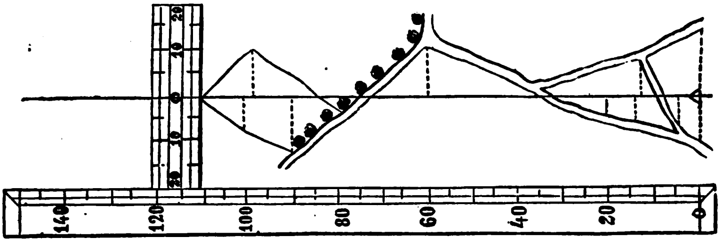
Fig. 70.



When offsets from lines measured around a field are taken inside of these bounding lines, they are sometimes distinguished as *Insets*.

(119) Platting. The most rapid method of platting the offsets, is by the use of a *Platting Scale* (described in Art. 49) and an *Offset Scale*, which is a short scale divided on its edges like a platting scale, but having its zero in the middle, as in the figure.

Fig. 71



The platting scale is placed parallel to the line, with its zero point opposite to the beginning of the line. The offset scale is slid along the platting scale, till its edge comes to a distance on the latter at which an offset had been taken, the length of which is marked off with a needle point from the offset scale. This is then slid on to the next distance, and the operation is repeated. If one person reads off the field-notes, and another plats, the operation will be greatly facilitated. The points thus obtained are joined by straight lines, and a miniature copy of the curved line is thus obtained; all the operations of the platting being merely repetitions of the measurements made on the ground.

If no offset scale is at hand, make one of a strip of thick drawing paper, or pasteboard; or use the platting scale itself, turned crossways, having previously marked off from it the points from which the offsets had been taken.

In plats made on a small scale, the shorter offsets are best estimated by eye.

On the Ordnance Survey of Ireland, the platting of offsets is facilitated by the use of a combination of the offset scale and the platting scale, the former being made to slide in a groove in the latter, at right angles to it.

(120) Calculating Content. When the crooked line determined by offsets is the boundary of a field, the content, enclosed between it and the straight line surveyed, must be determined, that it may be added to, or subtracted from, the content of the field bounded by the straight lines. There are various methods of effecting this.

The area enclosed between the straight and the crooked lines is divided up by the offsets into *triangles and trapezoids*, the content of which may be calculated separately by Arts. (65) and (67), and then added together. The content of the plat on page 75, will, therefore, be 1500 + 4125 + 625 = 6250 square links = 0.625 square chain. The content of the plat on page 76, will in like manner be found to be, on the left of the straight line 30,000 square links, and on its right 5,000 square links.

(121) *When the offsets have been taken at equal distances*, the content may be more easily obtained by adding together half of the first and of the last offset, and all the intermediate ones, and multiplying the sum by one of the equal distances between the offsets. This rule is merely an abbreviation of the preceding one.

Thus, in the plat of page 76, the distances being equal, the content of the offsets on the left of the straight line will be 120 × 250 = 30,000 square links, and on the right 20 × 250 = 5,000 square links; the same results as before.

When the line determined by the offsets is a curved line, "Simpson's rule" gives the content more accurately. To employ it, an *even* number of *equal* distances must have been measured in the part to be calculated. Then add together the first and last offset, four times the sum of the even offsets, (i. e. the 2d, 4th, 6th, &c.,) and twice the sum of the odd offsets, (i. e. the 3d, 5th, 7th, &c.,) not including the first and the last. Multiply the sum by one of the equal distances between the offsets, and divide by 3. The quotient will be the area.

Example 12. The offsets from a straight line to a curved fence, were 8, 9, 11, 15, 16, 14, 9, links, at equal distances of 5 links. What was the content included between the curved fence and the straight line? *Ans.* 371.666

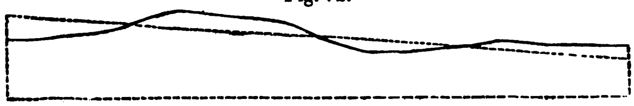
(122) Many erroneous rules have been given on this part of the subject. *One rule* directs the surveyor to divide the sum of all the offsets by one less than their number, and multiply the quotient by the whole length of the straight line; or, what is the same thing, to multiply the sum of all the offsets by the common distance between them. This will be correct only when the offsets at each end of the line are nothing, i. e. when the curved line starts from the straight line and returns to it at the beginning and end of one of the equal distances. In all other cases it will give too much. *A second rule* directs the surveyor to divide the sum of all the offsets by their number, and then to multiply the quotient by the whole straight line. This may give too much, or too little, according to circumstances.

Suppose offsets of 10, 30, 20, 80, 50, 30, links, to have been taken at equal distances of a chain. The correct content of the enclosed space is 200 × 100 = 2 square chains. The first of the above rules would give 2.2 square chains, and the second would give 1.8333 chains.

(123) *Reducing to one triangle* the many-sided figure which is formed by the offsets, is the method of calculation sometimes adopted. This has been fully explained in Part I, Art. (78), &c. The method of Art. (83) is best adapted for this purpose.

(124) *Equalizing*, or *giving and taking*, is an approximate mode of calculation much used by practical surveyors. A crooked line, determined by offsets, having been platted, a straight line is drawn on the plat, across the crooked line, leaving as much space outside of the straight line as inside of it, as nearly as can be estimated by the eye, "Equalizing" it, or "Giving and taking" equal

Fig. 72.



portions. The straight line is best determined by laying across the irregular outline the straight edge of a piece of transparent horn, or tracing paper, or glass, or a fine thread or horse-hair stretched straight by a light bow of whalebone. In practical hands, this method is sufficiently accurate in most cases. The student will do well to try it on figures, the content of which he has previously ascertained by perfectly accurate methods.

Sometimes this method may be advantageously combined with the preceding; short lengths of the croooked boundary being "Equalized," and the fewer resulting zigzags reduced to one line by the method of Art. (78), &c.

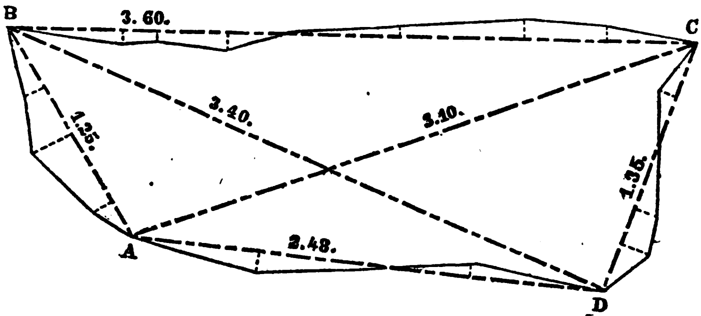
CHAPTER IV.

SURVEYING BY THE PRECEDING METHODS COMBINED.

125) All the methods which have been explained in the three preceding chapters—Surveying by *Diagonals*, by *Tie-lines*, and by *Perpendiculars*, particularly in the form of offsets—are frequently required in the same survey. The method by *Diagonals* should be the leading one; in some parts of the survey, obstacles to the measurement of diagonals may require the use of *Tie-lines;* and if the fences are crooked, straight lines are to be measured near them, and their crooks determined by *Offsets.*

(126) *Offsets* are necessary additions to almost every other method of surveying. In the smallest field, surveyed by diagonals, unless all the fences are perfectly straight lines, their bends must be determined by offsets. The plat (scale of 1 chain to 1 inch), and field-notes, of such a case are given below. A sufficient number of the sides, diagonals, and proof-lines, to prove the work, should be platted before platting the offsets.

Fig. 73.



SIDE.		C	
	0	360	
	6	315	
	10	275	
	5	215	
	0	150	0
		115	10
		80	5
		65	8
		B	0 ┌

SIDE.		B	
	0	125	
	11	90	
	23	62	
	12	22	
	0	A	

PROOF LINE.		B	
		340	
		D	

DIAG'L.		C	
		310	
		A	┌

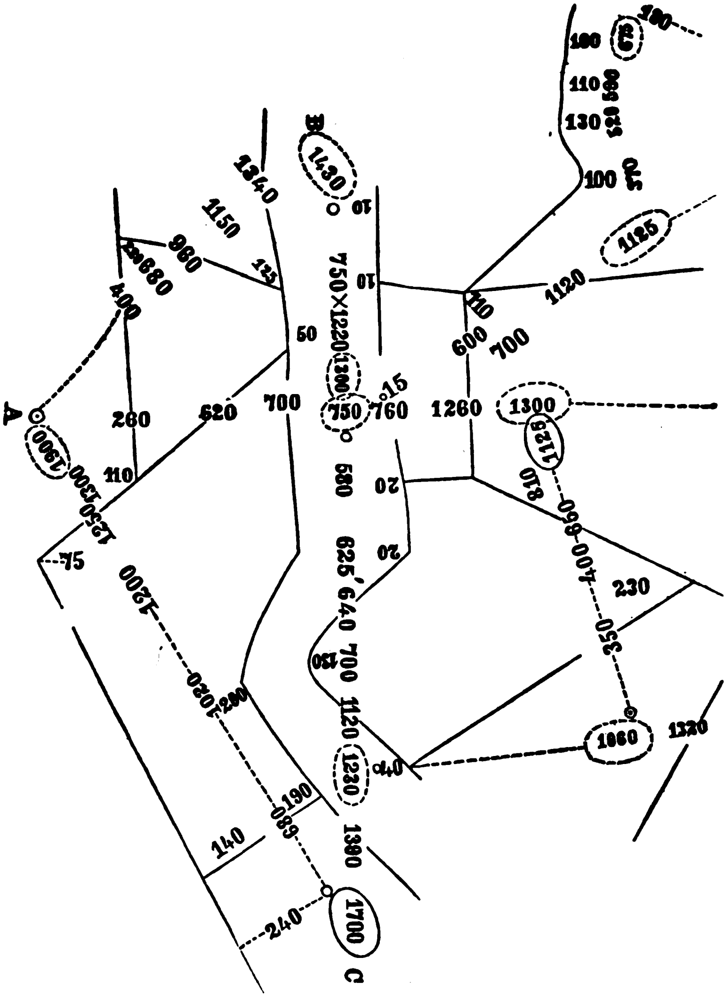
SIDE.		A	
	0	248	
	11	180	
	0	105	0
		65	5
		D	0 ┌

SIDE.		D	
	0	135	
	15	110	
	13	90	
	0	50	0
		30	9
		C	0 ┌

Example 13. Required the content of the above field. *Ans.*

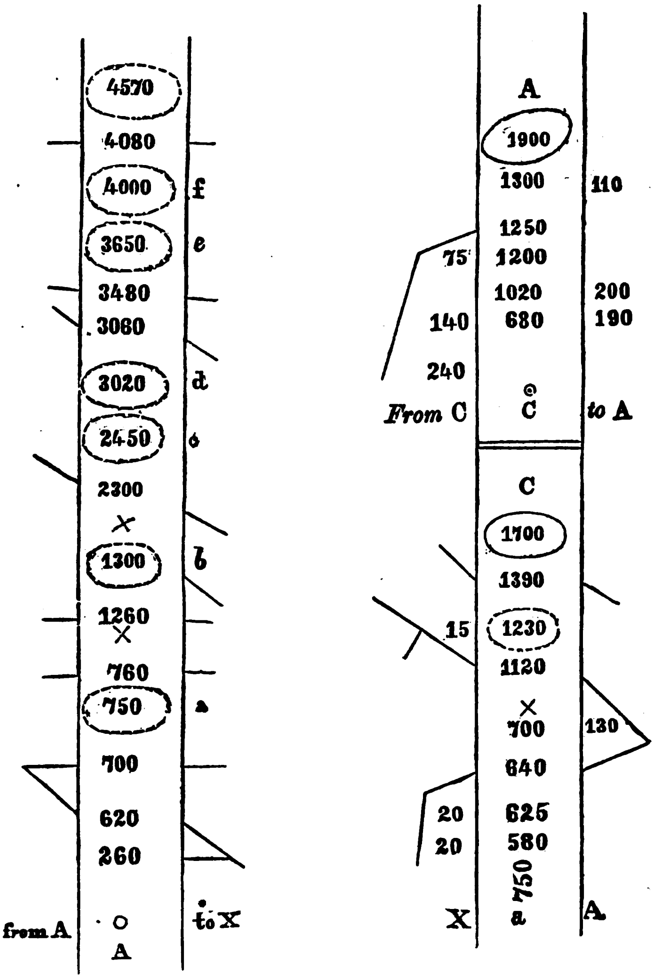
(127) Field-books. The difficulty and the importance of keeping the Field-notes clearly and distinctly, increase with each new combination of methods. For this reason, three different methods of keeping the Field-notes of the same survey will now be given, (from Bourns' Surveying), and a careful comparison by the student of the corresponding portions of each will be very profitable to him.

FIELD-BOOK No. 1.



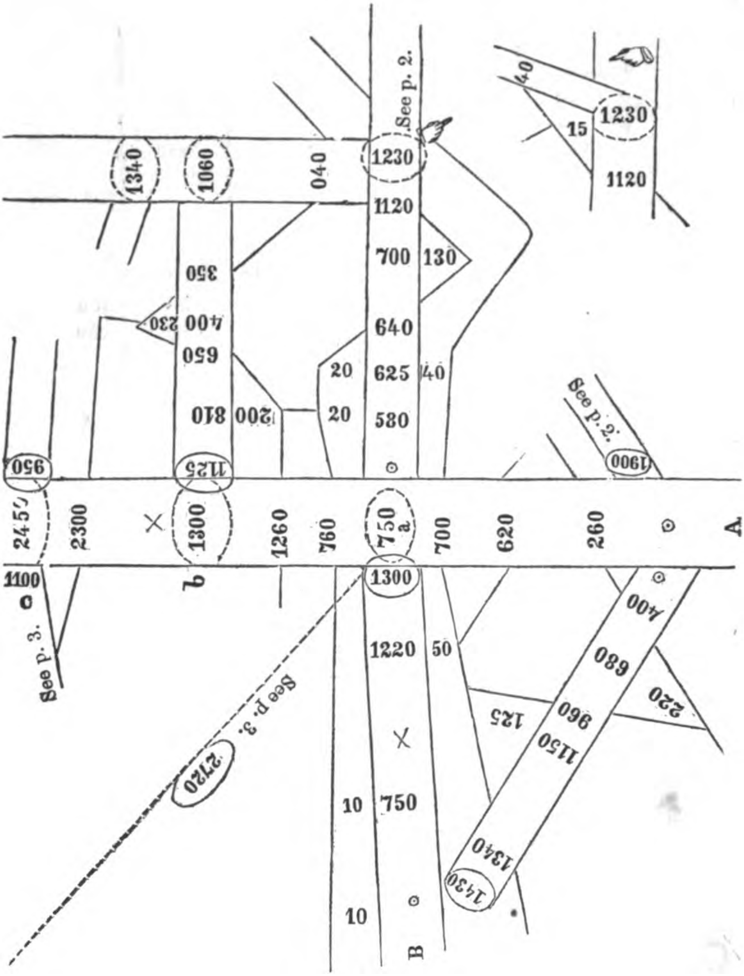
Field-Book No. 1 (Fig. 74) shews the Sketch method, explained in Art. (94).

FIELD-BOOK No. 2.



Field-Book No. 2 (Fig. 75) shews the Column method, explained in Art. (95).

FIELD-BOOK No. 3.

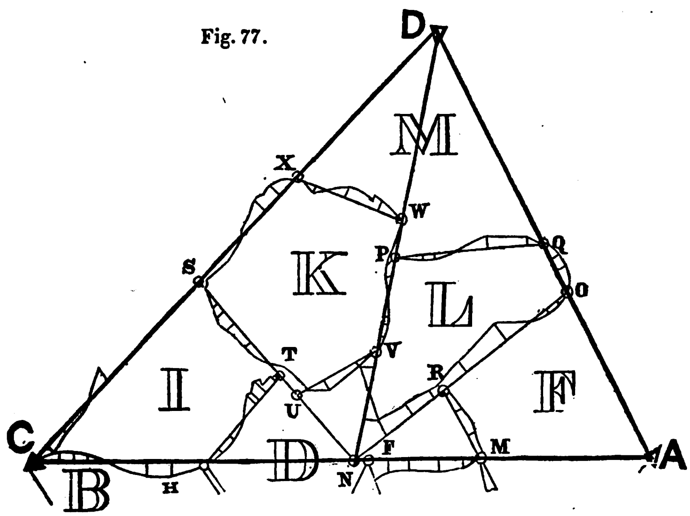


Field Book No. 3 (Fig. 76) is a convenient combination of the two preceding methods. The bottom of the Book is at the side of this figure, at A.

(128) It will easily appear from the sketch of Field-book No. 1, how much time and labor may be saved, or lost, by the manner of doing the work. Thus, beginning at A, and measuring 750 links, a pole should be left there, and the line to the right measured to 17 chains, or C, leaving a pole at 12.30 as a new starting point by and by. Then from C measure 19 chains to A again; then measure from A to B, and from B back to the pole left at 7.50 on the main line.

(129) The example which will now be given shows part of the Field-notes, the plat, (on a scale of 6 inches to 1 mile [1:10,560]), and a partial calculation of the "Filling up" of a large triangle, the angular points of which are supposed to have been determined by the methods of Geodesic Surveying. They should be well studied.*

Fig. 77.



* Capt. FROME, in his "Trigonometrical Survey," from which this example has been condensed, remarks, "It may, perhaps, be thought that too much stress is laid on *forms*; but *method* is a most essential part of an undertaking of magnitude: and without excellent preliminary arrangements to ensure uniformity in all the most trifling details, the work never could go on creditably."

	2564	80
F	2452	
N	2324	
H	1264	0
	1240	52
	1140	86
	950	100
	772	60
0	604	0
84	502	
50	450	
70	342	
82	220	
From C	△	*to* A
	C	

	D	
	△	
	3398	
Q	1700	0
	1530	84
	1420	40
O	1340	0
From A	△	*to* D
	A	

	A	
	△	
	4454	
M	3296	0
	3275	54
	3120	62
	2940	85
	2572	60

	C	
	△	
	4250	
100	4050	
62	3890	
42	3730	
0	3540	0
	3420	30
72	2484	S
40	2332	
60	2206	
0	2056	0
	1805	40
	1550	50
X	1442	0
From D	△	*to* C.
	D	

In the above specimen of a field-book, (which resembles that on page 85), all offsets, except those having relation to the boundary lines, are purposely omitted, to prevent confusion, the example being given solely to illustrate the method of calculating these larger divisions. Rough diagrams are drawn in the field-book not to any scale, but merely bearing some sort of resemblance to the lines measured on the ground, for the purpose of showing, at any period of the work, their directions and how they are to be connected; and also of eventually assisting in laying down the diagram and content plat. On these rough diagrams are written the distinctive letters by which each line is marked in the field-book, and also its length, and the distances between points marked upon it, from which other measurements branch off to connect the interior portions of the district surveyed.

(130) Calculations. The calculations of one of the figures, M, is given below in detail. It is composed of the triangle DPQ, with offsets along the sides PQ; and of the triangle DWX, with offsets along the sides PW and WX. From the content thus obtained must be subtracted the offsets on PQ, belonging to the figure 𝔏, and those on WX belonging to the figure 𝔎. When the offsets are triangles, (right angled, of course), the base and perpendicular are put down as two sides; when they are trapezoids, the two parallel sides and the distance between them occupy the columns of "sides."

DIVISION.	TRIANGLE OR TRAPEZOID.	1ST SIDE.	2D SIDE.	3D SIDE.	CONTENT IN CHAINS.
	DPQ	1680	1698	1078	86.2650
	PQ	—	52	250	.6500
		52	30	80	.3280
		30	—	216	.3240
					1.3020
𝔐 *Additives.*	DWX	1370	1442	770	51.8339
	PW	30	—	310	.4650
	WX	—	56	114	.3192
		56	36	104	.4784
		36	—	90	.1620
					.9596
		Total Additives,			140.8255
	PQ	—	50	174	.4350
		50	30	292	1.1680
		30	—	66	.0990
					1.7020
𝔐 *Subtractives.*	WX	—	52	142	.3692
		52	64	232	1.3456
		64	—	88	.2816
					1.9964
		Total Subtractives,			3.6984
		Total Additives,			140.8255
		Difference,			137.1271

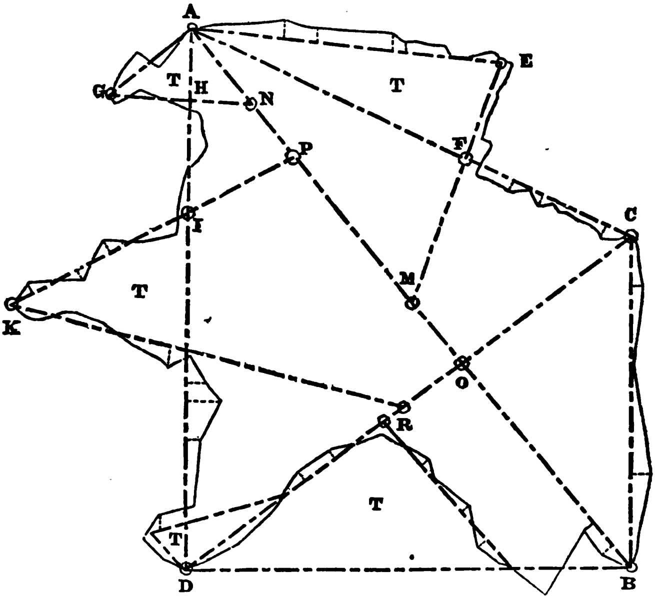
The other figures, comprised within the large triangle, are recorded and calculated in a similar manner. An abridged register of the results is given below.

DIVISION.	ADDITIVES.	SUBTRACTIVE.	DIFFERENCE IN SQUARE CHAINS.
K	DNS and offsets.	DWX NUV and offsets.	140.4893
L	DNO and offsets.	DPQ and offsets.	100.1882
F	ANO and offsets.	NRM and offsets.	103.9778
D	HTN NUV NRM and offsets.	Offsets.	81.6307
I	CNS and offsets.	HTN and offsets.	109.5064
M	DPQ DWX and offsets.	Offsets.	137.1271
Total,	- - -	- - -	672.9195

The accuracy of the preceding calculations of the separate figures must now be tested by comparing the sum of their areas with that of the large triangle ACD, which comprises them all. Their area must previously be increased by the offsets on the lines CS and CH, which had been deducted from I, and which amount respectively to 3.5270 and 2.8690. The total areas will then equal 679.3155 square chains. That of the triangle ACD is 679.5032; a difference of less than a fifth of a square chain, or a fiftieth of an acre; or about one-fortieth of one per cent. on the total area.

(131) The six lines. In most cases, great or small, *six fundamental lines* will need to be measured; viz. four approximate boundary lines, forming a quadrilateral, and its two diagonals. Small triangles, to determine prominent points, can be formed within and without these main lines by the FIRST METHOD, Art. (5), and the lesser irregularities can be determined by offsets.

Fig. 78.



Thus, in the above figure, two straight lines AB and CD are measured through the entire length and breadth of the farm, or township, which is to be surveyed. The connecting lines AC, CB, BD and DA are also measured, uniting the extremities of the first two lines. The last four lines thus form a quadrilateral, which is divided into two triangles by one of the first measured lines, while the second serves as a proof-line. The distance from the intersection of the two diagonals to the extremities of each, being measured on the ground and on the plat, affords an additional test.

Other points of the district surveyed (as E, G, K., &c., in the figure,) are determined by measuring the distances from them to known points (as M, N, P, R, &c., in the figure) situated on some of the six fundamental lines, thus forming the triangles T, T.

The intersection O of the main diagonals, and also the intersections of the various minor lines with the main lines and with each other, should all be carefully noted, as additional checks when the work comes to be platted.

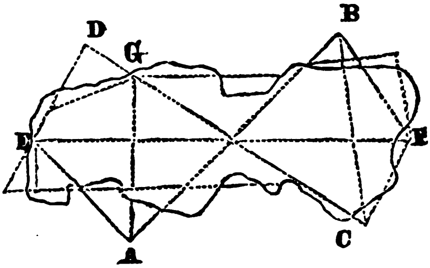
The larger figures are determined first, and the smaller ones based upon them, in accordance with this important principle in all surveying operations, always to work from the whole to the parts, and from greater to less. The unavoidable inaccuracies are thus subdivided and diminished. The opposite course would accumulate and magnify them.

These additional lines, which form secondary triangles, should be so chosen and ranged as to pass through and near as many objects as possible, in order to require as few and as short offsets as the position of the lines will permit; the smaller irregularities being determined by offsets as usual. It is better to measure too many lines than too few, and to establish unnecessary "false stations," rather than not to have enough.

(132) Exceptional cases. The preceding arrangement of lines, though in most cases the best, may sometimes be varied with advantage. Unless the farm surveyed be of a shape nearly as broad as long, the two diagonals will cross each other obliquely, instead of nearly at right angles, as is desirable.

When the farm is much longer than it is wide, two systems, of six lines each, may be used with much advantage, as in Fig. 79. Several such may be combined when necessary.

Fig. 79.



In a case like that in Fig. 80, five lines will be better than six, and will tie one another together, their points of intersection being carefully noted.

Fig. 80.

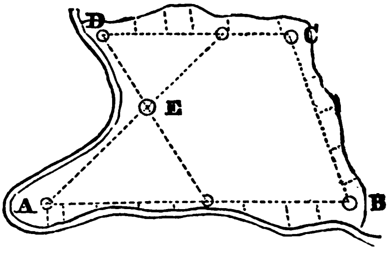
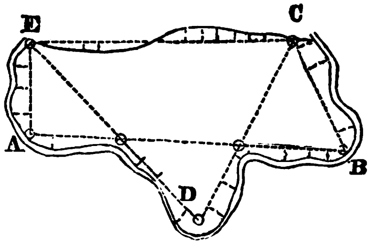


Fig. 81.



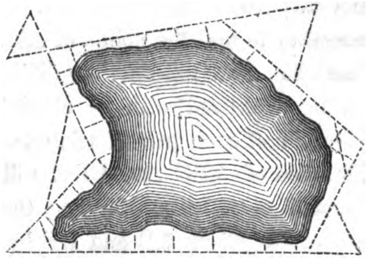
In the farm represented in Fig. 81, the system of lines there shown is the best, and they will also tie one another.

(133) Much difficulty will often be found in ranging and measuring the long lines required by this method in extensive surveys. Various contrivances for overcoming the obstacles which may be met with, will be explained in the following chapter. It will often be convenient to measure the minor lines along roads, lanes, paths, &c., although they may not lie in the most desirable directions. Steeples, chimneys, remarkable trees, and other objects of that character, may often be sighted to, and the line measured towards them, with much saving of time and labor. The point where the measured lines cross one another should always be noted, and they will thus form a very complete series of tie-lines.*

A view of the district to be surveyed, taken from some elevated position, will be of much assistance in planning the general direction of the lines to be measured.

(134) Inaccessible Areas. A combination of offsets and tie-lines supplies an easy method of surveying an *inaccessible area*, such as a pond, swamp, forest, block of houses, &c., as appears from the figure; in which external bounding lines are taken at will and

Fig. 82.



* To find the exact point of intersection of these lines, which are only "visual lines," (explained in Art. (19),) three persons are necessary: one stands at some point of one of the lines and sights to some other point on it; a second does the same on the second line; by signs they direct, to right or left, the movements of a third person, who holds a rod, till he is placed in both of the lines and thus at their intersection, on the principle of Art. (11).

measured, and tied by "tie-lines" measured between these lines, prolonged when necessary, as in Art. (101), while offsets from them determine the irregularities of the actual boundaries of the pond, &c.

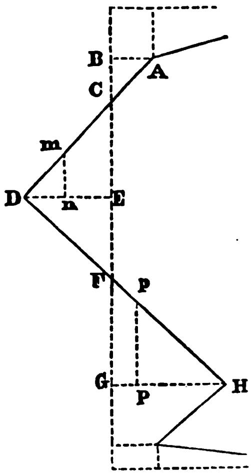
These *offsets* are *insets*, and their content is, of course, to be subtracted from the content of the principal figure.

Even a circular field might thus be approximately measured from the outside.

Fig. 83.

If the shape of the field admits of it, it will be preferable to measure four lines about the field in such directions as to enclose it in a rectangle, and to measure offsets from the sides of this to the angles of the field.

Fig. 84

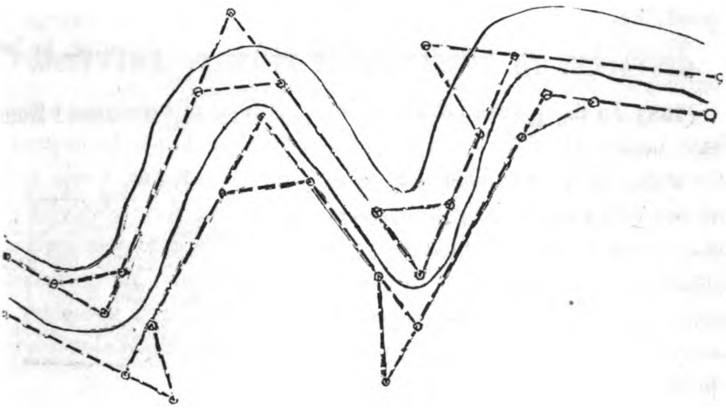


(135) When one of the lines with which an inaccessible field is surrounded, as in the last two figures, cuts a corner of the field, as in Fig. 84, the triangle ABC is to be deducted from the content of the enclosing figure, and the triangle CDE added to it. The triangle DEF is also to be added, and the triangle FGH deducted. To do this directly, it would be necessary to find the points of intersection C and F. But this may be difficult, and can be dispensed with by obtaining the difference of each pair of triangles. The difference of ABC and CDE will be obtained at once by multiplying the difference of the offsets AB and DE by half of BE; and the difference of DEF and FGH by multiplying the difference of DE and GH by half of EG.*

* For, making the triangle $Dmn = ABC$, then $mnEC = En \times \frac{1}{2}(mn + CE) = (DE - AB) \times \frac{1}{2} EB$; and so with the other pair of triangles.

(136) Roads. A winding Road may also be surveyed thus, as is shown in Fig. 85; straight lines being measured in the road,

Fig. 85.



their changes in direction determined by tie-lines, tying one line to the preceding one prolonged, as explained in Chapter II, of this Part, and points in the road-fences, on each side of these straight lines, being determined by offsets.

A **River** may also be supposed to be represented by the above winding lines; and the lower set of lines, tied to one another as before, and with offsets from them to the water's edge, will be sufficient for making an accurate survey of one side of the river.

(137) Towns. A town could be surveyed and mapped in the same manner, by measuring straight lines through all the streets, determining their angles by tie-lines, and taking offsets from them to the blocks of houses.

CHAPTER V.

OBSTACLES TO MEASUREMENT IN CHAIN SURVEYING.

(138) In the practice of the various methods of surveying which have been explained, the hills and vallies which are to be crossed, the sheets of water which are to be passed over, the woods and houses which are to be gone through—all these form *obstacles* to the measurement of the necessary lines which are to join certain points, or to be prolonged in the same direction. Many special precautions and contrivances are, therefore, rendered necessary; and the best methods to be employed, when the chain alone is to be used, will be given in the present chapter.

(139) The methods now to be given for overcoming the various obstacles met with in practice, constitute a LAND-GEOMETRY. Its problems are performed on the ground instead of on paper: its *compasses* are a chain fixed at one end and free to swing around with the other; its *scale* is the chain itself; and its *ruler* is the same chain stretched tight. Its advantages are that its single instrument, (or a substitute for it, such as a tape, a rope, &c.) can be found anywhere; and its only auxiliaries are equally easy to obtain, being a few straight and slender rods, and a plumb-line, for which a pebble suspended by a thread is a sufficient substitute.

Many of these problems require the employment of perpendicular and parallel lines. For this reason we will commence with this class of Problems.

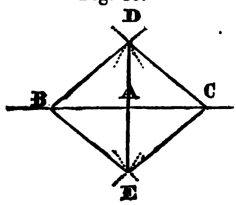
The Demonstrations of these problems will be placed in an Appendix to this volume, which will be the most convenient arrangement for the two great classes of students of surveying; those who wish merely the practice without the principles, and those who wish to secure both.

The elegant "Theory of Transversals" will be an important element in some of these demonstrations. All of them will constitute excellent exercises for students.

PROBLEMS ON PERPENDICULARS.*

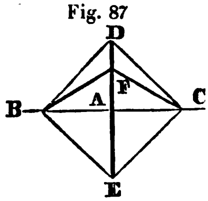
Problem 1. *To erect a perpendicular at any point of a line.*

Fig. 86.



(140) *First Method.* Let A be the point at which a perpendicular to the line is to be set out. Measure off equal distances AB, AC, on each side of the point. Take a portion of the chain not quite $1\frac{1}{2}$ times as long as AB or AC, fix one end of this at B, and describe an arc with the other end. Do the same from C. The intersection of these arcs will fix a point D. AD will be the perpendicular required. Repeat the operation on the other side of the line. If that is impossible, repeat it on the side with a different length of chain

Fig. 87



(141) *Second Method.* Measure off as before, equal distances AB, AC, but each about only one-third of the chain. Fasten the ends of the chain with two pins at B and C. Stretch it out one side of the line and put a pin at the middle of it, D. Do the same on the other side of the line, and set a pin at E. Then is DE a perpendicular to BC. If it is impossible to perform the operation on both sides of the line, repeat it on the same side with a different length of chain, as shown by the lines BF and CF in the figure, so as to get a second point.

(142) *Other Methods.* All the methods to be given for the next problem may be applied to this.

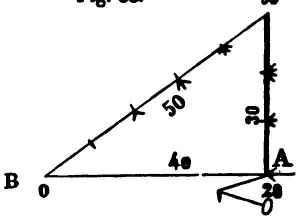
* Many of these methods would seldom be required in practice, but cases sometimes occur, as every surveyor of much experience in Field-work has found to his serious inconvenience, in which some peculiarity of the local circumstances forbids any of the usual methods being applied. In such cases the collection here given will be found of great value.

In all the figures, the given and measured lines are drawn with fine full lines, the visual lines, or lines of sight, with broken lines, and the lines of the result with heavy full lines. The points which are centres around which the chain is swung, are enclosed in circles. The alphabetical order of the letters attached to the points, shows in what order they are taken.

Problem 2. *To erect a perpendicular to a line at a given point, when the point is at or near the end of the line.*

(143) *First Method.* Measure 40 links along the line. Let one assistant hold one end of the chain at that point; let a second hold the 20 link mark which is nearest the other end, at the given point A, and let a third take the 50 link mark, and tighten the chain, drawing equally on both portions of it. Then will the 50 link mark be in the perpendicular desired. Repeat the operation on the other side of the line so as to test the work.

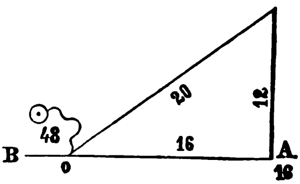
Fig. 88.



The above numbers are the most easily remembered, but the longer the lines measured the better; and nearly the whole chain may be used, thus: Fix down the 36th link from one end at A, and the 4th link from the same end on the line at B. Fix the other end of the chain also at B. Take the 40th link mark from this last end, and draw the chain tight, and this mark will be in the perpendicular desired. The sides of the triangle formed by the chain will be 24, 32 and 40.

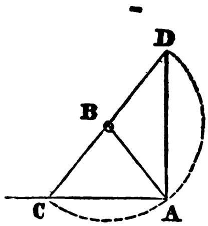
(144) Otherwise: using a 50 feet tape, hold the 16 feet mark at A; hold the 48 feet mark and the ring-end of the tape together on the line; take the 28 feet mark of the tape, and draw it tight; then will the 28 feet mark be in the perpendicular desired.

Fig. 89.



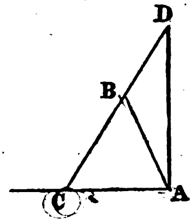
(145) *Second Method.* Hold one end of the chain at A and fix the other end at a point B, taken at will. Swing the chain around B as a centre, till it again meets the line at C. Then carry the same end around (the other end remaining at B) till it comes in the line of CB at D. AD is the perpendicular required.

Fig. 90.



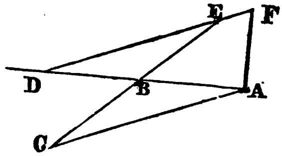
(146) *Third Method.* Let A be the given point. Choose any point B. Measure BA. Set off, on the given line, AC = AB. On CB produced set off from C, a distance $= \frac{2\ AC^2}{CB}$. This will fix the point D, and AD will be the perpendicular required.

Fig. 91.



(147) *Fourth Method.* From the given point A set off on the given line any distance AB. From B, in any convenient direction, set off BC = AB. Then on the given line, set off AD = AC. On CB prolonged, set off CE = AD. Join DE; and on DE, from D, set off DF = 2 AB. Then will the line AF be perpendicular to the line AD at the point A.

Fig. 92.



Problem 3. *To erect a perpendicular to an inaccessible line, at a given point of it.*

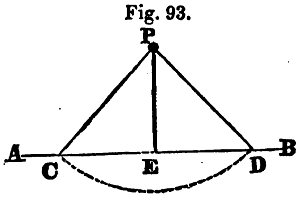
(148) *First Method.* Get points in the direction of the inaccessible line prolonged, and from them set out a parallel to the line, by methods which are given in Art. (165), &c. Find by trial the point in which a perpendicular to this second line (and therefore to the first line) will pass through the required point.

(149) *Second Method.* If the line is not only inaccessible, but cannot have its direction prolonged, the desired perpendicular can be obtained only by a complicated trigonometrical operation.

Problem 4. *To let fall a perpendicular from a given point to a given line.*

(150) *First Method.* Let P be the given point, and AB the given line. Measure some distance, a chain or less, from C to P, and then fix one end of the chain at P, and swing it around till the same distance meets

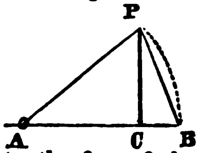
Fig. 93.



the line at some point D. The middle point E of the distance CD will be the required point, at which the perpendicular from P would meet the line.

(151) *Second Method.* Stretch a chain, or a portion of it, from the given point P, to some point, as A, of the given line. Hold the end of the distance at A, and swing round the other end of the chain from P, so as to set off the same distance along the given line from A to some point B. Measure BP. Then will the distance BC from B to the foot of the desired perpendicular $= \frac{BP^2}{2\,AB}$.

Fig. 94.

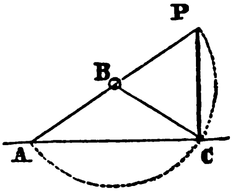


(152) *Other Methods.* All the methods given in the next problem can be applied to this one.

Problem 5. *To let fall a perpendicular to a line, from a point nearly opposite to the end of the line.*

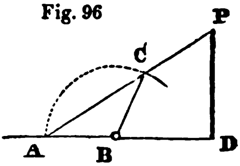
(153) *First Method.* Stretch a chain from the given point P, to some point, as A, of the given line. Fix to the ground the middle point B of the chain AP, and swing around the end which was at P, or at A, till it meets the given line in a point C, which will be the foot of the required perpendicular.

Fig. 95.



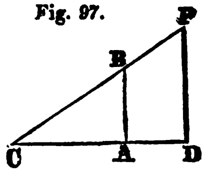
(154) *Second Method.* Take any point, as A, on the given line. Measure a distance AB. Let the end of this distance on the chain be held at B, and swing around the end of the chain, till it comes in the line of AP at some point C, thus making BC = AB. Measure AC and AP. Then the distance AD, from A to the foot of the perpendicular required $= \frac{AP \times AC}{2\,AB}$.

Fig. 96



(155) *Third Method.* At any convenient point, as A, of the given line, erect a perpendicular, of any convenient length, as AB, and mark a point C on the given line, in the line of P and B. Measure CA, CB and CP. Then the distance from C to the foot of the perpendicular, i. e. $CD = \frac{CA \times CP}{CB}$.

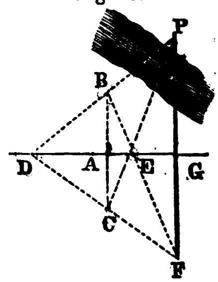
Fig. 97.



Problem 6. *To let fall a perpendicular to a line, from an inaccessible point.*

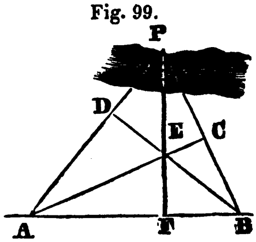
(156) *First Method.* Let P be the given point. At any point A, on the given line, set out a perpendicular AB of any convenient length. Prolong it on the other side of the line the same distance. Mark on the given line a point D in the line of PB; and a point E in the line of PC. Mark the point F at the intersection of DC and BE prolonged. The line FP is the line required, being perpendicular to the given line at the point G.

Fig. 98.



(157) *Second Method.* Let A and B be two points of the given line. From A let fall a perpendicular, AC, to the visual line BP; and from B let fall a perpendicular, BD, to the visual line AP. Find the point at which these perpendiculars intersect, as at E (see Art. (133)), and the line PE, prolonged to F, will give the perpendicular required.

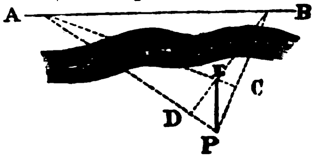
Fig. 99.



Problem 7. *To let fall a perpendicular from a given point to an inaccessible line.*

(158) *First Method.* Let P be the given point and AB the given line. By the preceding problem, let fall perpendiculars from A to BP, at C; and from B to AP, at D; the line PE, passing from the given point to the intersection of these perpendiculars, is the desired perpendicular to the inaccessible line AB.

Fig. 100.



This method will apply when only two points of the line are visible.

(159) *Second Method.* Through the given point, set out, by the methods of Art. (**165**), &c., a line parallel to the inaccessible line. At the given point erect a perpendicular to the parallel line, and it will be the required perpendicular to the inaccessible line.

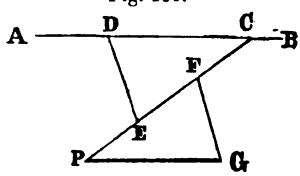
PROBLEMS ON PARALLELS.

Problem 1. *To run a line, from a given point, parallel to a given line.*

(160) *First Method.* Let fall a perpendicular from the point to the line. At another point of the line, as far off as possible, erect a perpendicular, equal in length to the one just let fall. The line joining the end of this line to the given point will be the parallel required.

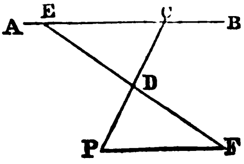
(161) *Second Method.* Let AB be the given line, and P the given point. Take any point, as C, on the given line, and from it set off equal distances, as long as possible, CD on the given line, and CE, on the line CP. Measure DE. From P set off PF = CE; and from F, with a distance = DE, and from P, with a distance = CD, describe arcs intersecting in G. PG will be the parallel required. If it is more convenient, PC may be prolonged, and the equal triangle, CDE, be formed on the other side of the line AB.

Fig. 101.



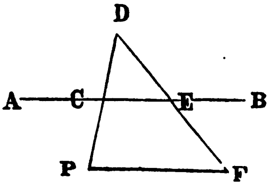
(162) *Third Method.* Measure from P to any point, as C, of the given line, and put a mark at the middle point, D, of that line. From any point, as E, of the given line, measure a line to the point D, and continue it till DF = DE. Then will the line PF be parallel to AB.

Fig. 102.



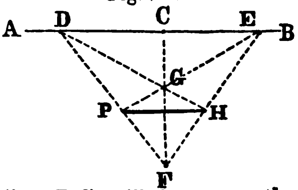
(163) *Fourth Method.* Measure from P to any point C, of the given line, and continue the measurement till CD = CP. From D measure to any point E of the given line, and continue the measurement till EF = ED. Then will the line PF be parallel to AB. If more convenient, CD may be made one-half, or any other fraction, of CP, and EF be then made twice, &c., DE.

Fig. 103.



(164) *Fifth Method.* From any point, as C, of the line, set off equal distances along the line, to D and E. Take a point F, in the line of PD. Stake out the lines FC and FE, and also the line EP, crossing the line CF in the point G. Lastly, prolong the line DG, till it meets the line EF in the point H. PH is the parallel required.

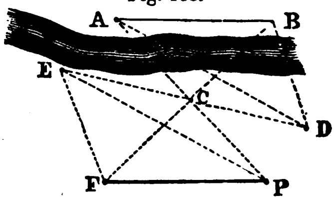
Fig. 104.



Problem 2. *To run a line from a given point parallel to an inaccessible line.*

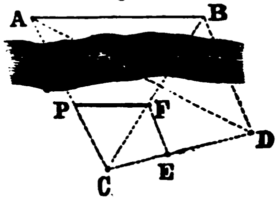
(165) *First Method.* Let AB be the given line, and P the given point. Set a stake at C, in the line of PA, and another at any convenient point, D. Through P, set out, by the preceding problem, a parallel to DA, and set a stake at the point, as E, where this parallel intersects DC prolonged. Through E

Fig. 105.



set out a parallel to BD, and set a stake at the point F, where this parallel intersects BC prolonged. PF is the parallel required.

Fig. 106.



(166) *Second Method.* Set a stake at any point, C, in the line of AP, and another at any convenient place, as at D. Through P set out a parallel to AD, (not shewn in the figure,) intersecting CD in E. Through E set out a parallel to DB, intersecting CB in F. The line PF will be the parallel required.

(167) Alinement and Measurement. We are now prepared, having secured a variety of methods for setting out Perpendiculars and Parallels in every probable case, to take up the general subject of overcoming Obstacles to Measurement.

Before a line can be measured, its direction must be determined. This operation is called *Ranging* the line; or *Alining* it; or *Boning* it.* The word *Alinement*† will be found very convenient for expressing the direction of a line on the ground, whether between two points, or in their direction prolonged.

This branch of our subject naturally divides itself into two parts, the first of which is preliminary to the second; viz:

I. Of Obstacles to Alinement; *or how to establish the direction of a line in any situation.*

II. Of Obstacles to Measurement; *or how to find the length of a line which cannot be actually measured.*

I. OBSTACLES TO ALINEMENT.

(168) All the cases which can occur under this head, may be reduced to two; viz:

A. To find points in a line *beyond* the given points, i. e. to *prolong* the line.

B. To find points in a line *between* two given points of it, i. e. to *interpolate* points in the line.

* This word, like many others used in Engineering, is derived from a French word, *Borner*, to mark out, or limit; indicating that the Normans introduced the art of Surveying into England.

† Slightly modified from the French *Alignement*.

A. TO PROLONG A LINE

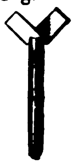
(169) By Ranging with rods. When two points in a line are given, and it is desired to prolong the line by ranging it out with rods, three persons are required, each furnished with a straight slender rod, and with a plumb-line, or other means of keeping their rods vertical. One holds his rod at one of the given points, A, in the figure, and another at B. A third, C, goes forward as far as he can without losing sight of the first two rods, and then, looking back, puts himself "in line" with A and B, i. e. so that when his eye is placed at C, the rod at B hides or covers the rod at A. This he can do most accurately by holding a plumb-line before his eye, so that it shall cover the first two rods. The lower end of the plumb-bob will then indicate the point where the third rod should be placed; and so with the rest. The first man, at A, is then signalled, and comes forward, passes both the others, and puts himself at D, "in line" with C and B. The man at B, then goes on to E, and "lines" himself with D and C: and so they proceed, in this "hand over hand" operation, as far as is desired. Stakes are driven at each point in the line, as soon as it is determined.

Fig. 107.

(170) The rods should be perfectly straight, either cylindrical or polygonal, and as slender as they can be without bending. They should be painted in alternate bands of red and white, each a foot, or link in length. Their lower ends should be pointed with iron, and a projecting bolt of iron will enable them to be pressed down by the foot into the earth, so that they can stand alone. When this is done, one man can range out a line. A rod can be set perfectly vertical, by holding a plumb-line before the eye at some distance from the rod, and adjusting the rod so that the plumb-line covers it from top to bottom; and then repeating the operation in a direction at right angles to the former. A stone dropped from top to bottom of the rods will approximately attain the same end.

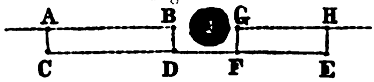
When the lines to be ranged are long, and great accuracy is required, the rods may have attached to them plates of tin with openings cut out of them, and black horse-hairs stretched from top to bottom of the openings. A small telescope must then be used for ranging these hairs in line. In a hasty survey, straight twigs, with their tops split to receive a paper folded as in the figure, may be used.

Fig. 108.



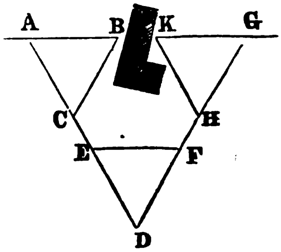
(171) By Perpendiculars. The straight line, AB in the figure, is supposed to be stopped by a tree, a house, or other obstacle, and it is desired to prolong the line beyond this obstacle. From any two points, as A and B, of the line, set off (by some of the methods which have been given) equal perpendiculars, AC and BD, long enough to pass the obstacle. Prolong this line beyond the obstacle, and from any two points in it, as E and F, measure the perpendiculars EH and FG, equal to the first two, but in a contrary direction. Then will G and H be two points in the line AB prolonged, which can be continued by the method of the last article. The points A and B should be taken as far apart as possible, as should also the points E and F. Three or more perpendiculars, on each side of the obstacle, may be set off, in order to increase the accuracy of the operation. The same thing may also be done on the other side of the line, as another confirmation, or test, of the accuracy of the prolonged line.

Fig. 109.



(172) By equilateral triangles. The obstacles, noticed in the last article, may also be overcome by means of three equilateral triangles, formed by the chain. Fix one end of the chain, and also the end of the first link from its other end, at B; fix the end of the 33d link at A; take hold of the 66th link, and draw the chain tight, pulling equally on each part, and put a pin at the point thus found, C, in the figure. An equilateral triangle will thus be formed, each side being 33 links. Prolong the line AC, past the obstacle, to some point, as D. Make another

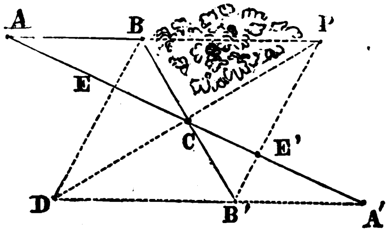
Fig. 110



equilateral triangle, DEF, as before, and thus fix the point F. Prolong DF, to a length equal to that of AD, and thus fix a point G. At G form a third equilateral triangle GHK, and thus fix a point K. Then will KG give the direction of AB prolonged.

(173) By symmetrical triangles. Let AB be the line to be prolonged. Take any convenient point, as C. Range out the line AC, to a point A′, such that CA′ = CA. Range out CB, so that CB′ = CB. Range backwards A′B′, to some point D, such that DC prolonged will pass the obstacle. Find, by ranging, the intersection, at E, of DB and AC. From C, measure, on CA′, the distance CE′ = CE. Then range out DC and B′E′ to their intersection in P, which will be a required point in the direction of AB prolonged. The symmetrical points are marked by corresponding letters. Several other points should be obtained in the same manner.

Fig. 111.

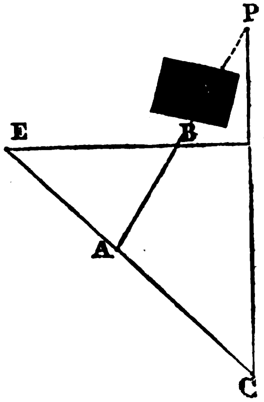


In this, as in all similar operations, very acute intersections should be avoided as far as possible.

(174) By transversals. Let AB be the given line. Take any two points C and D, such that the line CD will pass the obstacle. Take another point, E, in the intersection of CA and DB. Measure AE, AC, CD, BD and BE. Then the distance from D to P, a point in the required prolongation, will be

$$DP = \frac{CD \times BD \times AE}{BE \times AC - BD \times AE}.$$

Fig. 112.



Other points in the prolongation may be obtained in the same manner, by merely moving the single point C, in the line of EA; in which case the new distances CA and CD will alone require to be measured.

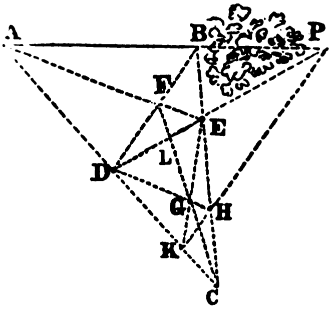
If AE be made equal to AC, then is $DP = \frac{CD \times BD}{BE - BD}$.

If BE be made equal to BD, then is $DP = \frac{CD \times AE}{AC - AE}$.

The *minus* sign in the denominators must be understood as only meaning that the difference of the two terms is to be taken, without regard to which is the greater.

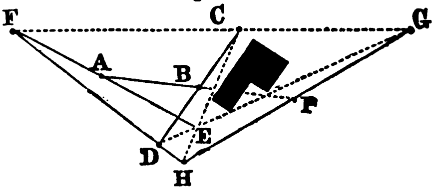
(175) By harmonic conjugates. Let AB be the given line. Set a stake at any point C. Set stakes at points, D, on the line CA, and at E, on the line CB; these points, D and E, being so chosen that the line DE will pass beyond the obstacle. Set a fourth stake, F, at the intersection of the lines AE and DB. Set a fifth stake, G, anywhere in the line CF; a sixth stake, H, at the intersection of CB and DG prolonged; and a seventh, K, at the intersection of CA and EG prolonged. Finally, range out the lines DE and KH, and their intersection at P, will be in the line AB prolonged.

Fig. 113.



(176) By the complete Quadrilateral. Let AB be the given line. Take any convenient point C; measure from it to B, and onward, in the same line prolonged, an equal distance to D. Take any other convenient point, E, such that CE and DE produced will clear the obstacle. Measure from E to A, and onward, an equal distance, to F. Range out the lines FC and DE to their intersection in G. Range out FD and CE to intersect in H. Measure GH. Its middle point, P, is the required point in the line of AB prolonged. The unavoidable acute intersections in this construction are objectionable.

Fig. 114



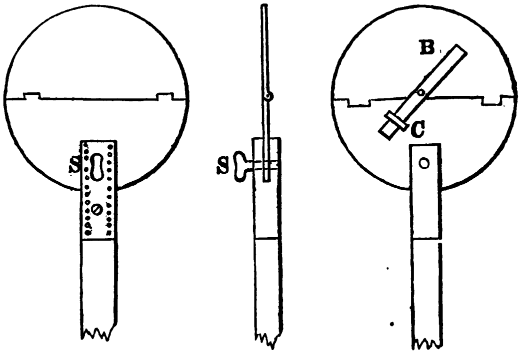
B. TO INTERPOLATE POINTS IN A LINE.

(177) The most distant given point of the line must be made as conspicuous as possible, by any efficient means, such as placing there a staff, bearing a flag; red and white, if seen against woods, or other dark back-ground; and red and green, if seen against the sky.

A convenient and portable signal is shown in the figure.

Fig. 115.

Front View. *Side View.* *Back View.*



The figure represents a disc of tin, about six inches in diameter, painted white and hinged in the middle, to make it more portable. It is kept open by the bar, B, being turned into the catch, C. A screw, S, holds the disc in a slit in the top of the pole.

Another contrivance is a strip of tin, which has its ends bent horizontally in contrary directions. As the wind will take strongest hold of the side which is concave towards it, the bent strip will continually revolve, and thus be very conspicuous. Its upper half should be painted red and its lower half white.

A bright tin cone set on the staff, can be seen at a great distance when the sun is shining.

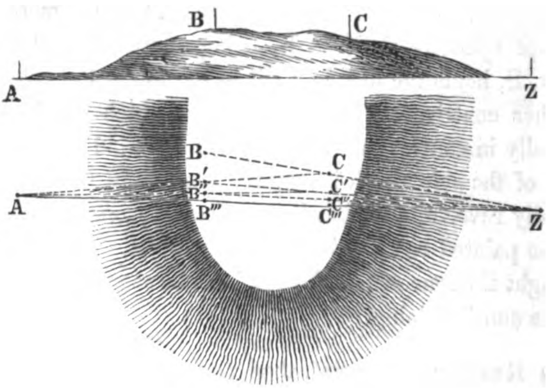
178) Ranging to a point, thus made conspicuous, is very simple when the ground is level. The surveyor places his eye at the nearest end of the line, or stands a little behind a rod placed on it, and by signs moves an assistant, holding a rod at some point as nearly in the desired line as he can guess, to the right or left, till his rod appears to cover the distant point.

(179) Across a Valley. When a valley, or low spot, intervenes between the two ends of the line, A and Z in the figure, a rod held in the low place, as at B, would seldom be high enough to be seen, from A, to cover the distant rod at Z. In such a case, the surveyor at A should hold up a plumb-line over the point, at arm's length, and place his eye so that the plumb-line covers the rod at Z. He should then direct the rod held at B to be moved till it too is covered by the plumb-line. The point B is then said to be "in line" between A and Z. In geometrical language, B has now been placed in the vertical plane determined by the vertical plumb-line and the point Z. Any number of intermediate points can thus be "interpolated," or placed in line between A and Z.

Fig. 116.

(180) Over a Hill. When a hill rises between two points and prevents one being seen from the other, as in the figure, (the upper

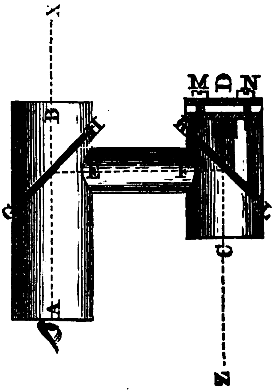
Fig. 117.



of which shows the hill in "Elevation," and the lower part in "Plan"), two observers, B and C, each holding a rod, may place themselves on the ridge, in the line between the two points, as nearly as they can guess, and so that each can at once see the other and the point beyond him. B looks to Z, and by signals puts C "in line." C then looks to A, and puts B in line at B′. B repeats his operation from B′, putting C at C′, and is then himself moved to B″, and so they alternately "line" each other, continually approximating to the straight line between A and Z, till they at last find themselves both exactly in it, at B‴ and C‴.

(181) A single person may put himself in line between two points, on the same principle, by laying a straight stick on some support, going to each end of it in turn, and making it point successively to each end of the line. The "Surveyor's Cross," Art. **(104)**, is convenient for this purpose, when set up between the two given points, and moved again and again, until, by repeated trials, one of its slits sights to the given points when looked through in either direction.

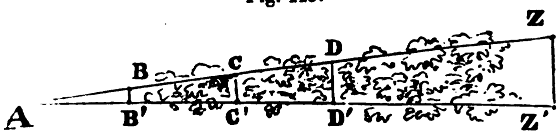
Fig. 118.



(182) On Water. A simple instrument for the same object, is represented in the figure. AB and CD are two tubes, about $1\frac{1}{2}$ inches in diameter, connected by a smaller tube EF. A piece of looking-glass, GH, is placed in the lower part of the tube AB, and another, KL, in the tube CD. The planes of the two mirrors are at right angles to each other. The eye is placed at A, and the tube AB is directed to any distant object, as X, and any other object behind the observer, as Z, will be seen, apparently under the first object in the mirror GH, by reflection from the mirror KL, when the observer has succeeded in getting in line between the two objects. M, N, are screws by which the mirror KL may be adjusted. The distance between the two tubes will cause a small parallax, which will, however, be insensible except when the two objects are near together.

(183) Through a Wood. When a wood intervenes between any two given points, preventing one from being seen from the other, as in the figure, in which A and Z are the given points, proceed thus:

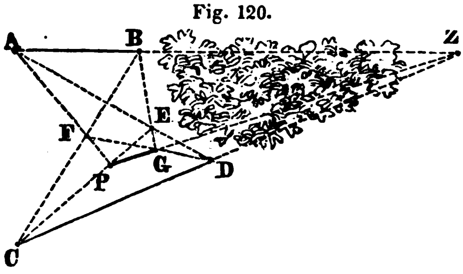
Fig. 119.



Hold a rod at some point B′ as nearly in the desired line from A as can be guessed at, and as far from A as possible. To approximate to the proper direction, an assistant may be sent to the other end of the line, and his shouts will indicate the direction; or a gun may be fired there; or, if very distant, a rocket may be sent up after dark. Then range out the "random line" AB′, by the method given in Art. (169), noting also the distance from A to each point found, till you arrive at a point Z′, opposite to the point Z, i. e. at that point of the line from which a perpendicular there erected would strike the point Z. Measure Z′Z. Then move each of the stakes, perpendicularly from the line AZ′, a distance proportional to their distances from A. Thus, if AZ′ be 1000 links, and Z′Z be 10 links, then a stake B′, 200 links from A, should be moved 2 links to a point B, which will be in the desired straight line AZ; if C′ be 400 links from A, it should be moved 4 links to C, and so with the rest. The line should then be cleared, and the accuracy of the position, of these stakes tested by ranging from A to Z.

(184) To an invisible intersection. Let AB and CD be two lines, which, if prolonged, would meet in a point Z, invisible from either of them; and let P be a point, from which a line is required to be set out, tending to this invisible intersection.

Fig. 120.



Set stakes at the five given points, A, B, C, D, P. Set a sixth stake at E, in the alinements of AD and CP; and a seventh stake at F, in the alinements of BC and AP. Then set an eighth stake at G, in the alinements of BE and DF. PG will be the required line.

Otherwise; Through P range out a parallel to the line BD. Note the points where this parallel meets AB and CD, and call these points Q and R. Then the distance from B, on the line BD, to a point which shall be in the required line running from P to the invisible point, will be $= \frac{BD \times QP}{QR}$.

II. OBSTACLES TO MEASUREMENT.

(185) The cases, in which the direct measurement of a line is prevented by various obstacles, may be reduced to three.

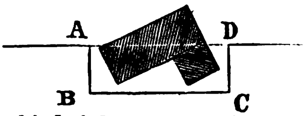
A. When both ends of the line are accessible.

B. When one end of it is inaccessible.

C. When both ends of it are inaccessible.

A. WHEN BOTH ENDS OF THE LINE ARE ACCESSIBLE.

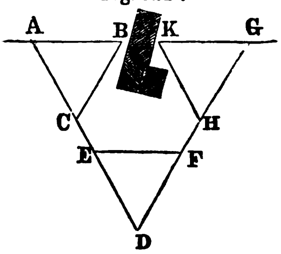
Fig. 121.



(186) By Perpendiculars. On reaching the obstacle, as at A in the figure, set off a perpendicular, AB; turn a second right angle at B, and measure past the obstacle; turn a third right angle at C; and measure to the original line at D. Then will the measured distance, BC, be equal to the desired distance, AD.

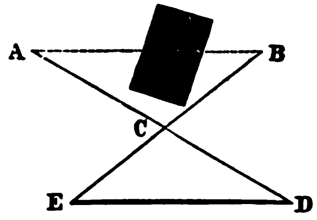
If the direction of the line is also unknown, it will be most easily obtained by the additional perpendiculars shown in Fig. 109, of Art. **(171)**.

Fig. 121′.



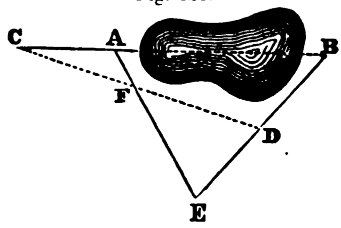
(187) By equilateral triangles. The method given in Art. **(172)**, for determining the direction of a line through an obstacle, will also give its length; for in Fig. 121′ (Fig. 110 repeated) the desired distance AG is equal to the measured distances AD, or DG.

Fig. 122.



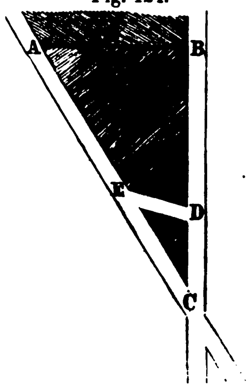
(188) By symmetrical triangles. Let AB be the distance required. Measure from A obliquely to some point C, past the obstacle. Measure onward, in the same line, till CD is as long as AC. Place stakes at C and D. From B measure to C, and from C measure onward, in the same line, till CE is equal to CB. Measure ED, and it will be equal to AB, the distance required. If more convenient, make CD and CE equal, respectively, to half of AC and CB; then will AB be equal to twice DE.

Fig. 123.



(189) By transversals. Let AB be the required distance. Set a stake, C, in the line prolonged; set another stake, D, so that C and B can be seen from it; and a third stake, E, in the line of BD prolonged, and at a distance from D equal to the distance from D to B. Set a fourth stake, F, at the intersection of EA and CD. Measure AC, AF and FE. Then is $AB = \frac{AC}{AF}(FE - AF)$.

Fig. 124.



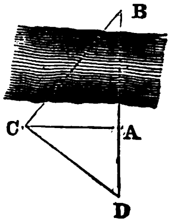
(190) In a Town. Cases may occur, in the streets of a compactly built town, in which it is impossible to measure along any other lines than those of the streets. The figure represents such a case, in which is required the distance, AB, between points situated on two streets which meet at the point C, and between which runs a cross-street, DE. In this case measure AC, CE, CD, DE and CB. Then is the required distance

$$AB = \sqrt{\left\{(AC - BC)^2 + \left[DE^2 - (CE - CD)^2\right]\frac{AC \times BC}{CD \times CE}\right\}}$$

As this expression is somewhat complicated, an example will be given: Let $AC = 100$, $CE = 40$, $CD = 30$, $DE = 21$, and $CB = 80$; then will $AB = 51.7$.

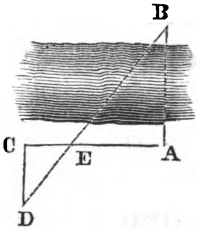
B. WHEN ONE END OF THE LINE IS INACCESSIBLE.

Fig. 125.



(191) By Perpendiculars. This principle may be applied in a variety of ways. In Fig. 125, let AB be the required distance. At the point A, set off AC, perpendicular to AB, and of any convenient length. At C, set off a perpendicular to CB, and continue it to a point, D, in the line of A and B. Measure DA. Then is $AB = \frac{AC^2}{AD}$.

Fig. 126.

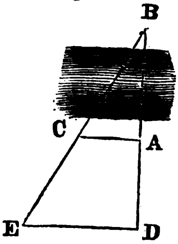


(192) *Otherwise:* At the point A, in Fig. 126, set off a perpendicular, AC. At C set off another perpendicular, CD. Find a point, E, in the line of AC, and BD. Measure AE and EC. Then is $AB = \frac{AE \times CD}{CE}$.

If EC be made equal to AE, and D be set in the line of BE, and also in the perpendicular from C, then will CD be equal to AB.

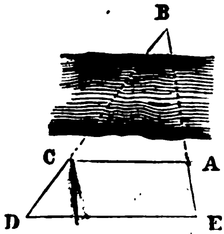
If $EC = \frac{1}{2} AE$, then $CD = \frac{1}{2} AB$.

Fig. 127.



(193) *Otherwise:* At A, in Fig. 127, measure a perpendicular, AC, to the line AB; and at any point, as D, in this line, set off a perpendicular to DB, and continue it to a point E, in the line of CB. Measure DE and also DA. Then is $AB = \frac{AC \times AD}{DE - AC}$.

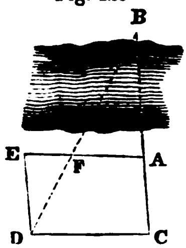
Fig. 128.



(194) By Parallels. From A measure AC, in any convenient direction. From a point D, in the line of BC, measure a line parallel to CA, to a point E, in the line of AB. Measure also AE.

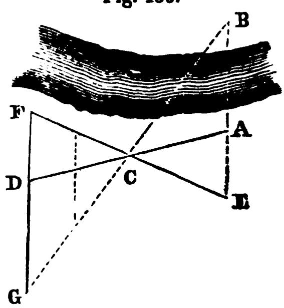
Then is $AB = \frac{AC \times AE}{DE - AC}$.

Fig. 129



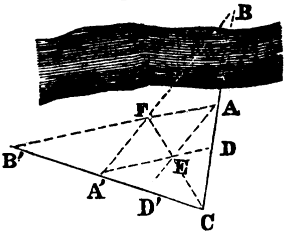
(195) By a Parallelogram. Set a stake, C, in the line of A and B, and set another stake, D, wherever convenient. With a distance equal to CD, describe from A, an arc on the ground; and, with a distance equal to AC, describe another arc from D, intersecting the first arc in E. Or, take AC and CD, so that together they make one chain; fix the ends of the chain at A and D; take hold of the chain at such a link, that one part of it equals AC, and the other CD, and draw it tight to fix the point E. Set a stake at F, in the intersection of AE and DB. Measure AF and EF. Then is $AB = \frac{AC \times AF}{EF}$; or, $CB = \frac{AC \times CD}{EF}$.

Fig. 130.



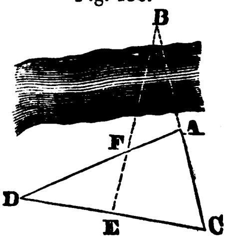
(196) By Symmetrical Triangles. Let AB be the required distance. From A measure a line, in any convenient direction, as AC, and measure onward, in the same direction, till $CD = AC$. Take any point E in the line of A and B. Measure from E to C, and onward in the same line, till $CF = CE$. Then find by trial a point G, which shall be at the same time in the line of C and B, and in the line of D and F. Measure the distance from G to D, and it will be equal to the required distance from A to B. If more convenient, make $CD = \frac{1}{2} AC$, and $CF = \frac{1}{2} CE$, as shown by the finely dotted lines in the figure. Then will $DG = \frac{1}{2} AB$.

Fig. 131.



(197) *Otherwise:* Prolong BA to some point C. Range out any convenient line CA′, and measure CA′ = CA. The triangle CA′B, is now to be reproduced in a symmetrical triangle, situated on the accessible ground. For this object, take, on AC, some point D, and measure CD′ = CD. Find the point E, at the intersection of AD′ and A′D. Find the point F, at the intersection of A′B and CE. Lastly, find the point B′, at the intersection of AF and CA′. Then will A′B′ = AB. The symmetrical points have corresponding letters affixed to them.

Fig. 132.



(198) By transversals. Set a stake, C, in the alinement of BA; a second, D, at any convenient point; a third, E, in the line CD; and a fourth, F, at the intersection of the alinements of DA and EB. Measure AC, CE, ED, DF and FA. Then is

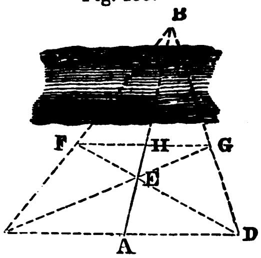
$$AB = \frac{AC \times AF \times DE}{CE \times DF - AF \times DE}.$$

If the point E be taken in the middle of CD, (as it is in the figure) then $AB = \frac{AC \times AF}{DF - AF}$.

If the point F be taken in the middle of AD, then $AB = \frac{AC \times DE}{CE - DE}$.

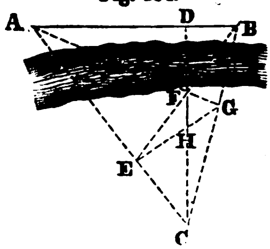
The *minus* signs must be interpreted as in Art. (174).

Fig. 133.



(199) By harmonic division. Set stakes, C and D, on each side of A, and so that the three are in the same straight line. Set a third stake at any point, E, of the line AB. Set a fourth, F, at the intersection of CB and DE; and a fifth, G, at the intersection of DB and CE. Set a sixth stake, H, at the intersection of AB and FG. Measure AE and EH. Then is $AB = \frac{AE \times AH}{AE - EH}$

Fig. 134.



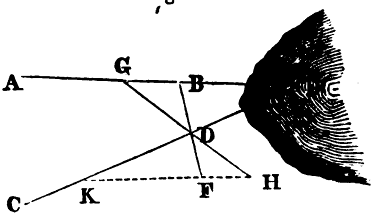
(200) To an inaccessible line. The shortest distance, CD, from a given point, C, to an inaccessible straight line AB, is required. From C let fall a perpendicular to AB, by the method of Art. (158). Then set a stake at any point, E, on the line AC; set a second, F, at the intersection of EB and CD; a third, G, at the intersection of AF and CB; and a fourth, H, at the intersection of EG and CD. Measure CH and HF. Then is

$$CD = \frac{CH \times CF}{CH - HF}\,;\ \text{or, } CD = CH \cdot \frac{CH + HF}{CH - HF}\,;\ \text{or, } CD = \frac{CH \times CF}{2CH - CF}$$

Otherwise; When the inaccessible line is determined by the method of Art. (205) or (206), the distance from any point to it, can be at once measured to its symmetrical representative.

(201) To an inaccessible intersection. When two lines (as AB, CD, in the figure) meet in a river, a building, or any other inaccessible point, the distance from any point of either to their intersection, DE, for example, may be found thus. From any point B, on one line, measure BD, and continue it, till DF = DB. From any other point, G, of the former line, measure GD, and continue the line till DH = GD. Continue HF to meet DC in some point K. Measure KD. KD will be equal to the desired distance DE.

Fig. 135.



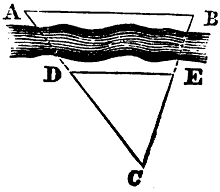
BE can be found by measuring FK, which is equal to it.

If DF and DH, be made respectively equal to one-half, or one-third, &c., of DB and DG, then will KD and KF be respectively equal to one-half or one-third, &c., of DE and BE.

C. WHEN BOTH ENDS OF THE LINE ARE INACCESSIBLE.

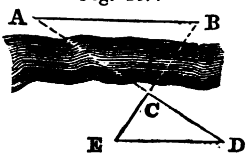
(202) By similar triangles. Let AB be the inaccessible distance. Set a stake at any convenient point C, and find the distances CA and CB, by any of the methods just given. Set a second stake at any point, D, on the line CA. Measure a distance, equal to $\frac{CB \times CD}{CA}$, from C, on the line CB, to some point E. Measure DE. Then is $AB = \frac{AC \times DE}{CD}$.

Fig. 136.



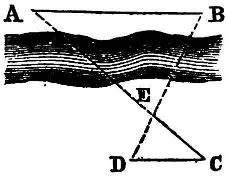
If more convenient, measure CD in the contrary direction from the river, as in Fig. 137, instead of towards it, and in other respects proceed as before.

Fig. 137.



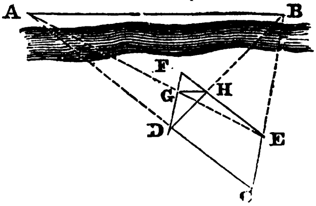
(203) By parallels. Let AB be the inaccessible distance. From any point, as C, range out a parallel to AB, as in Art. (**165**), &c. Find the distance CA, by Art. (**191**), &c. Set a stake at the point E, the intersection of CA and DB, and measure CE. Then is $AB = \frac{CD \times (AC - CE)}{CE}$.

Fig. 138.



(204) By a parallelogram. Set a stake at any convenient point C. Set stakes D and E, anywhere in the alinements CA and CB. With D as a centre, and a length of the chain equal to CE, describe an arc; and with E as a centre, and a length of the chain equal to CD, describe another arc, intersecting the former one at F. A parallelogram, CDEF, will thus be formed. Set stakes at G and H, where the alinements DB and EA intersect the sides of this parallelogram. Measure CD, DF, GF, FH,

Fig. 139.

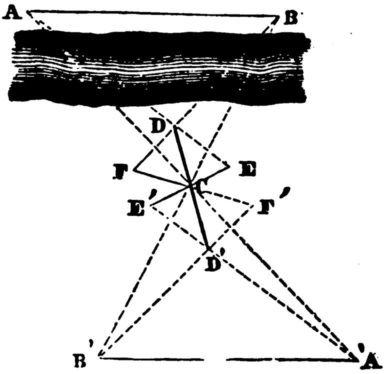


and HG. The inaccessible distance $AB = \dfrac{CD \times DF \times GH}{FG \times FH}$.

If $CD = CE$, then $AB = \dfrac{CD^2 \times GH}{FG \times FH}$.

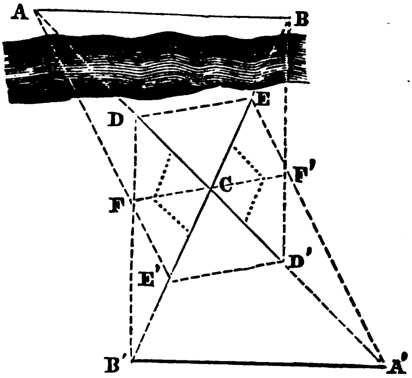
(205) By symmetrical triangles. Take any convenient point, as C. Set stakes at two other points, D and D′, in the same line, and at equal distances from C. Take a point E, in the line of AD; measure from it to C, and onward till CE′ = CE. Take a point F in the line of BD; measure from it to C, and onward till CF′ = CF. Range out the lines AC and E′D′, and set a stake at their intersection, A′. Range out the lines BC and F′D′, and set a stake at their intersection, B′. Measure A′B′. It will be equal to the desired distance AB.

Fig. 140.



(206) *Otherwise:* Take any convenient point, as C, and set off equal distances on each side of it, in the line of CA, to D and D′. Set off the same distances from C, in the line of CB, to E and E′. Through C, set out a parallel to DE, or D′E′, and set stakes at the points F and F′ where this parallel intersects AE′ and BD′. Range out the lines AD′ and EF′, and set a stake at their intersection A′. Range out the lines BE′ and DF, and set a stake at their intersection B′. Measure A′B′, and it will be equal to the desired distance AB.

Fig. 141.

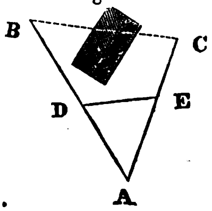


The easiest method of setting out the parallel in the above case, is to fix the middle of the chain at the point C, and its ends on the lines CD, CE′; then carry the middle of the chain from C towards F, and mark the point to which it reaches; and repeat this on the other side of C, as shown by the finely dotted lines in the figure.

INACCESSIBLE AREAS.

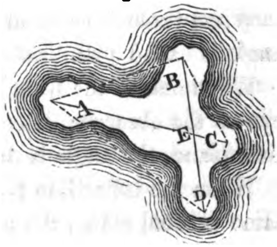
(207) Triangles. In the case of a triangular field, in which one side cannot be measured, or determined by any of the methods just given, the two accessible sides may be prolonged to their full length, and an equal symmetrical triangle formed, all of whose sides can be measured. Thus in Fig. 102, page 103, if CDE be the original triangle, of which the side EC is inaccessible, DFP will be equal to it. But if this also be impossible, portions of the sides may be measured, as AD, AE, in the figure in the margin, and also DE, and the area of this triangle found by any of the methods which have been given. Then is the desired area of the triangle ABC = area of ADE $\times \frac{AB \times AC}{AD \times AE}$.

Fig. 142.



(208) Quadrilaterals. In the case of a four-sided field, whose sides cannot be measured, or prolonged, but whose diagonals can be measured, the area may be obtained thus. Measure the diagonals AC and BD; and also the portions AE, EC, into which one of them is divided by the other. Calculate the area of the triangle BCE, by the preceding method, or any of those heretofore given. Then the area of the quadrilateral ABCD = area of BCE $\times \frac{AC \times BD}{BE \times CE}$.

Fig. 143.



(209) Polygons. Methods for obtaining the areas of inaccessible fields of more than four sides, have been given in Arts. (101,) &c.

PART III.

COMPASS SURVEYING;

OR

By the Third Method

CHAPTER I.

ANGULAR SURVEYING IN GENERAL.

(210) *Angular Surveying* determines the relative positions of points, and therefore of lines, on the THIRD PRINCIPLE, as explained in Art. (7), which should now be referred to.

(211) When the two lines which form an angle lie in the same horizontal or level plane, the angle is called a *horizontal angle.**

When these lines lie in a plane perpendicular to the former, the angle is called a *vertical angle.*

When one of the lines is horizontal and the other line from the eye of the observer passes above the former, and in the same vertical plane, the angle is called an *angle of elevation.*

When the latter line passes below the horizontal line, and in the same vertical plane, the angle is called an *angle of depression.*

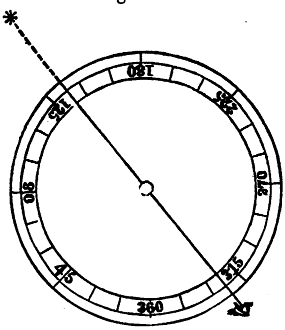
When the two lines which form an angle, lie in other planes which make oblique angles with each of the former planes, the angle is called an *oblique angle.*

Horizontal angles are the only angles employed in common land surveying.

* A plane is said to be *horizontal*, or *level*, when it is parallel to the surface of standing water, or perpendicular to a plumb-line. A line is horizontal when it lies in a horizontal plane.

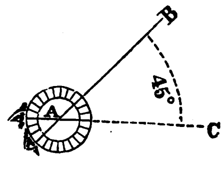
(212) The angles between the directions of two lines, which it is necessary to measure, may be obtained by a great variety of instruments. All of them are in substance mere modifications of the very simple one which will now be described, and which any one can make for himself.

Fig. 144.



(213) Provide a circular piece of wood, and divide its circumference (by any of the methods of Geometrical Drafting) into three hundred and sixty equal parts, or "Degrees," and number them as in the figure. The divisions will be like those of a watch face, but six times as many. These divisions are termed *graduations*. The figure shows only every fifteenth one. In the centre of the circle, fix a needle, or sharp-pointed wire, and upon this fix a straight stick, or thin ruler placed edge-wise, (called an *alidade*), so that it may turn freely on this point and nearly touch the graduations of the circle. Fasten the circle on a staff, pointed at the other end, and long enough to bring the alidade to the height of the eyes. The instrument is now complete. It may be called a *Goniometer*, or Angle-measurer.

Fig. 145.



(214) Now let it be required to measure the angle between the lines AB and AC. Fix the staff in the ground, so that its centre shall be exactly over the intersection of the two lines. Turn the alidade, so that it points, (as determined by sighting along it) to a rod, or other mark at B, a point on one of the lines, and note what degree it covers, i. e. "The Reading." Then, without disturbing the circle, turn the alidade till it points to C, a point on the other line. Note the new reading. The difference of these readings, (in the figure, 45 degrees), is the difference in the directions of the two lines, or is the angle which one makes with the other. If the distance from A to C be now measured, the point C is "determined," with respect to the points A and B, on the *Third Principle*. Any number of points may be thus determined.

(215) Instead of the very simple and rude alidade, which has been supposed to be used, needles may be fixed on each end of the alidade; or sights may be added, such as those described in Art. (168); or a small straight tube may be used, one end being covered with a piece of pasteboard in which a very small eye hole is pierced, and threads, called "cross-hairs," being stretched across the other end of it, as in the figure; so that their intersection may give a more precise line for determining the direction of any point.

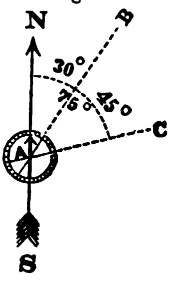
Fig. 146.

(216) When a telescope is substituted for this tube, and supported in such a way that it can turn over, so as to look both backwards and forwards, the instrument (with various other additions, which however do not affect the principle), is called the *Engineer's Transit*.

With the addition of a level, and a vertical circle, for measuring vertical angles, the instrument becomes a *Theodolite;* in which, however, the telescope does not usually admit of being turned over.

(217) The *Compass* differs from the instruments which have been described, in the following respect. They all measure the angle which one line makes with another. The compass measures the angle which each of these lines makes with a third line, viz: that shown by the magnetic needle, which always points (approximately) in the same direction, i. e. North and South, in the *Magnetic Meridian*. Thus, in the figure, the line AB makes an angle of 30 degrees with the line AN, and the line AC makes an angle of 75 degrees with AN. The difference of these angles, or 45 degrees, is the angle which AC makes with AB, agreeing with the result obtained in Art. (214).

Fig. 147.



(218) Surveying with the compass is, therefore, a less direct operation than surveying with the Transit or Theodolite. But as the use of the compass is much more rapid and easy (only one sight and reading at each station being necessary, instead of two, as in the former case), for this reason, as well as for its smaller cost, it is the instrument most commonly employed in land surveying in this country, in spite of its imperfections and inaccuracies.

As many may wish to learn "Surveying with the Compass," without being obliged to previously learn "Surveying with the Transit," (which properly, being more simple in principle, though less so in practice, should precede it, but which will be considered in Part IV), we will first take up COMPASS SURVEYING.

(219) Angular Surveying in general, and therefore Compass Surveying, may employ either of the 3d, 4th and 5th methods of determining the position of a point, given in Part I; that is, any instrument which measures angles may be employed for *Polar*, *Triangular*, or *Trilinear* Surveying. The first of these, Polar Surveying, is the one most commonly adopted for the compass, and is therefore the one which will be specially explained in this part.

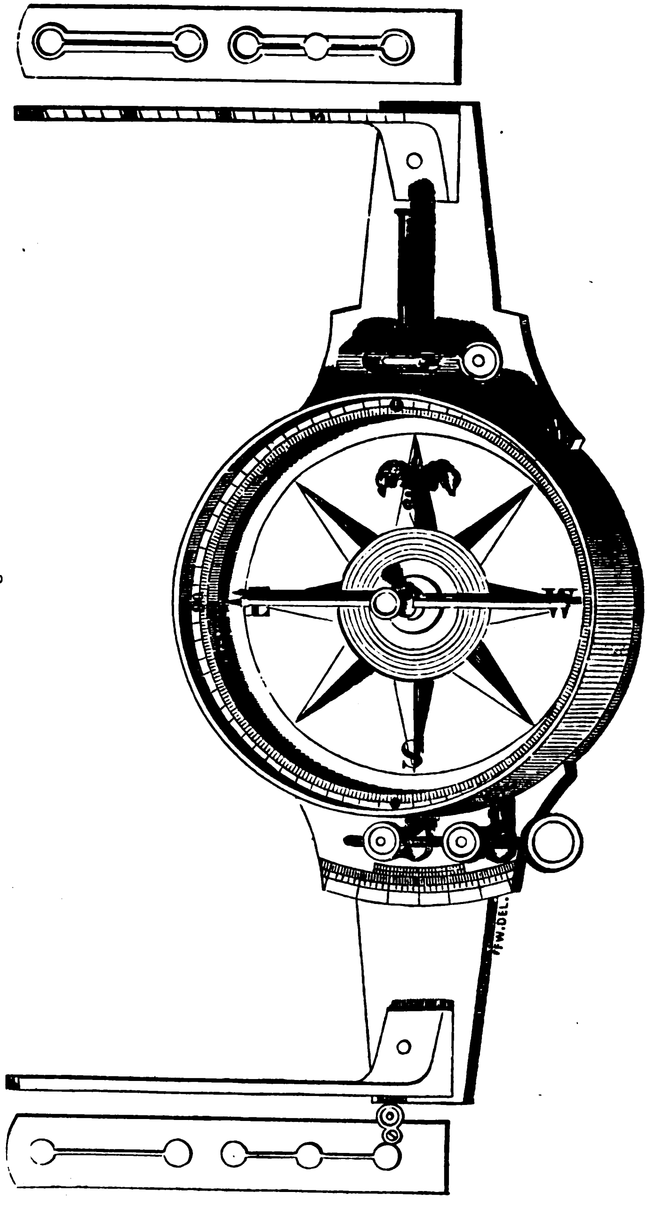
The same method, as employed with the Transit and Theodolite, will be explained in the following part.

The 4th and 5th methods will be explained in the next two parts.

(220) The method of *Polar Surveying* embraces two minor methods. The most usual one consists in going around the field with the instrument, setting it at each corner and measuring there the angle which each side makes with its neighbor, as well as the length of each side. This method is called by the French the method of *Cheminement.* It has no special name in English, but may be called (from the American verb, To progress), the *Method of Progression.* The other system, the *Method of Radiation*, consists in setting the instrument at one point, and thence measuring the direction and distance of each corner of the field, or other object. The corresponding name of what we have called Triangular Surveying is the *Method of Intersections;* since it determines points by the intersections of straight lines.

THE COMPASS.

Fig. 198.



CHAPTER II.

THE COMPASS.

(221) The Needle. The most essential part of the compass is the magnetic needle. It is a slender bar of steel, usually five or six inches long, strongly magnetized, and balanced on a pivot, so that it may turn freely, and thus be enabled to continue pointing in the same direction (that of the "*Magnetic Meridian*," approximately North and South) however much the "Compass Box," to which the pivot is attached, may be turned around.

As it is important that the needle should move with the least possible friction, the pivot should be of the hardest steel ground to a very sharp point; and in the centre of the needle, which is to rest on the pivot, should be inserted a cap of agate, or other hard material. Iridium for the pivot, and ruby for the cap, are still better.

If the needle be balanced on its pivot before being magnetized, one end will sink, or "Dip," after the needle is magnetized. To bring it to a level, several coils of wire are wound around the needle so that they can be slid along it, to adjust the weight of its two ends, and balance it more perfectly.

The North end of the needle is usually cut into a more ornamental form than the South end, for the sake of distinction.

The principal requisites of a compass needle are, intensity of directive force and susceptibility. "Shear steel" was found by Capt. Kater to be the kind capable of receiving the greatest magnetic force. The best form is that of a rhomboid, or lozenge, cut out in the middle, so as to diminish the extent of surface in proportion to the mass, as it is the latter on which the directive force depends. Beyond a certain limit, say five inches, no additional power is gained by increasing the length of the needle. On the contrary, longer ones are apt to have their strength diminished by several consecutive poles being formed. Short needles, made very hard, are therefore to be preferred.

Fig. 149.

The needle should not come to rest very quickly. If it does, it indicates either that it is weakly magnetized, or that the friction on the pivot is great. Its sensitiveness is indicated by the number of vibrations which it makes in a small space before coming to rest.

A screw, with a milled head, on the under side of the plate which supports the pivot, is used to raise the needle off this pivot, when the instrument is carried about, to prevent the point being dulled by unnecessary friction.

(222) **The Sights.** Next after the needle, which gives the direction of the fixed line, whose angles with the lines to be surveyed, are to be measured, should be noticed the Sights, which show the directions of these last lines. At each end of a line passing through the pivot is placed a "Sight," consisting of an upright bar of brass, with openings in it of various forms; usually either slits, with a circular aperture at their top and bottom*; or of the form described in Art. (**106**); all these arrangements being intended to enable the line of sight to be directed to any desired object, with precision.

(223) A Telescope which can move up and down in a vertical plane, i. e. a plunging telescope, or one which can turn completely over, is sometimes substituted for the sights. It has the great advantage of giving more distinct vision at long distances, and of admitting of sights up and down very steep slopes. Its accuracy of vision is however rendered nugatory by the want of precision in the readings of the needle. If a telescope is applied to the compass, a graduated scale and index should also be added, thus converting the compass into a "Transit." The Telescope will be found minutely described in Part IV, "Transit Surveying."

(224) **The divided circle.** We now have the means of indicating the directions of the two lines whose angle is to be measured. The number of degrees contained in it is to be read from a circle, divided into degrees, in the centre of which is fixed the

* An inside and an outside view, or "Elevation," of such sights, are given on each side of the figure of the Compass, on page 126. It is itself drawn in "Military Perspective."

pivot bearing the needle. The graduations are usually made to half a degree, and a quarter of a degree or less can then be "estimated." The pivot and needle are sunk in a circular box, so that its top may be on a level with the needle. The graduations are usually made on the top of the surrounding rim of the box, but should also be continued down its inside circumference so that it may be easier to see with what division the ends of the needle coincide.

The degrees are not numbered consecutively from 0° around to 360°; but run from 0° to 90°, both ways from the two diametrically opposite points at which a line, passing through the slits in the middle of the sights, would meet the divided circle.

The lettering of the Surveyor's Compass has one important difference from that of the Mariner's Compass.

When we stand facing the North, the East is on our right hand, and the West on our left. The graduated card of the Mariner's Compass which is fastened to the needle, and turns with it, is marked accordingly. But, in the Surveyor's compass, one of the 0 points being marked N, or North, (or indicated by a fleur-de-lis,) and the opposite one S, or South, the 90-degrees-point on the *right* of this line, as you stand at the S end and look towards the N, is marked W, or West; and the *left* hand 90-degrees-point is marked E, or East. The reason of this will be seen when the method of using the compass comes to be explained in the following chapter.

(225) The Points. In ordinary land surveying, only four points of the compass have names, viz: North, South, East and West; the direction of a line being described by the angle which it makes with a North and South line, to its East or to its West. But for nautical purposes, the circle of the compass is divided into 32 points, the names of which are shown in

Fig. 150.

the figure. Two rules embrace all the cases. 1° When the letters indicating two points are joined together, the point half way between the two is meant; thus, N. E. is half way between North and East; and N. N. E. is half way between North and North East. 2° When the letters of two points are joined together with the intermediate word *by*, it indicates the point which comes next after the first, in going towards the second; thus, N. by E, is the point which follows North in going towards the East; S. E. *by* S. is the next point from South East, going towards the South.

(226) Eccentricity. The centre-pin, or pivot of the needle, ought to be exactly in the centre of the graduated circle; the needle ought to be straight; and the line of the sights ought to pass exactly through this centre and through the 0 points of the circle. If this is not the case, there will be an error in every observation. This is called the *error of eccentricity*.

When the maker of a compass is about to fix the pivot in place, he is in doubt of two things; whether the needle is perfectly straight, and whether the pivot is exactly in the centre. In figures 151 and 152, both of these are represented as being excessively in error.

Fig. 151.

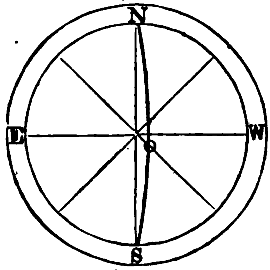
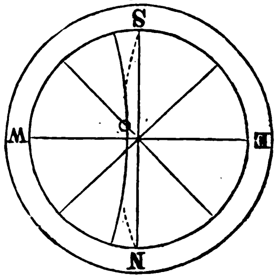


Fig. 152.

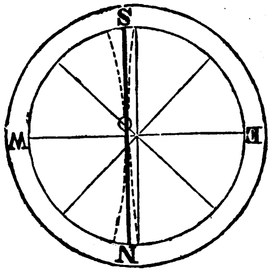


Firstly, to examine if the needle be straight. Fix the pivot temporarily, so that the ends of the needle may cut opposite degrees, i. e. degrees differing by 180°. The condition of things at this stage of progress, will be represented by Fig. 151. Then turn the compass-box half way around. The error will now be doubled, as is shown by Fig. 152, in which the former position of the needle is indicated by a dotted line.* Now bend the needle, as in Fig. 153, till it cuts divisions midway between those cut by it in

* This is another example of the fruitful principle of *Reversion*, first noticed in Art. (105).

its present and in its former position This makes it certain that the needle is straight, or that its two ends and its centre lie in the same straight line.

Fig 153.



Secondly, to put the pivot in the centre. Move it till the straightened needle cuts opposite divisions. It is then certain that the direction of the needle passes through the centre. Turn the compass box one-quarter around, and if the needle does not then cut opposite divisions, move the pivot till it does. Repeat the operation in various positions of the box. It will be a sufficient test if it cuts the opposite divisions of 0°, 45° and 90°.

To fix the sights precisely in line, draw a hair through their slits and move them till the hair passes over the 0 points on the circle.

The surveyor can also examine for himself, by the principle of Reversion, whether the line of the sights passes through the centre or not. Sight to any very near object. Read off the number of degrees indicated by one end of the needle. Then turn the compass half around, and sight to the same object. If the two readings do not agree, there is an error of eccentricity, and the arithmetical mean, or half sum of the two readings is the correct one.

Fig. 154.

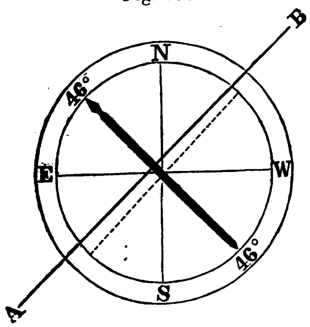
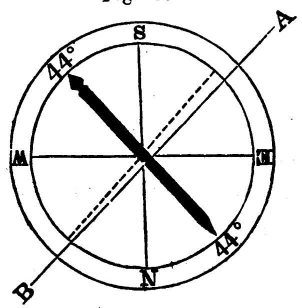


Fig. 155.



In Fig. 154, the line of sight AB is represented as passing to one side of the centre, and the needle as pointing to 46°. In Fig. 155, the compass is supposed to have been turned half around and the other end of the sights to be directed to the same object. Suppose that the needle would have pointed to 45°, if the line of sight had passed through the centre. The needle will now point to 44°, the error being doubled by the reversion, and the true reading being the mean.

This does not, however, make it certain that the line of the sights passes through the 0 points, which can only be tested by the hair, as mentioned above.

(227) Levels. On the compass plate are two small spirit levels. They consist of glass tubes, slightly curved upwards, and nearly filled with alcohol, leaving a bubble of air within them. They are so adjusted that when the bubbles are in the centres of the tubes, the plate of the compass shall be level. One of them lies in the direction of the sights, and the other at right angles to this direction.

(228) Tangent Scale. This is a convenient, though not essential, addition to the compass, for the purpose of measuring the slopes of ground, so that the proper allowance in chaining may be made. In the figure of the compass, page 126, may be seen, on the edge of the left hand sight, a small projection of brass with a hole through it. On the edge of the other sight are engraved lines numbered from 0° to 20°, the 0° being of the same height above the compass plate that the eye-hole is. To use this, set the compass at the bottom of a slope, and at the top set a signal of exactly the height of the eye-hole from the ground. Level the compass very carefully, particularly by the level which lies lengthwise, and, with the eye at the eye-hole, look to the signal and note the number of the division on the farther sight which is cut by the visual ray. That will be the angle of the slope; the distances of the engraved lines from the 0° line being tangents (for the radius equal to the distance between the sights) of the angles corresponding to the numbers of the lines.

(229) Vernier. The compass box is connected with the plate, which carries it and the sights, so that it can turn around on this plate. This motion is given to it by a screw, (called a slow-motion, or Tangent screw), the head of which is the nearest one in the figure on page 126. If two marks be made opposite to each other, one on the projecting part of the compass box, and the other on the plate to which the sights are fastened, these marks will separate when the slow-motion screw is turned. Their distance apart (in angular measurement, i. e. fractions of a circle), in any position, is measured by a contrivance called a *Vernier*, which is the minutely divided arc of a circle seen between the left hand sight and the compass box. It will be better to defer explaining the mode of reading the vernier for the present, since it is rarely used with the compass, and an entire chapter will be given to it in Part IV. Its principle is similar to that of the Vernier Scale, described in Art. (50). Its applications in "Field-work" will be noticed under that head.

(230) Tripod. The compass, like most surveying instruments, is usually supported on a Tripod, consisting of three legs, shod with iron, and so connected at top as to be movable in any direction. There are many forms of these. Lightness and stiffness are the qualities desired. The most usual form is shewn in the figures of the Transit and the Theodolite, at the beginning of Part IV. Of the two represented in Figs. 156 and 157, the first has the advantage of being very easily and cheaply made; and the second that of being light and yet capable of very firmly resisting horizontal torsion.

Fig. 156.

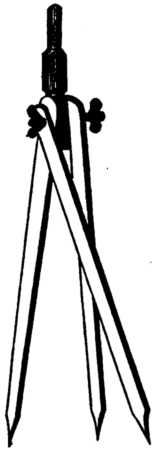
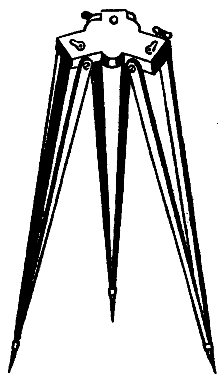


Fig. 157.



The joints, by which the instrument is connected with the tripod, are also various. Fig. 158 is the "Ball-and-socket joint," most usual in this country. It takes its name from the ball, in which

Fig. 158.

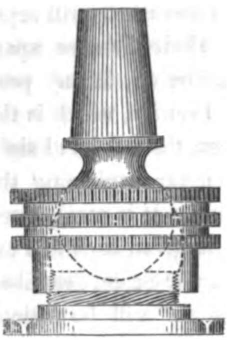


Fig. 159.

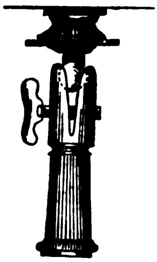
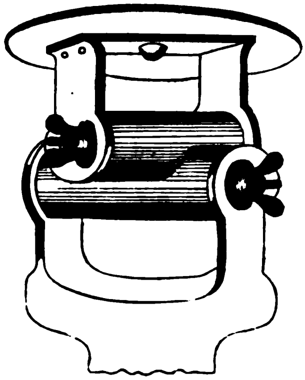


Fig. 160.

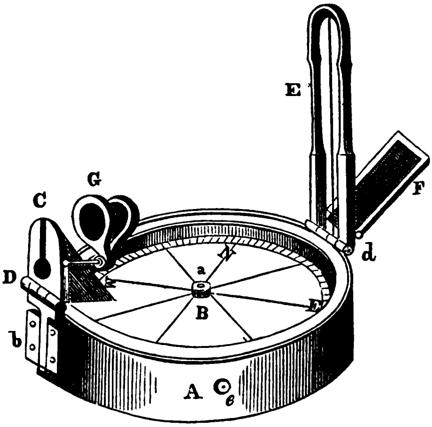


terminates the covered spindle which enters a corresponding cavity under the compass plate, and the socket in which this ball turns. It admits of motion in any direction, and can be tightened or loosened by turning the upper half of the hollow piece enclosing it, which is screwed on the lower half. Fig. 159 is called the "Shell-joint." In it the two shell-shaped pieces enclosing the ball are tightened by a thumb-screw. Fig. 160, is "Cugnot's joint." It consists of two cylinders, placed at right angles to each other, and through the axes of which pass bolts, which turn freely in the cylinder and can be tightened or loosened by thumb-screws at their ends. The combination of the two motions which this joint permits, enables the instrument which it carries, to be placed in any desired position. This joint is much the most stable of the three.

(231) Jacob's Staff. A single leg, called a "Jacob's Staff," has some advantages, as it is lighter to carry in the field, and can be made of any wood, on the spot where it is to be used, thus saving the expense of a tripod and the trouble of its transportation. Its upper end is fitted into the lower end of a brass head which has a ball and socket joint, and axis above. Its lower end should be shod with iron, and a spike running through it is useful for pressing it into the ground with the foot. Of course it cannot be conveniently used on frozen ground, or on pavements. It may, however, be set before or behind the spot at which the angle is to be measured, provided that it is placed very precisely in the line of direction from that station to the one to which a sight is to be taken.

(232) The Prismatic Compass. The peculiarity of this instrument (often called Schmalcalder's) is that a glass triangular prism is substituted for one of the sights. Such a prism has this peculiar property that at the same time, it can be seen through, so that a sight can be taken through it, and that its upper surface reflects like a mirror, so that the numbers of the degrees immediately under it, can be read off at the same time that a sight to any object is taken. Another peculiarity, necessary for profiting by the last one, is, that the divided circle is not fixed, but is a card fastened to the needle and moving around with it, as in the Mariner's Compass. The minute description, which follows, is condensed from Simms.

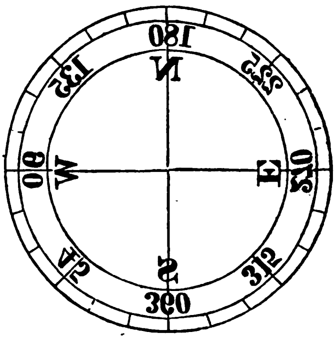
Fig. 161.



In the figure, A represents the compass box, and B the card, which, being attached to the magnetic needle, moves as *it* moves, around the agate centre, *a*, on which it is suspended. The circumference of the card is usually divided to $\frac{1}{4}$ or $\frac{1}{2}$ of a degree. C is a prism, which the observer looks through. The perpendicular thread of the sight-vane, E, and the divisions on the card, appear *together* on looking through the prism, and the division with which the thread coincides, when the needle is at rest, is the "Bearing" of whatever object the thread may bisect, i. e. is the angle which the line of sight makes with the direction of the needle. The prism is mounted with a hinge joint, D. The sight-vane has a fine thread stretched along its opening, in the direction of its length, which is brought to bisect any object, by turning the box around horizontally. F is a mirror, made to slide on or off the sight-vane, E; and it may be reversed at pleasure, that is, turned face downwards; it can also be inclined at any angle, by means of its joint, *d*; and it will remain stationary on any part of the vane, by the friction of its slides. Its use is to reflect the image of an object to the eye of an observer when the object is much above or below the horizontal plane. The colored glasses represented at G, are intended for observing the sun. At *e*, is shown a spring, which being pressed by the finger at the time of observation, and then released, checks the vibrations of the card, and brings it more speedily to rest. A stop is likewise fixed to the other side of the box, by which the needle may be thrown off its centre.

The method of using this instrument is very simple. First raise the prism in its socket, *b*, until you obtain a distinct view of the divisions on the card. Then, standing over the point where the angles are to be taken, hold the instrument to the eye, and, looking through the slit, *C*, turn around till the thread in the sight-vane bisects one of the objects whose bearing is required; then by touching the spring, *e*, bring the needle to rest, and the division on the card which coincides with the thread on the vane, will be the bearing of the object from the north or south points of the magnetic meridian. Then turn to any other object, and repeat the operation; the difference between the bearing of this object and that of the former, will be the angular distance of the objects in question. Thus, suppose the former bearing to be 40° 30′, and the latter 10° 15′, both east, or both west, from the north or south, the angle will be 30° 15′. The divisions are generally numbered 5°, 10°, 15°, &c. around the circle to 360°.

Fig. 162.



The figures on the compass card are reversed, or written upside down, as in the figure (in which only every fifteenth degree is marked), because they are again reversed by the prism.

(233) The prismatic compass is generally held in the hand, the bearing being caught, as it were, in passing; but more accurate readings would of course be obtained if it rested on a support, such as a stake cut flat on its top.

In the former mode, the needle never comes completely to rest, particularly in the wind. In such cases, observe the extreme divisions between which the needle vibrates, and take their arithmetical mean.

(234) **Defects of compass.** The compass is deficient in both precision and correctness.*

The former defect arises from the indefiniteness of its mode of indicating the part of the circle to which it points. The point of the needle has considerable thickness; it cannot quite touch the divided circle; and these divisions are made only to whole or half degrees, though a fraction of a division may be estimated, or guessed at. The Vernier does not much better this, as we shall see when explaining its use. Now an inaccuracy of one quarter of a degree in an angle, i. e. in the difference of the directions of two lines, causes them to separate from each other $5\frac{1}{4}$ inches at the end of 100 feet; at the end of 1000 feet nearly $4\frac{1}{2}$ feet; and at the end of a mile, 23 feet. A difference of only one-tenth of a degree, or six minutes, would produce a difference of $1\frac{3}{4}$ feet at the end of 1000 feet; and $9\frac{1}{4}$ feet at the distance of a mile. Such are the differences which may result from the want of *precision* in the indications of the compass.

But a more serious defect is the want of *correctness* in the compass. Its not pointing exactly to the true north does not indeed affect the correctness of the angles measured by it. But it does not point in the same or in a parallel direction, during even the same day, but changes its direction between sunrise and noon nearly a quarter of a degree, as will be fully explained in Chapter VIII. The effect of such a difference we have just seen. This direction

* The student must not confound these two qualities. To say that the sun appears to rise in the eastern quarter of the heavens and to set in the western, is *correct*, but not *precise*. A watch with a second hand indicates the time of day *precisely*, but not always *correctly*. The statement that two and two make five, is *precise*, but is not usually regarded as *correct*.

may also be greatly altered in a moment, without the knowledge of the surveyor, by a piece of iron being brought near to the compass, or by some other local attraction, as will be noticed hereafter. This is the weak point in the compass.

Notwithstanding these defects, the compass is a very valuable instrument, from its simplicity, rapidity and convenience in use; and though never precise, and seldom correct, it is generally not *very* wrong.

CHAPTER III.

THE FIELD WORK.

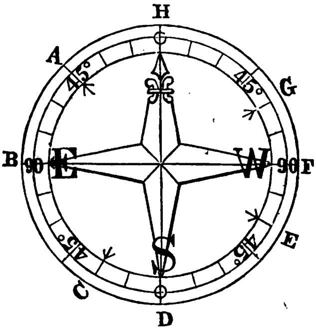
(235) Taking Bearings. The "Bearing" of a line is the angle which it makes with the direction of the needle. Thus, in Fig. 147, page 124, the angle NAB is the Bearing of the line AB, and NAC is the Bearing of AC. The Bearing and length of a line are named collectively the *Course.*

To take the Bearing of any line, set the compass exactly over any point of it by a plumb-line suspended from beneath the centre of the compass, or, approximately, by dropping a stone. Level the compass by bringing the air bubbles to the middle of the level tubes. Direct the sights to a rod held truly vertical, or "plumb," at another point of the line, the more distant the better. The two ends are usually taken. Sight to the lowest visible point of the rod. When the needle comes to rest, note what division on the circle it points to; taking the one indicated by the North end of the needle, if the North point on the circle is farthest from you, and *vice versa.*

In reading the division to which one end of the needle points, the eye should be placed over the other end, to avoid the error which might result from the "parallax," or apparent change of place, of the end read from, when looked at obliquely.

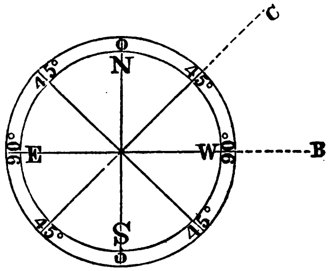
The bearing is read and recorded by noting between what letters the end of the needle comes, and to what number; naming, or writing down, *firstly*, that letter, N or S, which is at the 0° point nearest to that end of the needle from which you are reading; *secondly*, the number of degrees to which it points, and *thirdly*, the letter, E or W, of the 90° point which is nearest to the same end of the needle. Thus, in the figure, if when the sights were directed along a line, (the North point of the compass being most distant from the observer), the North end of the needle was at the point A, the bearing of the line sighted on, would be North 45° East; if the end of the needle was at B, the bearing would be *East;* if at C, S. 30° E; if at D, *South;* if at E, S. 60° W; if at F, West; if at G, N. 60° W; if at H, *North.*

Fig. 163.



(236) We can now understand why W is on the right hand of the compass-box, and E on the left. Let the direction from the centre of the compass to the point B in the figure, be required, and suppose the sights in the first place to be pointing in the direction of the needle, S N, and the North sight to be ahead. When the sights (and the circle to which they are fastened) have been turned so as to point in the direction of B, the point of the circle marked E, will have come round to the North end of the needle, (*since the needle remains immovable,*) and the reading will therefore be "East," as it should be. The effect on the reading is the same as if the needle had moved to the left the same quantity which the sights have moved to the right, and the left side is therefore properly marked "East," and *vice versa.* So, too, if the bearing of the line to C be desired, half-way between North and

Fig. 164.



East, i. e. N. 45° E.; when the sights and the circle have turned 45 degrees to the right, the needle, really standing still, has apparently arrived at the point half-way between N. and E., i. e. N. 45° E.

Some surveyors' compasses are marked the reverse of this, the E on the right and the W on the left. These letters must then be reversed in the mind before the bearing is noted down.

(237) Reading with Vernier. When the needle does not point precisely to one of the division marks on the circle, the fractional part of the smallest space is usually estimated by the eye, as has been explained. But this fractional part may be measured by the Vernier, described in Art. (229), as follows. Suppose the needle to point between N. 31° E. and N. 31½° E. Turn the tangent screw, which moves the compass-box, till the smaller division (in this case 31°) has come round to the needle. The Vernier will then indicate through what space the compass-box has moved, and therefore how much must be added to the reading of the needle. Suppose it indicates 10 minutes of a degree. Then the bearing is N. 31° 10′ E. It is, however, so difficult to move the Vernier without disturbing the whole instrument, that this is seldom resorted to in practice. The chief use of the Vernier is to set the instrument for running lines and making an allowance for the variation of the needle, as will be explained in the proper place. A Vernier-arc is sometimes attached to one end of the needle and carried around by it.

(238) Practical Hints. Mark every station, or spot, at which the compass is set, by driving a stake, or digging up a sod, or piling up stones, or otherwise, so that it can be found if any error, or other cause, makes it necessary to repeat the survey.

Very often when the line of which the bearing is required, is a fence, &c., the compass cannot be set upon it. In such cases, set the compass so that its centre is a foot or two from the line, and set the flag-staff at precisely the same distance from the line at the other end of it. The bearing of the flag-staff from the compass will be the same as that of the fence, the two lines being parallel.

The distances should be measured on the real line. If more convenient the compass may be set at some point on the line prolonged, or at some intermediate point of the line, "in line" between its extremities.

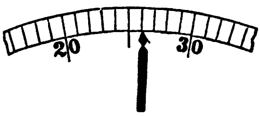
In setting the compass level, it is more important to have it level crossways of the sights than in their direction; since if it be not so, on looking up or down hill through the upper part of one sight and the lower part of the other, the line of sight will not be parallel to the N and S, or zero line, on the compass, and an incorrect bearing will therefore be obtained.

The compass should *not* be levelled by the needle, as some books recommend, i. e. so levelled that the ends of the needle shall be at equal distances below the glass. The needle should be brought so originally by the maker, but if so adjusted in the morning, it will not be so at noon, owing to the daily variation in the *dip*. If then the compass be levelled by it, the lines of sight will generally be more or less oblique, and therefore erroneous. If the needle touches the glass, when the compass is levelled, balance it by sliding the coil of wire along it.

The same end of the compass should always go ahead. The North end is preferable. The South end will then be nearest to the observer. Attention to this and to the caution in the next paragraph, will prevent any confusion in the bearings.

Always take the readings from the same end of the needle; from the North end, if the North end of the compass goes ahead; and *vice versa*. This is necessary, because the two ends will not always cut opposite degrees. With this precaution, however, the angle of two meeting lines can be obtained correctly from either end, provided the same one is used in taking the bearings of both the lines.

Fig. 165.



Guard against a very frequent source of error with beginners, in reading from the wrong number of the two between which the needle points, such as reading 34° for 26°, in a case like that in the figure.

Check the vibrations of the needle by gently raising it off the pivot so as to touch the glass, and letting it down again, by the screw on the under side of the box.

The compass should be smartly tapped after the needle has settled, to destroy the effect of any adhesion to the pivot, or friction of dust upon it.

All iron, such as the chain, &c., must be kept at a distance from the compass, or it will attract the needle, and cause it to deviate from its proper direction.

The surveyor is sometimes troubled by the needle refusing to traverse and adhering to the glass of the compass, after he has briskly wiped this off with a silk handkerchief, or it has been carried so as to rub against his clothes. The cause is the electricity excited by the friction. It is at once discharged by applying a wet finger to the glass.

A compass should be carried with its face resting against the side of the surveyor, and one of the sights hooked over his arm.

In distant surveys an extra centre pin should be carried, (as it is very liable to injury, and its perfection is most essential), and, also, an extra needle. When two such are carried, they should be placed so that the north pole of one rests against the south pole of the other.

(239) When the magnetism of the needle is lessened or destroyed by time, it may be renewed as follows. Obtain two bar magnets. Provide a board with a hole to admit of the axis, so that its collar may fit fairly, and that the needle may rest flat on it, without bearing at the centre. Place the board before you, with the north end of the needle to your right. Take a magnet in each hand, the left holding the North end of the bar, or that which has the mark across, downwards; and the right holding the same mark upwards. Bring the bars over the axis, about a foot above it, without approaching each other within two inches:—bring them down vertically on the needle, (the marks as directed) about an inch on each side of its axis; slide them outwards to its ends with slight pressure; raise them up; bring them to their former position, and repeat this a number of times.

(240) Back Sights. To test the accuracy of the bearing of a line, taken at one end of it, set up the compass at the other end, or point sighted to, and look back to a rod held at the first station, or point where the compass had been placed originally. The reading of the needle should now be the same as before.

If the position of the sights had been reversed, the reading would be the *Reverse Bearing;* a former bearing of N. 30° E. would then be S. 30° W., and so on.

(241) Local attraction. If the Back-sight does not agree with the first or forward sight, this latter must be taken over again. If the same difference is again found, this shows that there is *local attraction* at one of the stations; i. e. some influence, such as a mass of iron ore, ferruginous rocks, &c., under the surface, which attracts the needle, and makes it deviate from its usual direction. Any high object, such as a house, a tree, &c., has recently been found to produce a similar effect.

To discover at which station the attraction exists, set the compass at several intermediate points in the line which joins the two stations, and at points in the line prolonged, and take the bearing of the line at each of these points. The agreement of several of these bearings, taken at distant points, will prove their correctness. Otherwise, set the compass at a third station; sight to each of the two doubtful ones, and then from them back to this third station. This will show which is correct.

When the difference occurs in a series of lines, such as around a field, or along a road, proceed thus. Let C be the station at which the back-sight to B differs from the foresight from B to C. Since the back-sight from B to A is supposed to have agreed with the foresight from A to B, the local attraction must be at C, and the forward bearing must be corrected by the difference just found between the fore and back sights, adding or subtracting it, according to circumstances. An easy method is to draw a

Fig. 166.

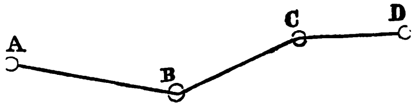


Fig. 167.

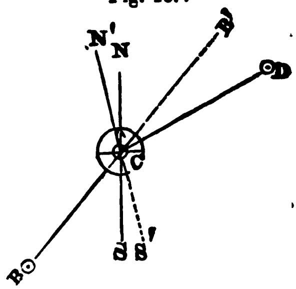


figure for the case, as in Fig. 167. In it, suppose the true bearing of BC, as given by a fore-sight from B to C, to be N. 40° E., but that there is local attraction at C, so that the needle is drawn aside 10°, and points in the direction S'N', instead of SN. The back-sight from C to B will then give a bearing of N. 50° E.; a difference, or correction for the next fore-sight, of 10°. If the next fore-sight, from C to D, be N. 70° E, this 10° must be subtracted from it, making the true fore-sight N. 60° E.

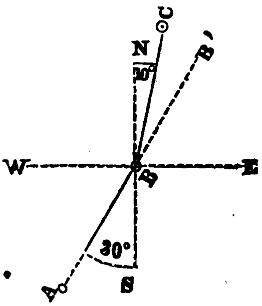
A general rule may also be given. *When the back-sight is greater than the fore-sight*, as in this case, subtract the difference from the next fore-sight, if that course and the preceding one have both their letters the same (as in this case, both being N. and E.), or both their letters different; or add the difference if either the first or last letters of the two courses are different. *When the back-sight is less than the fore-sight*, add the difference in the case in which it has just been directed to subtract it, and subtract it where it was before directed to add it.

(242) Angles of deflection. When the compass indicates much local attraction, the difference between the directions of two meeting lines, (or the "*angle of deflection*" of one from the other), can still be correctly measured, by taking the difference of the bearings of the two lines, as observed at the same point. For, the error caused by the local attraction, whatever it may be, affects both bearings equally, inasmuch as a "Bearing" is the angle which a line makes with the direction of the needle, and that here remains fixed in some one direction, no matter what, during the taking of the two bearings. Thus, in Fig. 167, let the true bearing of BC, i. e. the angle which it makes with the line SN, be, as before, N. 40° E., and that of CD N. 60° E. The true "angle of deflection" of these lines, or the angle B'CD, is therefore 20°. Now, if local attraction at C causes the needle to point in the direc- S'N', 10° to the left of its proper direction, BC will bear N. 50°

E., and CD N. 70° E., and the difference of these bearings, i. e. the angle of deflection, will be the same as before.

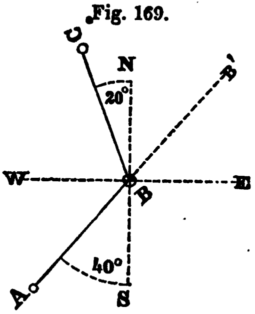
(243) Angles between Courses. To determine the angle of deflection of two courses meeting at any point, the following simple rules, the reasons of which will appear from the accompanying figures, are sufficient.

Fig. 168.



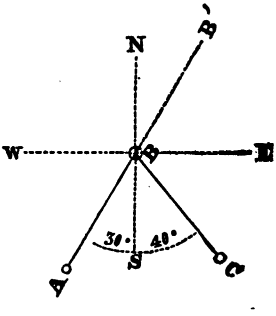
Case 1. When the first letters of the bearing are alike, (i. e. both N. or both S.), and the last letters also alike, (i. e. both E. or both W.), take the difference of the bearings. *Example.* If AB bears N. 30° E. and BC bears N. 10° E., the angle of deflection CBB′ is 20°.

Fig. 169.



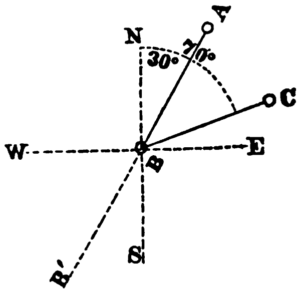
Case 2. When the first letters are alike and the last letters different; take the sum of the bearings. *Ex.* If AB bears N. 40° E. and BC bears N. 20° W.; the angle CBB′ is 60°.

Fig. 170.



Case 3. When the first letters are different and the last letters alike, subtract the sum of the bearings from 180°. *Ex.* If AB bears N. 30° E. and BC bears S. 40° E.; the angle CBB′ is 110°.

Fig. 170.



Case 4. When both the first and last letters are different, subtract the difference of the bearings from 180°. *Ex.* If AB bears S. 30° W. and BC bears N. 70° E.; the angle CBB′ is 140°.

If the angles included between the courses are desired, they will be at once found by reversing one bearing, and then applying the above rules; or by subtracting the results obtained as above from 180°; or an analogous set of rules could be formed for them.

(244) To change Bearings. It is convenient in certain calculations to suppose one of the lines of a survey to change its direction so as to become due North and South; that is, to become a new Meridian line. It is then necessary to determine what the bearings of the other lines will be, supposing them to change with it. The subject may be made plain by supposing the survey to be platted in the usual way, with the North uppermost, and the plat to be then turned around, till the line to be changed is in the desired direction. The effect of this on the other lines will be readily seen. A *General Rule* can also be formed.

Take the *difference* between the original bearing of the side which becomes a Meridian and each of those bearings which have both their letters the same as it, or both different from it. The changed bearings of these lines retain the same letters as before, if they were originally greater than the original bearing of the new Meridian line; but, if they were less, they are thrown on the other side of the N. and S. line, and their last letters are changed; E. being put for W. and W for E.

Take the *sum* of the original bearing of the new Meridian line, and each of those bearings which have one letter the same as one letter of the former bearing, and one different. If this sum exceeds

90°, this shews that the line is thrown on the other side of the East or West point, and the difference between this sum and 180° will be the new bearing and the first letter will be changed, N. being put for S. and S. for N.

Example. Let the Bearings of the sides of a field be as follows: N. 32° E.; N. 80° E.; S. 48° E.; S. 18° W.; N. 73½° W.; North. Suppose the first side to become due North; the changed bearings will then be as follows: North; N. 48° E.; S. 80° E.; S. 14° E.; S. 74½° W.; N. 32° W.

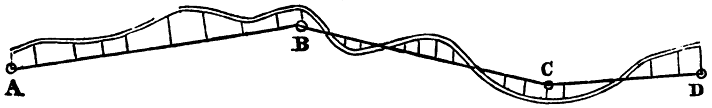
To apply the rule to the "North" course, as above, it must be called N. 0° W.; and then by the Rule, 32° must be added to it.

The true bearings can of course be obtained from the changed bearings, by reversing the operation, taking the sum instead of the difference, and *vice versa.*

(245) Line Surveying. This name may be given to surveys of lines, such as the windings of a brook, the curves of a road, &c., by way of distinction from *Farm Surveying*, in which the lines surveyed enclose a space.

To survey a *brook*, or any similar line, set the compass at, or near, one end of it, and take the bearing of an imaginary or visual line, running in the general average direction of the brook,

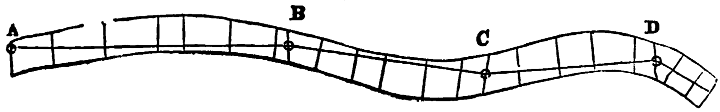
Fig. 172.



such as AB in the figure. Measure this line, taking offsets to the various bends of the brook, as to the fence explained in Art. (**115**). Then set the compass at B, and take a back-sight to A, and if they agree, take a fore-sight to C, and proceed as before, noting particularly the points where the line crosses the brook.

To survey a *road*, take the bearings and lengths of the lines

Fig. 173.



which can be most conveniently measured in the road, and measure offsets on each side, to the outside of the road.

When the line of a new road is surveyed, the bearings and lengths of the various portions of its intended centre line should be measured, and the distance which it runs through each man's land should be noted. Stones should be set in the ground at recorded distances from each angle of the line, or in each line prolonged a known distance, so as not to be disturbed in making the road.

In surveying a wide river, one bank may be surveyed by the method just given, and points on the opposite banks, as trees, &c., may be fixed by the method of intersections, founded on the Fourth Method of determining the position of a point; and fully explained in Part IV.

(246) Checks by intersecting bearings. At each station at which the compass is set, take bearings to some remarkable object, such as a church steeple, a distant house, a high tree, &c. At least three bearings should be taken to each object to make it of any use: since two are necessary to determine it, (by our Fourth Method), and, till thus determined, it can be no check. When the line is platted, by the methods to be explained in the next chapter, plat also the lines given by these bearings. If those taken to the same object from three different stations, intersect in the same point, this proves that there has been no mistake in the survey or platting of those stations.

If any bearing does not intersect a point fixed by previous bearings, it shows that there has been an error, either between the last station and one of those which fixed the point, or in the last bearing to the point. To discover which it was, plat the following line of the survey, and, at its extremity, set off the bearing from it to the point; and if the line thus platted passes through the point, it proves that there was no error in the line, but only in the bearing to the point. If otherwise, the error was somewhere in the line between the stations from which the bearings to that point were taken.

(247) Keeping the Field-notes. The simplest and easiest method for a beginner is to make a rough sketch of the survey by eye, and write down on the lines their bearings and lengths.

An improvement on this is to actually lay down the precise bearings and lengths of the lines in the field-book in the manner to be explained in the chapter on Platting, Art. (269).

(248) A *second* method is to draw a straight line up the page of the field-book, and to write on it the bearings and lengths of the lines. The only advantage of this method is that the line will not run off the side of the page, as it is apt to do in the preceding method.

(249) A *third* method is to represent the line surveyed, by a double column, as in Part II, Chapter I, Art. (95), which should be now referred to. The bearings are written obliquely up the columns. At the end of each course, its length is written in the column, and a line drawn across it. Dotted lines are drawn across the column at any intermediate measurement. Offsets are noted as explained in Art. (114).

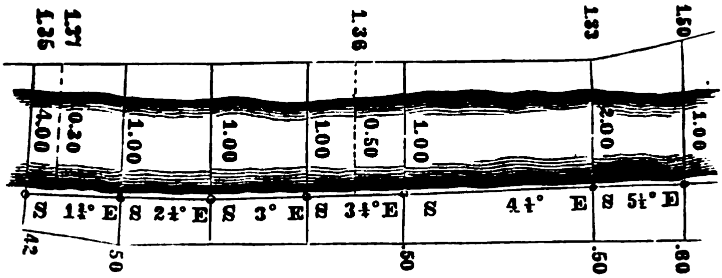
The intersection-bearings, described in Art. (246), should be entered in the field-book *before* the bearings of the line, in order to avoid mistakes of platting, in setting off the measured distances on the wrong line.

(250) A *fourth* method is to write the Stations, Bearings, and Distances in three columns. This is compact, and has the advantage, when applied to farm surveying, of presenting a form suitable for the subsequent calculations of Content, but does not give facilities for noting offsets.

Examples of these four methods are given in Art. (254); which contains the field-notes of the lines bounding a field.

(251) New-York Canal Maps. The following is a description of the original maps of the survey of the line of the New-York Erie Canal, as published by the Canal Commissioners. The figure represents a portion of such a map; but, necessarily, with all its lines black; red and blue lines being used on the real map.

Fig. 174.



"The Red Line described along the inner edge of the towing path is the *base line*, upon which all the measurements in the direction of the length of the canal were made. The *bearings* refer to the magnetic meridian at the time of the survey. The *lengths* of the several portions are inserted at the *end* of each, in chains and links. The *offsets* at each station are represented by red lines drawn across the canal in such a direction as to bisect the angles formed by the two contiguous portions of the red or base line, upon the towing path. The intermediate offsets are set off at right angles to the base line; and the distances on both are given from it in links. The *intermediate offsets* are represented by *red dotted lines*, and the distances to them upon the base line are reckoned, in each case, from the last preceding station. The same is likewise done with the other distances upon the base line; those to the *Bridges* being taken to the lines joining the nearest angles, or corner posts of their abutments; those to the *Locks* extending to the lines passing through the centres of the two nearest quoin posts; and those to the *Aqueducts*, to the faces of their abutments. The space enclosed by the Blue Lines represents the portion embraced within the limits of the survey as belonging to the state; and the names of the adjoining proprietors are given as they stood at the time of executing the survey. The distances are projected upon a *scale* of two chains to the inch."

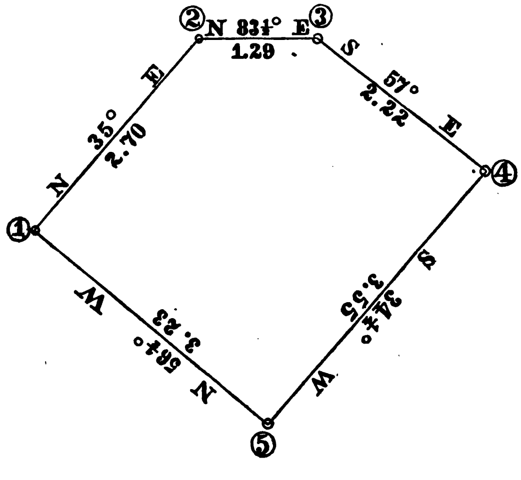
(252) Farm Surveying. A farm, or field, or other space included within known lines, is usually surveyed by the compass thus. Begin by walking around the boundary lines, and setting stakes at all the corners, which the flag-man should specially note, so that he may readily find them again. Then set the compass at any corner, and send the flag-man to the next corner. Take the bearing of the bounding line, running from corner to corner, which is usually a fence. Measure its length, taking offsets if necessary. Note where any other fence, or road, or other line, crosses or meets it, and take their bearings. Take the compass to the end of this first bounding line; sight back, and if the back-sight agrees, take the bearing and distance of the next bounding line; and so proceed till you have got back to the point of starting.

(253) Where speed is more important than accuracy in a survey, whether of a line or a farm, the compass need be set only at every other station, taking a forward sight, from the 1st station to the 2d; then setting the compass at the 3d station, taking a back-sight to the 2d station (but with the north point of the compass always ahead), and a fore-sight to the 4th; then going to the 5th, and so on. This is, however, not to be recommended.

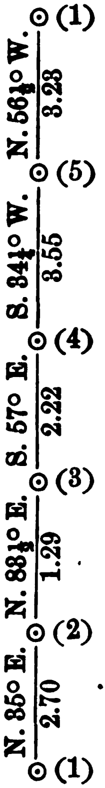
(254) Field-notes. The Field-notes of a Farm survey may be kept by any of the methods which have been described with reference to a Line survey. Below are given the Field-notes of the same field recorded by each of the methods.

First Method.

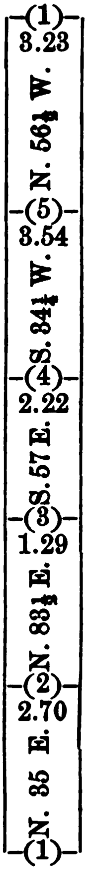
Fig. 175.



Second Method.



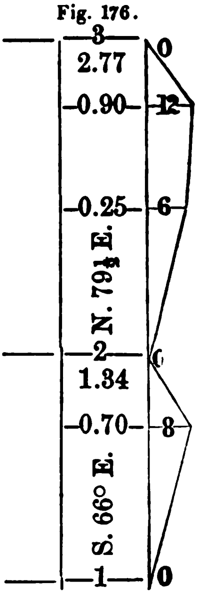
Third Method.*



Fourth Method.

STATIONS.	BEARINGS.	DISTANCES.
1	N. 35° E.	2.70
2	N. 83½° E.	1.29
3	S. 57° E.	2.22
4	S. 34¼° W.	3.55
5	N. 56½° W.	3.23

Fig. 176.



(255) The Field-notes of a field, in which offsets occur, may be most easily recorded by the Third Method; as in Fig. 176.

When the Field-notes are recorded by the Fourth Method, the offsets may be kept in a separate Table; in which the 1st column will contain the stations from which the measurements are made, the 2d column the distances at which they occur, the 3d

* In the "Third Method," the bearings should be written obliquely upward, as directed in Art. (249), but are not so printed here, from typographical difficulties.

column the lengths of the offsets, and the 4th column the side of the line, "Right," or "Left," on which they lie.

For calculation, four more columns may be added to the table, containing the intervals between the offsets; the sums of the adjoining pairs; and the products of the numbers in the two preceding columns, separated into Right and Left, one being additive to the field, and the other subtractive.

(256) Tests of accuracy. 1st. The check of intersections described in Art. (246), may be employed to great advantage, when some conspicuous object near the centre of the farm can be seen from most of its corners.

2nd. When the survey is platted, if the last course meets the starting point, it proves the work, and the survey is then said to "close."

3d. Diagonal lines, running from corner to corner of the farm, like the "Proof-lines" in Chain Surveying, may be measured and their bearings taken. When these are laid down on the plat, their meeting the points to which they had been measured, proves the work.

4th. The only certain and precise test is, however, that by "Latitudes and Departures." This is fully explained in Chapter V, of this Part.

(257) A very fallacious test is recommended by several writers on this subject. It is a well-known proposition of Geometry, that in any figure bounded by straight lines, the sum of all the interior angles is equal to twice as many right angles, as the figure has sides less two; since the figure can be divided into that number of triangles. Hence this common rule. "Calculate [by the last paragraph of Art. (243)] the *interior* angles of the field or farm surveyed; add them together, and if their sum equals twice as many right angles as the figure has sides less two, the angles have been correctly measured." This rule is *not* applicable to a compass survey; for, in Fig. 167, page 144, the interior angle BCD will contain the same number of degrees (in that case 160°) whether the bearings of the sides have been noted correctly, as being the angles which they make with NS—or incorrectly, as being the angles which they make with N′S′. This rule would therefore prove the work in either case.

(258) Method of Radiation. *A field may be surveyed from one station, either within it or without it,* by taking the bearings and the distances from that point to each of the corners of the field. These corners are then "determined," by the 3d method, Art. (7). This modification of that method, we named, in Art. (220), the *Method of Radiation.* All our preceding surveys with the compass have been by the *Method of Progression.*

The compass may be set at one corner of the field, or at a point in one of its sides, and the same method of Radiation employed.

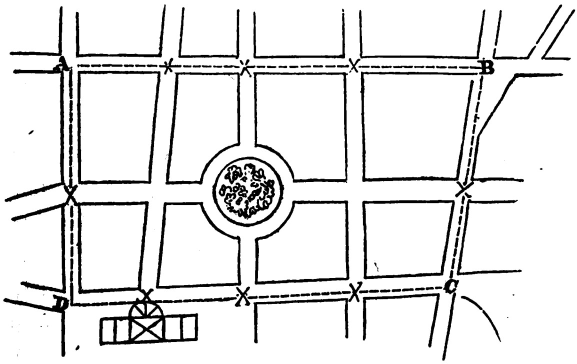
This method is seldom used however, since, unlike the method of Progression, its operations are not checks upon each other.

(259) Method of Intersection. A field may also be surveyed by measuring a *base line*, either within it or without it, setting the compass at each end of the base line, and taking, from each end, the bearings of each corner of the field; which will then be fixed and determined, by the 4th method, Art. (8). This mode of surveying is the *Method of Intersections*, noticed in Art. (220). It will be fully treated of in Part V, under the title of **Triangular Surveying.**

(260) Running out old lines. The original surveys of lands in the older States of the American Union, were exceedingly deficient in precision. This arose from two principal causes; the small value of land at the period of these surveys, and the want of skill in the surveyors. The effect at the present day is frequent dissatisfaction and litigation. Lots sometimes contain more acres than they were sold for, and sometimes less. Lines which are straight in the deed, and on the map, are found to be crooked on the ground. The recorded surveys of two adjoining farms often make one overlap the other, or leave a gore between them. The most difficult and delicate duty of the land-surveyor, is to run out these old boundary lines. In such cases, his first business is to find monuments, stones, marked trees, stumps, or any other old "corners," or landmarks. These are his starting points. The owners whose lands join at these corners should agree on them. Old fences must generally be accepted by right of possession; though such questions belong rather to the lawyer than to the surveyor.* His business is to mark out on the ground the lines given in the deed. When the bounds are given by compass-bearings, the surveyor must be reminded that these bearings are very far from being the same now as originally, having been changing every year. The method of determining this important change, and of making the proper allowance, will be found in Chapter VIII, of this Part.

(261) Town Surveying. Begin at the meeting of two or more of the principal streets, through which you can have the longest prospects. Having fixed the instrument at that point, and taken the bearings of all the streets issuing from it, measure all these lines with the chain, taking offsets to all the corners of streets, lanes, bendings, or windings; and to all remarkable objects, as churches, markets, public buildings, &c. Then remove the instrument to the next street, take its bearings, and measure along the street as before, taking offsets as you go along, with the offset-staff. Proceed in this manner from street to street, measuring the distances and offsets as you proceed.

Fig. 177.



* "In the description of land conveyed, the rule is, that known and fixed monuments control courses and distances. So, the certainty of metes and bounds will include and pass all the lands within them, though they vary from the given quantity expressed in the deed. In New-York, to remove, deface or alter landmarks maliciously, is an indictable offence."—*Kent's Commentaries*, IV, 515.

Thus, in the figure, fix the instrument at A, and measure lines in the direction of all the streets meeting there, noting their bearings; then measure AB, noting the streets at X, X. At the second station, B, take the bearings of all the streets which meet there; and measure from B to C, noting the places and the bearings of all the cross-streets as you pass them. Proceed in like manner from C to D, and from D to A, "closing" there, as in a farm survey. Having thus surveyed all the principal streets in a particular neighborhood, proceed then to survey the smaller intermediate streets, and last of all, the lanes, alleys, courts, yards, and every other place which it may be thought proper to represent in the plan. The several cross-streets answer as good check lines, to prove the accuracy of the work. In this manner you continue till you take in all the town or city.

(262) Obstacles in Compass Surveying. The various obstacles which may be met with in Compass Surveying, such as woods, water, houses, &c., can be overcome much more easily than in Chain Surveying. But as some of the best methods for effecting this involve principles which have not yet been fully developed, it will be better to postpone giving any of them, till they can be all treated of together; which will be done in Part VII.

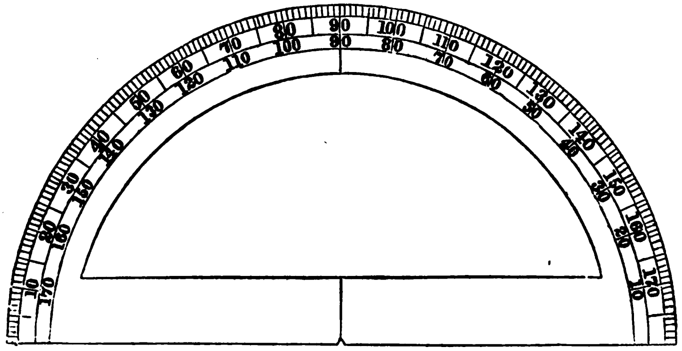
CHAPTER IV.

PLATTING THE SURVEY.

(263) The platting of a survey made with the compass, consists in drawing on paper the lines and the angles which have been measured on the ground. The lines are drawn "to scale," as has been fully explained in Part I, Chapter III. The manner of platting angles was referred to in Art. **(41)**, but its explanation has been reserved for this place.

(264) With a Protractor. A Protractor is an instrument made for this object, and is usually a semicircle of brass, as in the figure, with its semi-circumference divided into 180 equal parts, or

Fig. 178.

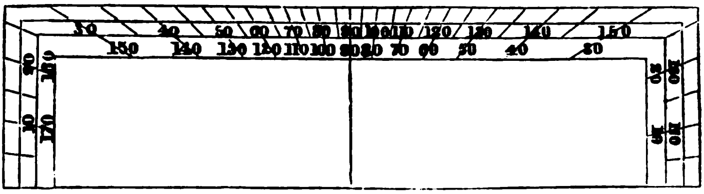


degrees, and numbered in both directions. It is, in fact, a miniature of the instrument, (or of half of it), with which the angles have been measured. To lay off any angle at any point of a straight line, place the Protractor so that its straight side, the diameter of the semi-circle, is on the given line, and the middle of this diameter, which is marked by a notch, is at the given point. With a needle, or sharp pencil, make a mark on the paper at the required number of degrees, and draw a line from the mark to the given point.

Sometimes the protractor has an arm turning on its centre, and extending beyond its circumference, so that a line can be at once drawn by it when it is set to the desired angle. A Vernier scale is sometimes added to it to increase its precision.

A Rectangular Protractor is sometimes used, the divisions of degrees being engraved along three edges of a plane scale. The semi-circular one is preferable. The objection to the rectangular protractor is that the division corresponding to a degree is very

Fig. 179.



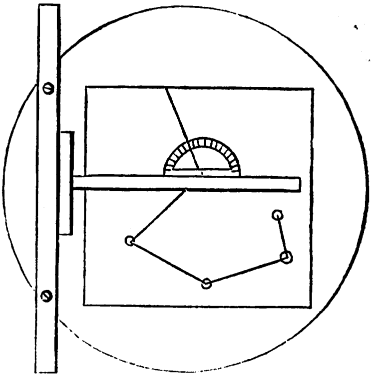
unequal on different parts of the scale, being usually two or three times as great at its ends as at its middle.

A Protractor embracing an entire circle, with arms carrying verniers, is also sometimes employed, for the sake of greater accuracy.

(265) Platting Bearings. Since "Bearings" taken with the Compass are the angles which the various lines make with the Magnetic Meridian, or the direction of the compass-needle, which, as we have seen, remains always (approximately) parallel to itself, it is necessary to draw these meridians through each station, before laying off the angles of the bearings.

The T square, shown in Fig. 14, is the most convenient instrument for this purpose. The paper on which the plat is to be made is fastened on the board so that the intended direction of the North and South line may be parallel to one of the sides of the board. The inner side of the stock of the T square being pressed against one of the other sides of the board and slid along, the edge of the long blade of the square will always be parallel to itself and to the first named side of the board, and will thus represent the meridian passing through any station.

Fig. 180.



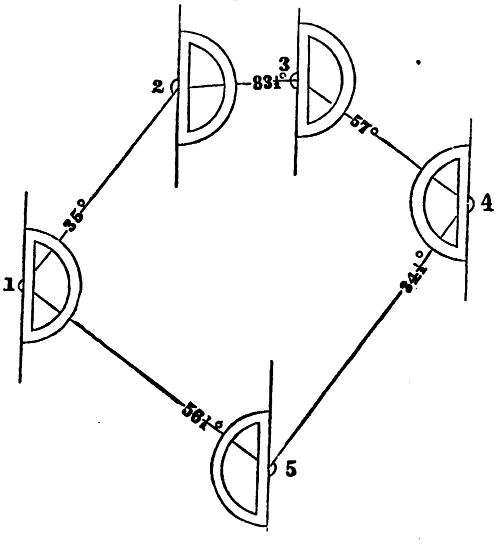
If a straight-edged drawing board or table cannot be procured, nail down on a table of any shape a straight-edged ruler, and slide along against it the outside of the stock of a T square, one side of the stock being flush with the blade.

A parallel ruler may also be used, one part of it being screwed down to the board in the proper position.

If none of these means are at hand, approximately parallel meridians may be drawn by the edges of a common ruler, at distances apart equal to its width, and the diameter of the protractor made parallel to them by measuring equal distances between it and them.

(266) To plat a survey with these instruments, mark, with a fine point enclosed in a circle, a convenient spot in the paper to represent the first station, 1 in the figure. Its place must be so chosen

Fig. 181.



that the plat may not "run off" the paper. With the T square draw a meridian through it. The top of the paper is usually, though not necessarily, called North. With the protractor lay off the angle of the first bearing, as directed in Art. (264). Set off the length of the first line, to the desired scale, by Art. (42), from 1 to 2. The line 1----2 represents the first course.

Through 2, draw another meridian, lay off the angle of the second course, and set off the length of this course, from 2 to 3.

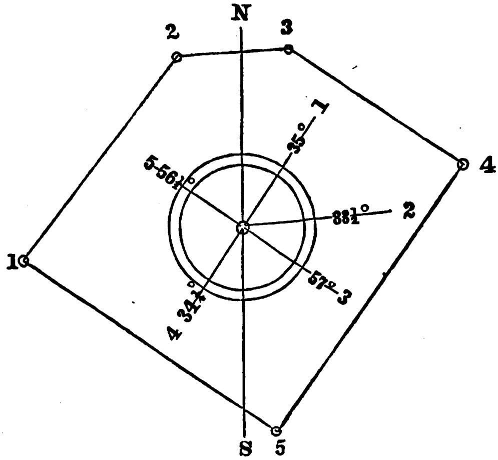
Proceed in like manner for each course. When the last course is platted, it should end precisely at the starting point, as the survey did, if it were a closed survey, as of a field. If the plat does not "close," or "come together," it shows some error or inaccuracy either in the original survey, if that have not been "tested" by Latitudes and Departures, or in the work of platting. A method of correction is explained in Art. (268). The plat here given is the same as that of Fig. 175, page 151.

This manner of laying down the directions of lines, by the angles which they make with a meridian line, has a great advantage, in both accuracy and rapidity, over the method of platting lines by the angles which each makes with the line which comes before it. In the latter method, any error in the direction of one line makes all that follow it also wrong in their directions. In the former, the *direction* of each line is independent of the preceding line, though its *position* would be changed by a previous error.

Instead of drawing a meridian through each station, sometimes only one is drawn, near the middle of the sheet, and all the bearings of the survey are laid off from some one point of it, as shown in the figure, and numbered to correspond with the stations from which these bearings were taken. The circular protractor is convenient for this. They are then transferred to the places where they are wanted, by a triangle or other parallel ruler, as explained on page 27. The figure at the top of the next page represents the same field platted by this method.

A semi-circular protractor is sometimes attached to the stock end of the T square, so that its blade may be set at any desired angle with the meridian, and any bearing be thus protracted without drawing a meridian. It has some inconveniences.

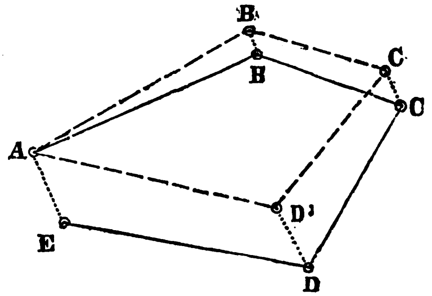
Fig. 182.



(267) The Compass itself may be used to plat bearings. For this purpose it must be attached to a square board so that the N and S line of the compass box may be parallel to two opposite edges of the board. This is placed on the paper, and the box is turned till the needle points as it did when the first bearing was taken. Then a line drawn by one edge of the board will be in a proper direction. Mark off its length, and plat the next and the succeeding bearings in the same manner.

(268) When the plat of a survey does not "close," it may be corrected as follows. Let ABCDE be the boundary lines platted according to the given bearings and distances, and suppose that the last course comes to E, instead of ending at A, as it should. Suppose also that there is no reason to suspect any single great error, and that no one of the lines was measured over very rough

Fig. 183.

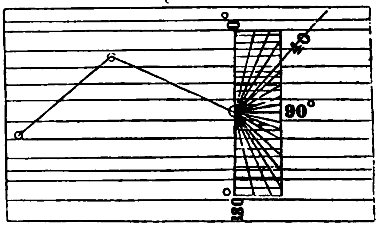


ground, or was specially uncertain in its direction when observed. The inaccuracy must then be distributed among all the lines *in proportion to their length.* Each point in the figure, B, C, D, E, must be moved in a direction parallel to EA, by a certain distance which is obtained thus. Multiply the distance EA by the distance AB, and divide by the sum of all the courses. The quotient will be the distance BB′. To get CC′, multiply EA by AB + BC, and divide the product by the same sum of all the courses. To get DD′, multiply EA by AB + BC + CD, and divide as before. So for any course, multiply by the sum of the lengths of that course and of all those preceding it, and divide as before. Join the points thus obtained, and the closed polygon AB′C′D′A will thus be formed, and will be the most *probable* plat of the given survey.*

The method of Latitudes and Departures, to be explained hereafter, is, however, the best for effecting this object.

(269) Field Platting. It is sometimes desirable to plat the courses of a survey in the field, as soon as they are taken, as was mentioned in Art. (247), under the head of "Keeping the field-notes." One method of doing this is, is to have the paper of the Field-book ruled with parallel lines, at *unequal* distances apart, and to use a rectangular protractor (which may be made of Bristol-board, or other stout drawing paper,) with lines ruled across it at equal distances of some fraction of an inch. A bearing having been taken and noted, the protractor is laid on the paper and its centre placed at the station where the bearing is to be laid off. It is then turned till one of its cross-lines coincides with some one of the lines on the paper, which represent East and West lines. The long side of the protractor will then be on a meridian and the proper angle (40° in the figure) can be at once marked off. The length of the course can also be set off by the equal spaces between the cross-lines, letting each space represent any convenient number of links.

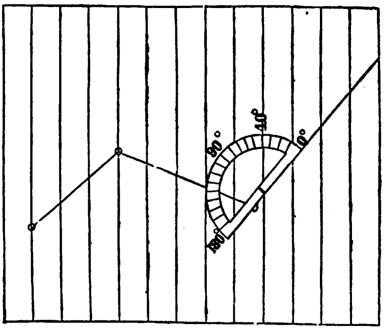
Fig. 184.



* This was demonstrated by Dr. Bowditch, in No. 4, of "The Analyst."

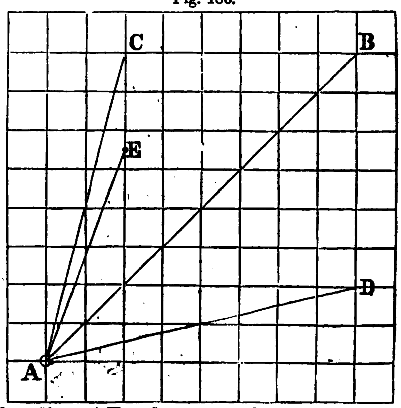
(270) A common rectangular protractor without any cross-lines, or a semi-circular one, can also be used for the same purpose. The parallel lines on the paper (which, in this method, may be equi-distant, as in common ruled writing paper) will now represent meridians. Place the centre of the protractor on the meridian nearest to the station at which the angle is to be laid off, and turn it till the given number of degrees is cut by the meridian. Slide the protractor up or down the meridian (which must continue to pass through the centre and the proper degree) till its edge passes through the station, and then draw by this edge a line, which will have the bearing required.

Fig. 185.



(271) *Paper ruled into squares*, (as are sometimes the right-hand pages of surveyors' field-books), may be used for platting bearings in the field. The lines running up the page may be called North and South lines, and those running across the page will then be East and West lines. Any course of the survey will be the hypothenuse of a right-angled triangle, and the ratio of its other two sides will determine the angle. Thus, if the ratio of the two sides of the right-angled triangle, of which the line AB in the figure is the hypothenuse, is 1, that line makes an angle of 45° with the meridian. If the ratio of the long to the short side of the right-angled triangle of which the line AC is the hypothenuse, is 4 to 1, the line AC makes an angle of 14° with the meridian. The line AD, the hypothenuse of an

Fig. 186.



equal triangle, which has its long side lying East and West, makes likewise an angle of 14° with that side, and therefore makes an angle of 76° with the meridian.*

To facilitate the use of this method, the following table has been prepared.

TABLE FOR PLATTING BY SQUARES.

Angle opposite short side	Ratio of long side to short side.	Angle opposite long side.	Angle opposite short side.	Ratio of long side to short side.	Angle opposite long side.	Angle opposite short side.	Ratio of long side to short side.	Angle opposite long side.
1°	57.3 to 1	89°	16°	3.49 to 1	74°	31°	1.664 to 1	59°
2°	28.6 to 1	88°	17°	3.27 to 1	73°	32°	1.600 to 1	58°
3°	19.1 to 1	87°	18°	3.08 to 1	72°	33°	1.540 to 1	57°
4°	14.3 to 1	86°	19°	2.90 to 1	71°	34°	1.483 to 1	56°
5°	11.4 to 1	85°	20°	2.75 to 1	70°	35°	1.428 to 1	55°
6°	9.5 to 1	84°	21°	2.61 to 1	69°	36°	1.376 to 1	54°
7°	8.1 to 1	83°	22°	2.48 to 1	68°	37°	1.327 to 1	53°
8°	7.1 to 1	82°	23°	2.36 to 1	67°	38°	1.280 to 1	52°
9°	6.3 to 1	81°	24°	2.25 to 1	66°	39°	1.235 to 1	51°
10°	5.7 to 1	80°	25°	2.14 to 1	65°	40°	1.192 to 1	50°
11°	5.1 to 1	79°	26°	2.05 to 1	64°	41°	1.150 to 1	49°
12°	4.9 to 1	78°	27°	1.96 to 1	63°	42°	1.111 to 1	48°
13°	4.3 to 1	77°	28°	1.88 to 1	62°	43°	1.072 to 1	47°
14°	4.0 to 1	76°	29°	1.80 to 1	61°	44°	1.036 to 1	46°
15°	3.7 to 1	75°	30°	1.73 to 1	60°	45°	1.000 to 1	45°

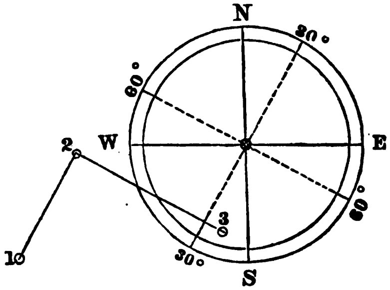
To use this table, find in it the ratio corresponding to the angle which you wish to plat. Then count, on the ruled paper, any number of squares to the right or to the left of the point which represents the station, according as your bearing was East or West; and count upward or downward according as your bearing was North or South, the number of squares given by multiplying the first number by the ratio of the Table. Thus; if the given bearing from A in the figure, was N. 20° E. and two squares were counted to the right, then $2 \times 2.75 = 5\frac{1}{2}$ squares, should be counted upward, to E, and AE would be the required course.

(272) With a paper protractor. Engraved paper protractors may be obtained from the instrument-makers, and are very conve-

* This and all the following ratios may be obtained directly from Trigonometrical Tables; for the ratio of the long side to the short side, the latter being taken as unity, is the natural cotangent of the angle.

nient. A circle of large size, divided into degrees and quarters, is engraved on copper, and impressions from it are taken on drawing paper. The divisions are not numbered. Draw a straight line to represent a meridian, through the centre of the circle, in any convenient direction. Number the degrees from 0 to 90°, each way from the ends of this meridian, as on the compass-plate. The protractor is now ready for use. Choose a convenient point for the first station. Suppose the first bearing to be N. 30° E. The line passing through the centre of the circle and through the opposite points N. 30° E. and S. 30° W. has the bearing required. But it does not pass through the station 1. Transfer it thither by drawing through station 1 a line parallel to it, which will be the course required, its proper length being set off on it from 1 to 2. Now suppose the bearing from 2 to be S. 60° E. Draw through 2 a line parallel to the line passing through the centre of the circle and through the opposite points S. 60° E., and N. 60° W., and it will be the line desired. On it set off the proper length from 2 to 3, and so proceed.

Fig. 187.



When the plat is completed, the engraved sheet is laid on a clean one, and the stations "pricked through," and the points thus obtained on the clean sheet are connected by straight lines. The pencilled plat is then rubbed off from the engraved sheet, which can be used for a great number of plates.

If the central circle be cut out, the plat, if not too large, can be made directly on the paper where it is to remain.

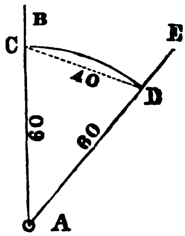
The surveyor can make such a paper protractor for himself, with great ease, by means of the *Table of Chords* at the end of this volume, the use of which is explained in Art. (275). The engraved ones may have shrunk after being printed.

Such a circle is sometimes drawn on the map itself. This will be particularly convenient if the bearings of any lines on the map, not taken on the ground, are likely to be required. If the map be very long, more than one may be needed.

(273) Drawing-Board Protractor. Such a divided circle, as has just been described, or a circular protractor, may be placed on a drawing board near its centre, and so that its 0° and 90° lines are parallel to the sides of the drawing board. Lines are then to be drawn, through the centre and opposite divisions, by a ruler long enough to reach the edges of the drawing board, on which they are to be cut in, and numbered. The drawing board thus becomes, in fact, a double rectangular protractor. A strip of white paper may have previously been pasted on the edges, or a narrow strip of white wood inlaid. When this is to be used for platting, a sheet of paper is put on the board as usual, and lines are drawn by a ruler laid across the 0° points and the 90° points, and the centre of the circle is at once found, and should be marked ⊙. The bearings are then platted as in the last method.

(274) With a scale of chords. On the plane scale contained in cases of mathematical drawing instruments will be found a series of divisions numbered from 0 to 90, and marked CH, or C. This is a scale of chords, and gives the lengths of the chords of any arc for a radius equal in length to the chord of 60° on the scale. To lay off an angle with this scale, as for example, to draw a line making at A an angle of 40° with AB, take, in the dividers, the distances from 0 to 60 on the scale of chords; with this for radius and A for centre, describe an indefinite arc CD. Take the distance from 0 to 40 on the same scale, and set it off on the arc as a chord, from C to some point D. Join AD, and prolong it. BAE is the angle required.

Fig. 188.



The Sector, represented on page 36, supplies a modification of this method, sometimes more convenient. On each of its legs is a scale marked C, or CH. Open it at pleasure; extend the compass from 60 to 60, one on each leg, and with this radius describe an arc. Then extend the compasses from 40 to 40, and the distance will be the chord of 40° to that radius. It can be set off as above.

The smallness of the scale renders the method with a scale of chords practically deficient in exactness; but it serves to illustrate the next and *best* method.

(275) **With a Table of chords.** At the end of this volume will be found a Table of the lengths of the chords of arcs for every degree and minute of the quadrant, calculated for a radius equal to 1.

To use it, take in the compasses one inch, one foot, or any other convenient distance (the longer the better) divided into tenths and hundredths, by a diagonal scale, or otherwise. With this as radius describe an arc as in the last case. Find in the table of chords the length of the chord of the desired angle. Take it from the scale just used, to the nearest decimal part which the scale will give. Set it off as a chord, as in the last figure, and join the point thus obtained to the starting point. This gives the angle desired.

The superiority of this method to that which employs a protractor, is due to the greater precision with which a straight line can be divided than can a circle.

A slight modification of this method is to take in the compasses 10 equal parts of any convenient length, inches, half inches, quarter inches, or any other at hand, and with this radius describe an arc as before, and set off a chord 10 times as great as the one found in the Table, i. e. imagine the decimal point moved one place to the right.

If the radius be 100 or 1000 equal parts, imagine the decimal point moved two, or three, places to the right.

Whatever radius may be taken or given, the product of that radius into a chord of the Table, will give the chord for that radius.

This gives an easy and exact method of getting a right angle; by describing an arc with a radius of 1, and setting off a chord equal to 1.4142.

If the angle to be constructed is more than 90°, construct on the other side of the given point, upon the given line prolonged, an angle equal to what the given angle wants of 180°; i. e. its *Supplement*, in the language of Trigonometry.

This same Table gives the means of measuring any angle. With the angular point for a centre, and 1, or 10, for a radius, describe an arc. Measure the length of the chord of the arc between the legs of the angle, find this length in the Table, and the angle corresponding to it is the one desired.*

(276) With a Table of natural sines. In the absence of a Table of chords, heretofore rare, a table of natural sines, which can be found anywhere, may be used as a less convenient substitute. Since the chord of any angle equals twice the sine of half the angle, divide the given angle by two; find in the table the natural sine of this half angle; double it, and the product is the chord of the whole angle. This can then be used precisely as was the chord in the preceding article.

An ingenious modification of this method has been much used. Describe an arc from the given point as centre, as in the last two articles, but with a radius of 5 equal parts. Take, from a Table, the length of the natural sine of half the given angle to a radius of 10. Set off this length as a chord on the arc just described, and join the point thus obtained to the given point.†

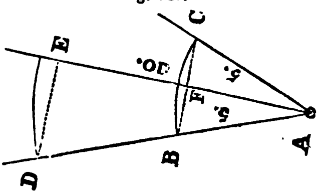
(277) By Latitudes and Departures. When the Latitudes and Departures of a survey have been obtained and corrected, (as explained in Chapter V), either to test its accuracy, or to obtain its content, they afford the easiest and best means of platting it. The description of this method will be given in Art. (285).

* This Table will also serve to find the *natural sine*, or *cosine*, of any angle. Multiply the given angle by two; find, in the Table, the chord of this double angle; and half of this chord will be the natural sine required. For, the chord of any angle is equal to twice the sine of half the angle. To find the *cosine*, proceed as above, with the angle which added to the given angle would make 90°.

Another use of this Table is to inscribe regular polygons in a circle by setting off the chords of the arcs which their sides subtend.

Still another use is to divide an arc or angle into any number of equal parts, by setting off the fractional arc or angle.

Fig. 189.



† The reason of this is apparent from the figure. DE is the sine of half the angle BAC, to a radius of 10 equal parts, and BC is the chord directed to be set off, to a radius of 5 equal parts. BC is equal to DE; for BC = 2.BF, by Trigonometry, and DE = 2.BF, by similar triangles; hence BC = DE.

CHAPTER V.

LATITUDES AND DEPARTURES.

(278) Definitions. The LATITUDE of a point is its distance North or South of some "*Parallel of Latitude*," or line running East or West. The LONGITUDE of a point is its distance East or West of some "*Meridian*," or line running North and South. In Compass-Surveying, the Magnetic Meridian, i. e. the direction in which the Magnetic Needle points, is the line from which the Longitudes of points are measured, or reckoned.

The distance which one end of a line is due North or South of the other end, is called the *Difference of Latitude* of the two ends of the line; or its *Northing* or *Southing;* or simply its *Latitude.*

The distance which one end of the line is due East or West of the other, is here called the *Difference of Longitude* of the two ends of the line; or its *Easting* or *Westing;* or its *Departure.*

Latitudes and *Departures* are the most usual terms, and will be generally used hereafter, for the sake of brevity.

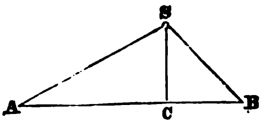
This subject may be illustrated geographically, by noticing that a traveller in going from New-York to Buffalo in a straight line, would go about 150 miles due north, and 250 miles due west. These distances would be the differences of Latitude and of Longitude between the two places, or his Northing and Westing. Returning from Buffalo to New-York, the same distances would be his Southing and Easting.*

In mathematical language, the operation of finding the Latitude and Longitude of a line from its Bearing and Length, would be called the transformation of Polar Co-ordinates into Rectangular Co-ordinates. It consists in determining, by our *Second Principle*, the position of a point which had originally been determined by the *Third Principle.* Thus, in the figure, (which is the same as

* It should be remembered that the following discussions of the Latitudes and Longitudes of the points of a survey will not always be fully applicable to those of distant places, such as the cities just named, in consequence of the surface of the earth not being a plane.

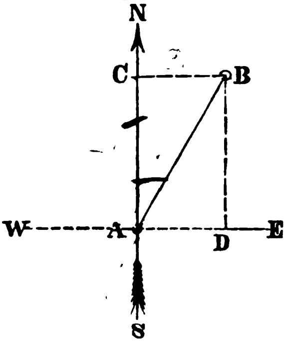
that of Art.(9)), the point S is determined by the angle SAC and by the distance AS. It is also determined by the distances AC and CS, measured at right angles to each other; and then, supposing CS to run due North and South, CS will be the *Latitude*, and AC the *Departure* of the line AS.

Fig. 190.



(279) Calculation of Latitudes and Departures. Let AB be a given line, of which the length AB, and the bearing (or angle, BAC, which it makes with the Magnetic Meridian), are known. It is required to find the differences of Latitude and of Longitude between its two extremities A and B: that is, to find AC and CB; or, what is the same thing, BD and DA.

Fig. 191.



It will be at once seen that AB is the hypothenuse of a right-angled triangle, in which the "Latitude" and the "Departure" are the sides about the right angle. We therefore know, from the principles of trigonometry, that

$$AC = AB \,.\, \cos. BAC,$$
$$BC = AB \,.\, \sin. BAC.$$

Hence, to find the *Latitude* of any course, multiply the natural cosine of the bearing by the length of the course; and to find the *Departure* of any course, multiply the natural sine of the bearing by the length of the course.

If the course be Northerly, the *Latitude* will be North, and will be marked with the algebraic sign +, *plus*, or additive; if it be Southerly, the Latitude will be South, and will be marked with the algebraic sign —, *minus*, or subtractive.

If the course be Easterly, the *Departure* will be East, and marked +, or additive; if the course be Westerly, the Departure will be West, and marked —, or subtractive.

(280) Formulas. The rules of the preceding article may be expressed thus;

$$\text{Latitude} = \text{Distance} \times \text{cos. Bearing},$$
$$\text{Departure} = \text{Distance} \times \text{sin. Bearing.*}$$

From these formulas may be obtained others, by which, when any two of the above four things are given, the remaining two can be found.

When the Bearing and Latitude are given;

$$\text{Distance} = \frac{\text{Latitude}}{\text{cos. Bearing}} = \text{Latitude} \times \text{ sec. of Bearing},$$
$$\text{Departure} = \text{Latitude} \times \text{tang. of Bearing.}$$

When the Bearing and Departure are given;

$$\text{Distance} = \frac{\text{Departure}}{\text{sin. Bearing}} = \text{Departure} \times \text{cosec. Bearing},$$
$$\text{Latitude} = \text{Departure} \times \text{cotang. Bearing.}$$

When the Distance and Latitude are given;

$$\text{Cos. Bearing} = \frac{\text{Latitude}}{\text{Distance}},$$
$$\text{Departure} = \text{Latitude} \times \text{tang. Bearing.}$$

When the Distance and Departure are given;

$$\text{Sin. Bearing} = \frac{\text{Departure}}{\text{Distance}},$$
$$\text{Latitude} = \text{Departure} \times \text{cotang. Bearing.}$$

When the Latitude and Departure are given;

$$\text{Tang. of Bearing} = \frac{\text{Departure}}{\text{Latitude}},$$
$$\text{Distance} = \text{Latitude} \times \text{sec. Bearing.}$$

Still more simply, any two of these three—Distance, Latitude and Departure—being given, we have

$$\text{Distance} = \sqrt{(\text{Latitude}^2 + \text{Departure}^2)}$$
$$\text{Latitude} = \sqrt{(\text{Distance}^2 - \text{Departure}^2)}$$
$$\text{Departure} = \sqrt{(\text{Distance}^2 - \text{Latitude}^2)}$$

(281) Traverse Tables. The Latitude and Departure of any distance, for any bearing, could be found by the method given in Art. (279), with the aid of a table of Natural Sines. But to

* Whenever sines, cosines, tangents, &c., are here named, they mean the natural sines, &c., of an arc described with a radius equal to *one*, or to the unit by which the sines, &c., are measured.

facilitate these calculations, which are of so frequent occurrence and of so great use, *Traverse Tables* have been prepared, originally for navigators, (whence the name *Traverse*), and subsequently for surveyors.*

The Traverse Table at the end of this volume gives the Latitude and Departure for any bearing, to each quarter of a degree, and for distances from 1 to 9.

To use it, find in it the number of degrees in the bearing, on the left hand side of the page, if it be less than 45°, or on the right hand side if it be more. The numbers on the same line running across the page,† are the Latitudes and Departures for that bearing, and for the respective distances—1, 2, 3, 4, 5, 6, 7, 8, 9,—which are at the top and bottom of the page, and which may represent chains, links, rods, feet, or any other unit. Thus, if the bearing be 15°, and the distance 1, the Latitude would be 0.966 and the Departure 0.259. For the same bearing, but a distance of 8, the Latitude would be 7.727, and the Departure 2.071.

Any distance, however great, can have its Latitude and Departure readily obtained from this table; since, for the same bearing, they are directly proportional to the distance, because of the similar triangles which they form. Therefore, to find the Latitude or Departure for 60, multiply that for 6 by 10, which merely moves the decimal point one place to the right; for 500, multiply the numbers found in the Table for 5, by 100, i. e. move the decimal point two places to the right, and so on. Merely moving the decimal point to the right, one, two, or more places, will therefore enable this Table to give the Latitude and Departure for any decimal multiple of the numbers in the Table.

For compound numbers, such as 873, it is only necessary to find separately the Latitudes and Departures of 800, of 70, and of 3, and add them together. But this may be done, with scarcely any risk of error, by the following simple rule.

* The first Traverse Table for Surveyors seems to have been published in 1791, by John Gale. The most extensive table is that of Capt. Boileau, of the British army, being calculated for every minute of bearing, and to five decimal places, for distances from 1 to 10. The Table in this volume was calculated for it, and then compared with the one just mentioned.

† In using this or any similar Table, lay a ruler across the page, just above or below the line to be followed out. This is a very valuable mechanical assistance.

Write down the Latitude and Departure for the first figure of the given number, as found in the Table, neglecting the decimal point; write under them the Latitude and Departure of the second figure, setting them one place farther to the right; under them write the Latitude and Departure of the third figure, setting them one place farther to the right, and so proceed with all the figures of the given number. Add up these Latitudes and Departures, and cut off the three right hand figures. The remaining figures will be the Latitude and Departure of the given number in links, or chains, or feet, or whatever unit it was given in.

For example; let the Latitude and Departure of a course having a distance of 873 links, and a bearing of 20°, be required. In the Table find 20°, and then take out the Latitude and Departure for 8, 7 and 3, in turn, placing them as above directed, thus:

Distances.	*Latitudes.*	*Departures.*
800	7518	2736
70	6578	2394
3	2819	1026
873	820.399	298.566

Taking the nearest whole numbers and rejecting the decimals, we find the desired Latitude and Departure to be 820 and 299.*

When a 0 occurs in the given number, the next figure must be set *two* places to the right, the reason of which will appear from the following example, in which the 0 is treated like any other number.

Given a bearing of 35°, and a distance of 3048 links.

Distances.	*Latitudes.*	*Departures.*
3000	2457	1721
000	0000	0000
40	3277	2294
8	6553	4589
3048	2496.323	1748.529

Here the Latitudes and Departures are 2496 and 1749 links.

* It is frequently doubtful, in many calculations, when the final decimal is 5, whether to increase the preceding figure by one or not. Thus, 43.5 may be called 43 or 44 with equal correctness. It is better in such cases not to increase the whole number, so as to escape the trouble of changing the original figure, and the increased chance of error. If, however, more than one such a case occurs in the same column to be added up, the larger and smaller number should be taken alternately.

When the bearing is over 45°, the names of the columns must be read from the bottom of the page, the Latitude of any bearing, as 50°, being the Departure of the complement of this bearing, or 40°, and the Departure of 40° being the Latitude of 50°, &c. The reason of this will be at once seen on inspecting the last figure, (page 170), and imagining the East and West line to become a Meridian. For, if AC be the magnetic meridian, as before, and therefore BAC be the bearing of the course AB, then is AC the Latitude, and CB the Departure of that course. But if AE be the meridian and BAD (the complement of BAC) be the bearing, then is AD (which is equal to CB) the Latitude, and DB, (which is equal to AC), the Departure.

As an example of this, let the bearing be $63\frac{1}{4}$°, and the distance 3469 links. Proceeding as before, we have

Distances.	*Latitudes.*	*Departures.*
3000	1350	2679
400	1800	3572
60	2701	5358
9	4051	8037
3469.	1561.061	3097.817

The required Latitude and Departure are 1561 and 3098 links.

In the few cases occurring in Compass-Surveying, in which the bearing is recorded as somewhere between the fractions of a degree given in the Table, its Latitude and Departure may be found by interpolation. Thus, if the bearing be $10\frac{3}{8}$°, take the half sum of the Latitudes and Departures for $10\frac{1}{4}$° and $10\frac{1}{2}$°. If it be 10° 20′, add one-third of the difference between the Lats. and Deps. for $10\frac{1}{4}$ and for $10\frac{1}{2}$°, to those opposite to $10\frac{1}{4}$°; and so in any similar case.

The uses of this table are very varied. The principal applications of it, which will now be explained, are to *Testing the accuracy of surveys;* to *Supplying omissions in them*; to *Platting them*, and to *Calculating their content.**

* The Traverse Table admits of many other minor uses. Thus, it may be used for solving, approximately, any right-angled triangle by mere inspection, the bearing being taken for one of the acute angles; the Latitude being the side adjacent, the Departure the side opposite, and the Distance the hypothenuse. Any two of these being given, the others are given by the Table. The Table will therefore serve to show the allowance to be made in chaining on slopes (see Art.

(282) Application to Testing a Survey. It is self-evident, that when the surveyor has gone completely around a field or farm, taking the bearings and distances of each boundary line, till he has got back to the starting point, that he has gone precisely as far South as North, and as far West as East. But the sum of the North Latitudes tells how far North he has gone, and the sum of the South Latitudes how far South he has gone. Hence these two sums will be equal to each other, if the survey has been correctly made. In like manner, the sums of the East and of the West Departures must also be equal to each other.

We will apply this principle to testing the accuracy of the survey of which Fig. 175, page 151, is a plat. Prepare seven columns, and head them as below. Find the Latitude and Departure of each course to the nearest link, and write them in their appropriate columns. Add up these columns. Then will the difference between the sums of the North and South Latitudes, and between the sums of the East and West Departures, indicate the degree of accuracy of the survey.

STATION.	BEARING.	DISTANCE.	LATITUDE.		DEPARTURE.	
			N.	S.	E.	W.
1	N. 35° E.	2.70	2.21		1.55	
2	N. $83\frac{1}{2}$° E.	1.29	.15		1.28	
3	S. 57° E.	2.22		1.21	1.86	
4	S. $34\frac{1}{4}$° W.	3.55		2.93		2.00
5	N. $56\frac{1}{2}$° W.	3.23	1.78			2.69
			4.14	4.14	4.69	4.69

The entire work of the above example is given below.

35°	1638	1147	$34\frac{1}{4}$°	2480	1688
	57340	40150	.	4133	2814
				4133	2814
270.	221.140	154.850			
			355.	293.463	199.754

(26)); for, look in the column of bearings for the slope of the ground, i. e. the angle it makes with the horizon, find the given distance, and the Latitude corresponding will be the desired horizontal measurement, and the difference between it and the Distance will be the allowance to be made.

83½°	113	994	56½°	1656	2502
	226	1987		1104	1668
	1019	8942		1656	2502
129.	14.579	128.212	323.	178.296	269.382

57°	1089	1677
	1089	1677
	1089	1677
222.	120.879	186.147

The nearest link is taken to be inserted in the Table, and the remaining Decimals are neglected.

In the preceding example the respective sums were found to be exactly equal. This, however, will rarely occur in an extensive survey. If the difference be great, it indicates some mistake, and the survey must be repeated with greater care; but if the difference be small it indicates, not absolute errors, but only inaccuracies, unavoidable in surveys with the compass, and the survey may be accepted.

How great a difference in the sums of the columns may be allowed, as not necessitating a new survey, is a dubious point. Some surveyors would admit a difference of 1 link for every 3 chains in the sum of the courses: others only 1 link for every 10 chains. One writer puts the limit at 5 links for each station; another at 25 links in a survey of 100 acres. But every practical surveyor soon learns how near to an equality his instrument and his skill will enable him to come in ordinary cases, and can therefore establish a standard for himself, by which he can judge whether the difference, in any survey of his own, is probably the result of an error, or only of his customary degree of inaccuracy, two things to be very carefully distinguished.*

(283) Application to supplying omissions. Any two omissions in the Field-notes can be supplied by a proper use of the method of Latitudes and Departures; as will be explained in Part VII, which treats of "Obstacles to Measurement," under which head this subject most appropriately belongs. But a knowledge of the fact that any two omissions can be supplied, should not lead

* A French writer fixes the allowable difference in chaining at 1-400 of level lines; 1-200 of lines on moderate slopes; 1-100 of lines on steep slopes.

the young surveyor to be negligent in making every possible measurement, since an omission renders it necessary to assume all the notes taken to be correct, the means of testing them no longer existing.

(284) Balancing a Survey. The subsequent applications of this method require the survey to be previously *Balanced.* This operation consists in *correcting* the Latitudes and Departures of the courses, so that their sums *shall* be equal, and thus "balance." This is usually done by distributing the differences of the sums among the courses in proportion to their length; saying, *As* the sum of the lengths of all the courses *Is to* the whole difference of the Latitudes, *So is* the length of each course *To* the correction of its Latitude. A similar proportion corrects the Departures.*

It is not often necessary to make the exact proportion, as the correction can usually be made, with sufficient accuracy, by noting how much per chain it should be, and correcting accordingly.

In the example given below, the differences have purposely been made considerable. The corrected Latitudes and Departures have been here inserted in four additional columns, but in practice they should be written *in red ink* over the original Latitudes and Departures, and the latter crossed out with red ink.

STA.	BEARING.	DIST.	LATITUDES.		DEP'TURES.		CORRECTED LATITUDES		CORRECTED DEP'TURES.	
			N.+	S.—	E.+	W.—	N.+	S.—	E.+	W.—
1	N. 52 E.	10.63	6.55		8.38		6.58		8.33	
2	S. 29¾ E.	4.10		3.55	2.02			3.54	2.00	
3	S. 31¾ W.	7.69		6.53		4.04		6.51		4.07
4	N. 61 W.	7.13	3.45			6.23	3.47			6.26
		29.65	10.00	10.08	10.40	10.27	10.05	10.05	10.33	10.33

The corrections are made by the following proportions.

For the Latitudes.	*For the Departures.*
29.65 : 10.63 :: 8 : 3	29.65 : 10.63 :: 13 : 5
29.65 : 4.10 :: 8 : 1	29.65 : 4.10 :: 13 : 2
29.65 : 7.69 :: 8 : 2	29.65 : 7.69 :: 13 : 3
29.65 : 7.13 :: 8 : 2	29.65 : 7.13 :: 13 : 3

* A demonstration of this principle was given by Dr. Bowditch, in No. 4. of "The Analyst."

This rule is not always to be strictly followed. If one line of a survey has been measured over very uneven and rough ground, or if its bearing has been taken with an indistinct sight, while the other lines have been measured over level and clear ground, it is probable that most of the error has occurred on that line, and the correction should be chiefly made on its Latitude and Departure.

If a slight change of the bearing of a long course will favor the Balancing, it should be so changed, since the compass is much more subject to error than the chain. So, too, if shortening any doubtful line will favor the Balancing, it should be done, since distances are generally measured too long.

(285) Application to Platting. Rule three columns; one for Stations; the next for total Latitudes; and the third for total Departures. Fill the last two columns by beginning at any convenient station (the extreme East or West is best) and adding up (algebraically) the Latitudes of the following stations, noticing that the South Latitudes are subtractive. Do the same for the Departures, observing that the Westerly ones are also subtractive.

Taking the example given on page 175, Art. (282), and beginning with Station 1, the following will be the results:

STA.	TOTAL LATITUDES FROM STATION 1.	TOTAL DEPARTURES FROM STATION 1.
1	0.00	0.00
2	+2.21 N.	+1.55 E.
3	+2.36 N.	+2.83 E.
4	+1.15 N.	+4.69 E.
5	—1.78 S.	+2.69 E.
1	0.00	0.00

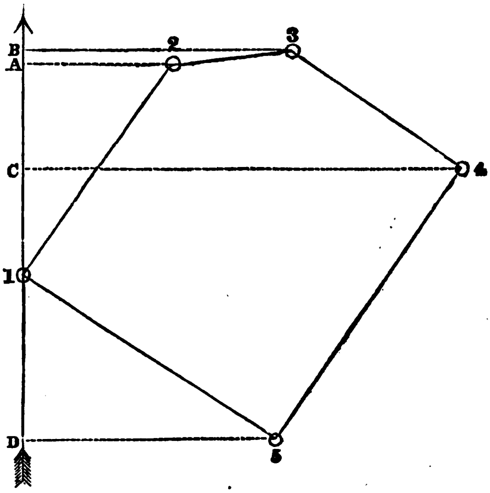
It will be seen that the work proves itself, by the total Latitudes and Departures for Station 1, again coming out equal to zero.

To use this table, draw a meridian through the point taken for Station 1, as in the figure on the following page. Set off, upward from this, along the meridian, the Latitude, 221 links, to A, and from A, to the right perpendicularly, set off the Departure, 155 links.* This gives the point 2. Join 1....2. From 1 again, set

* This is most easily done with the aid of a right-angled triangle, sliding one of the sides adjacent to the right angle along the blade of the square, to which the other side will then be perpendicular.

off, upward, 236 links, to B, and from B, to the right, perpendicularly, set off 283 links, which will fix the point 3. Join 2....3; and so proceed, setting off North Latitudes along the Meridian upwards, and South Latitudes along it downwards; East Departures perpendicularly to the right, and West Departures perpendicularly to the left.

Fig. 192.



The advantages of this method are its rapidity, ease and accuracy; the impossibility of any error in platting any one course affecting the following points; and the certainty of the plat "coming together," if the Latitudes and Departures have been "Balanced."

CHAPTER VI.

CALCULATING THE CONTENT.

(286) Methods. When a field has been platted, by whatever method it may have been surveyed, its content can be obtained from its plat by dividing it up into triangles, and measuring on the plat their bases and perpendiculars; or by any of the other means explained in Part I, Chapter IV.

But these are only approximate methods; their degree of accuracy depending on the largeness of scale of the plat, and the skill of the draftsman. The invaluable method of Latitudes and Departures gives another means, perfectly accurate, and not requiring the previous preparation of a plat. It is sometimes called the Rectangular, or the Pennsylvania, or Rittenhouse's, method of calculation.*

(287) Definitions. Imagine a Meridian line to pass through the extreme East or West corner of a field. According to the definitions established in Chapter V, Art. (278), (and here recapitulated for convenience of reference), the perpendicular distance of each Station from that Meridian, is the *Longitude* of that Station; additive, or *plus*, if East; subtractive, or *minus*, if West. The distance of the middle of any line, such as a side of the field, from the Meridian, is called the *Longitude* of that side.† The difference of the Longitudes of the two ends of a line is called the *Departure* of that line. The difference of the Latitudes of the two ends of a line is called the *Latitude* of the line.

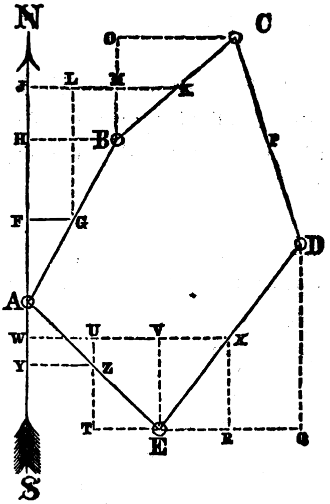
* It is, however, substantially the same as Mr. Thomas Burgh's "Method to determine the areas of right lined figures universally," published nearly a century ago.

† The phrase "Meridian Distance," is generally used for what is here called "Longitude"; but the analogy of "Differences of Longitude" with "Differences of Latitude," usually but anomalously united with the word "Departure," borrowed from Navigation, seems to put beyond all question the propriety of the innovation here introduced.

(288) **Longitudes.** To give more definiteness to the develop ment of this subject, the figure in the margin will be referred to, and may be considered to represent any space enclosed by straight lines.

Let NS be the Meridian passing through the extreme Westerly Station of the field ABCDE. From the middle and ends of each side draw perpendiculars to the Meridian. These perpendiculars will be the *Longitudes* and Departures of the respective sides. The Longitude, FG, of the *first course*, AB, is evidently equal to half its Departure HB. The Longitude, JK, of the second course, BC, is equal to JL + LM + MK, or equal to the Longitude of the preceding course, plus half its Departure, plus half the Departure of the course itself. The Longitude, YZ, of some other course, as EA, taken anywhere, is equal to WX — VX — UV, or equal to the Longitude of the preceding course, minus half its Departure, minus half the Departure of the course itself; i. e. equal to the *Algebraic* sum of these three parts, remembering that *Westerly* Departures are negative, and therefore to be subtracted when the directions are to make an *Algebraic* addition.

Fig. 193.



To avoid fractions, it will be better to double each of the preceding expressions. We shall then have a

GENERAL RULE FOR FINDING DOUBLE LONGITUDES.

The Double Longitude of the FIRST COURSE *is equal to its Departure.*

The Double Longitude of the SECOND COURSE *is equal to the Double Longitude of the first course, plus the Departure of that course, plus the Departure of the second course.*

The Double Longitude of the THIRD COURSE *is equal to the Double Longitude of the second course, plus the Departure of that course, plus the Departure of the course itself.*

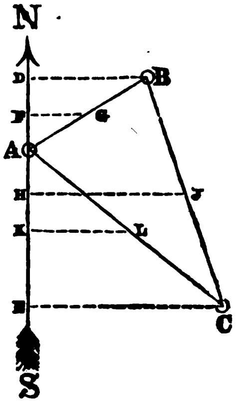
The Double Longitude of ANY *course is equal to the Double Longitude of the preceding course, plus the Departure of that course, plus the Departure of the course itself.**

The Double Longitude of the *last* course (as well as of the first) is equal to its Departure. Its "coming out" so, when obtained by the above rule, proves the accuracy of the calculation of all the preceding Double Longitudes.

(289) **Areas.** We will now proceed to find the Area, or Content of a field, by means of the "Double Longitudes" of its sides, which can be readily obtained by the preceding rule, whatever their number.

(290) Beginning with a *three-sided field*, ABC in the figure, draw a Meridian through A, and draw perpendiculars to it as in the last figure. It is plain that its content is equal to the difference of the areas of the Trapezoid DBCE, and of the Triangles ABD and ACE.

Fig. 194.



The area of the Triangle ABD is equal to the product of AD by half of DB, or to the product of AD by FG; i. e. equal to the product of the Latitude of the 1st course by its Longitude.

The area of the Trapezoid DBCE is equal to the product of DE by half the sum of DB and CE, or by HJ; i. e. to the product of the Latitude of the 2d course by its Longitude.

The area of the Triangle ACE is equal to the product of AE by half EC, or by KL; i. e. to the product of the Latitude of the 3d course by its Longitude.

Calling the products in which the Latitude was North, *North Products*, and the products in which the Latitude was South, *South Products*, we shall find the area of the Trapezoid to be a *South Product*, and the areas of the Triangles to be *North Pro-*

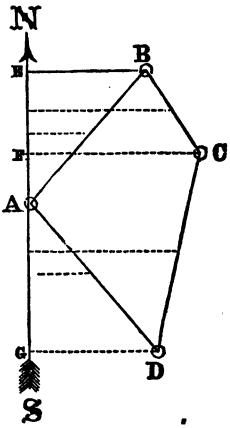
* The last course is a "preceding course" to the first course, as will appear on remembering that these two courses join each other on the ground.

ducts. The Difference of the North Products and the South Products is therefore the desired area of the three-sided field ABC.

Using the *Double* Longitudes, (in order to avoid fractions), in each of the preceding products, their difference will be the *double* area of the Triangle ABC.

(291) Taking now a *four-sided field,* ABCD in the figure, and drawing a Meridian and Longitudes as before, it is seen, on inspection, that its area would be obtained by taking the two Triangles, ABE, ADG, from the figure EBCDGE, or from the sum of the two Trapezoids EBCF and FCDG.

Fig. 195.



The area of the Triangle AEB will be found, as in the last article, to be equal to the product of the Latitude of the 1st course by its Longitude. The Product will be *North.*

The area of the Trapezoid EBCF will be found to equal the Latitude of the 2d course by its Longitude. The product will be *South.*

The area of the Trapezoid FCDG will be found to equal the product of the Latitude of the 3d course by its Longitude. The product will be *South.*

The area of the Triangle ADG will be found to equal the product of the Latitude of the 4th course by its Longitude. The product will be *North.*

The difference of the North and South products will therefore be the desired area of the four-sided field ABCD.

Using the *Double* Longitude as before, in each of the preceding products, their difference will be *double* the area of the field.

(292) Whatever the number or directions of the sides of a field, or of any space enclosed by straight lines, its area will always be equal to half of the difference of the North and South Products arising from multiplying together the Latitude and Double Longitude of each course or side.

We have therefore the following

GENERAL RULE FOR FINDING AREAS.

1. *Prepare ten columns, headed as in the example below, and in the first three write the Stations, Bearings and Distances.*

2. *Find the Latitudes and Departures of each course, by the Traverse Table, as directed in Art.* (281), *placing them in the four following columns.*

3. *Balance them, as in Art.* (284), *correcting them in red ink.*

4. *Find the Double Longitudes, as in Art.* (288), *with reference to a Meridian passing through the extreme East or West Station, and place them in the eighth column.*

5. *Multiply the Double Longitude of each course by the corrected Latitude of that course, placing the North Products in the ninth column, and the South Products in the tenth column.*

6. *Add up the last two columns, subtract the smaller sum from the larger, and divide the difference by two. The quotient will be the content desired.*

(293) To find the most Easterly or Westerly Station of a survey, without a plat, it is best to make a rough hand-sketch of the survey, drawing the lines in an approximation to their true directions, by drawing a North and South, and East and West lines, and considering the Bearings as fractional parts of a right angle, or 90°; a course N. 45° E. for example, being drawn about half way between a North and an East direction; a course N. 28° W. being not quite one-third of the way around from North to West; and so on, drawing them of approximately true proportional lengths.

(294) *Example* 1, given below, refers to the five-sided field, of which a plat is given in Fig. 175, page 151, and the Latitudes and Departures of which were calculated in Art. (282), page 175. Station 1 is the most Westerly Station, and the Meridian will be supposed to pass through it. The Double Longitudes are best

found by a continual addition and subtraction, as in the margin, where they are marked D. L. The Double Longitude of the last course comes out equal to its Departure, thus proving the work.

STA.		
1	+ 1.55	D. L.
	+ 1.55	
	+ 1.28	
2	+ 4.38	D. L.
	+ 1.28	
	+ 1.86	
3	+ 7.52	D. L.
	+ 1.86	
	– 2.00	
4	+ 7.38	D. L.
	– 2.00	
	– 2.69	
5	+ 2.69	D. L.

The Double Longitudes being thus obtained, are multiplied by the corresponding Latitudes, and the content of the field obtained as directed in the General Rule.

This example may serve as a pattern for the most compact manner of arranging the work.

STATION.	BEARINGS.	DISTANCES.	LATITUDES. N. +	LATITUDES. S. –	DEP'TURES E. +	DEP'TURES W. –	DOUBLE LONGITUDES.	DOUBLE AREAS. N. +	DOUBLE AREAS. S. –
1	N. 35° E.	2.70	2.21		1.55		+ 1.55	3.4255	
2	N. 83½° E.	1.29	.15		1.28		+ 4.38	0.6570	
3	S. 57° E.	2.22		1.21	1.86		+ 7.52		9.0992
4	S. 34¼° W.	3.55		2.93		2.00	+ 7.38		21.6234
5	N. 56½° W.	3.23	1.78			2.69	+ 2.69	4.7882	
			4.14	4.14	4.69	4.69		8.8707	30.7226
									8.8707
									2)21.8519

Content=1A. 0R. 15P.

Square Chains, 10.9259

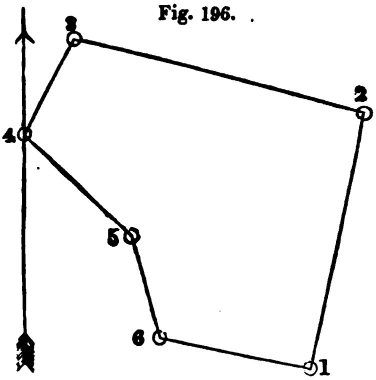
(295) The Meridian might equally well have been supposed to pass through the most Easterly station, 4 in the figure. The Double Longitudes could then have been calculated as in the margin. They will of course be all West, or minus. The products being then calculated, the sum of the North products will be found to be 29.9625, and of the South products 8.1106, and their difference to be 21.8519, the same result as before.

STA.		
4	– 2.00	D. L.
	– 2.00	
	– 2.69	
5	– 6.69	D. L.
	– 2.69	
	+ 1.55	
1	– 7.83	D. L.
	+ 1.55	
	+ 1.28	
2	– 5.00	D. L.
	+ 1.28	
	+ 1.86	
3	– 1.86	

(296) A number of examples, with and without answers, will now be given as exercises for the student, who should plat them by some of the methods given in the preceding chapter, using each of them at least once. He should then calculate their content by the method just given, and *check* it, by also calculating the area of the plat by some of the Geometrical or Instrumental methods given in Part I, Chapter IV; for no single calculation is ever reliable.

All the examples (except the last) are from the author's actual surveys.

Fig. 196.



Example 2, given below, is also fully worked out, as another pattern for the student, who need have no difficulty with any possible case if he strictly follows the directions which have been given. The plat is on a scale of 2 chains to 1 inch, (= 1:1584).

STATION.	BEARINGS.	DISTANCES.	LATITUDES. N. +	LATITUDES. S. −	DEP'TURES. E. +	DEP'TURES. W. −	DOUBLE LONGITUDES.	DOUBLE AREAS. N. +	DOUBLE AREAS. S. −
1	N. 12¼° E.	2.81	2.75		.60		+ 6.56	18.0400	
2	N. 76° W.	3.20	.77			3.11	+ 4.05	3.1185	
3	S. 24½° W.	1.14		1.04		.47	+ .47		.4888
4	S. 48° E.	1.53		1.02	1.14		+ 1.14		1.1628
5	S. 12½° E.	1.12		1.09	.24		+ 2.52		2.7468
6	S. 77° E.	1.64		.37	1.60		+ 4.36		1.6132
			3.52	3.52	3.58	3.58		21.1585	6.0116

6.0116
2)15.1469
Square Chains, 7.5734

Content = 0A. 3R. 1P.

Example 3.

STA.	BEARING.	DISTANCE.
1	N. 52° E.	10.64
2	S. 29¾° E.	4.09
3	S. 31¾° W.	7.68
4	N. 61° W.	7.24

Ans. 4A. 3R. 28P.

Example 4.

STA.	BEARING.	DISTANCE.
1	S. 21° W.	12.41
2	N. 83¼° E.	5.86
3	N. 12° E.	8.25
4	N. 47° W.	4.24

Ans. 4A. 2R. 37P.

Example 5.

STA.	BEARING.	DISTANCE.
1	N. 34¼° E.	2.73
2	N. 85° E.	1.28
3	S. 56¾° E.	2.20
4	S. 34¼° W.	3.53
5	N. 56½° W.	3.20

Ans. 1A. 0R. 14P.

Example 6.

STA.	BEARING.	DISTANCE.
1	N. 35° E.	6.49
2	S. 56¼° E.	14.15
3	S. 34° W.	5.10
4	N. 56° W.	5.84
5	S. 29½° W.	2.52
6	N. 48¼° W.	8.73

Example 7.

STA.	BEARING.	DISTANCE.
1	S. 21¼° W.	17.62
2	S. 34° W.	10.00
3	N. 56° W.	14.15
4	N. 34° E.	9.76
5	N. 67° E.	2.30
6	N. 23° E.	7.03
7	N. 18½° E.	4.43
8	S. 76½° E.	12.41

Example 8.

STA.	BEARING.	DISTANCE.
1	S. 65½° E.	4.98
2	S. 58° E.	8.56
3	S. 14¼° W.	20.69
4	S. 47° W.	0.60
5	S. 57¼° W.	8.98
6	N. 56° W.	12.90
7	N. 34° E.	10.00
8	N. 21¼ E.	17.62

Example 9.

STA.	BEARING.	DISTANCE.
1	S. 57° E.	5.77
2	S. 36¼° W.	2.25
3	S. 39¼° W.	1.00
4	S. 70¼° W.	1.04
5	N. 68¾° W.	1.23
6	N. 56° W.	2.19
7	N. 33¼° E.	1.05
8	N. 56½° W.	1.54
9	N. 33½° E.	3.18

Ans. 2A. 0R. 32P.

Example 10.

STA.	BEARING.	DISTANCE.
1	N. 63° 51′ W.	6.91
2	N. 63° 44′ W.	7.26
3	N. 69° 35′ W.	3.34
4	N. 77° 50′ W.	6.54
5	N. 31° 24′ E.	14.38
6	N. 31° 18′ E.	16.81
7	S. 68° 55′ E.	13.64
8	S. 68° 42′ E.	11.54
9	S. 33° 45′ W.	31.55

Ans. 74 Acres.

Example 11.

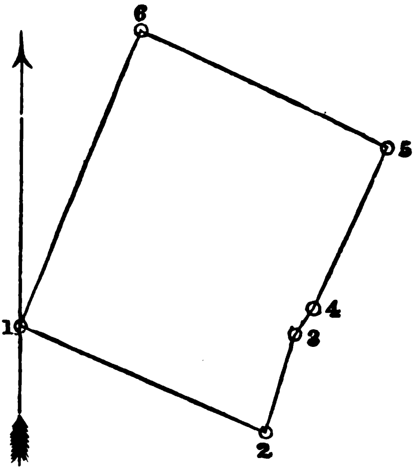
STA.	BEARING.	DISTANCE.
1	N. 18¾° E.	1.93
2	N. 9° W.	1.29
3	N. 14° W.	2.71
4	N. 74° E.	0.95
5	S. 48½° E.	1.59
6	S. 14¼° E.	1.14
7	S. 19½° E.	2.15
8	S. 23½° W.	1.22
9	S. 5° W.	1.40
10	S. 30° W.	1.02
11	S. 81½° W.	0.69
12	N. 32½° W.	1.98

Example 12.

STA.	BEARING.	DISTANCE.
1	N. 72¾° E.	0.88
2	S. 20½° E.	0.22
3	S. 63° E.	0.75
4	N. 51° E.	2.35
5	N. 44° E.	1.10
6	N. 25½° W.	1.96
7	N. 8½° W.	1.05
8	S. 29° W.	1.63
9	N. 71½° W.	0.81
10	N. 13½° W.	1.17
11	N. 63° W.	1.28
12	West.	1.68
13	N. 49° W.	0.80
14	S. 19½ E.	6.20

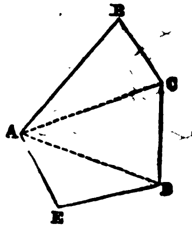
Example 18. A farm is described in an old Deed, as bounded thus. Beginning at a pile of stones, and running thence twenty-seven chains and seventy links South-Easterly sixty-six and a half degrees to a white-oak stump; thence eleven chains and sixteen links North-Easterly twenty and a half degrees to a hickory tree; thence two chains and thirty-five links North-Easterly thirty-six degrees to the South-Easterly corner of the homestead; thence nineteen chains and thirty-two links North-Easterly twenty-six degrees to a stone set in the ground; thence twenty-eight chains and eighty links North-Westerly sixty-six degrees to a pine stump; thence thirty-three chains and nineteen links South-Westerly twenty-two degrees to the place of beginning, containing ninety-two acres, be the same more or less. Required the exact content.

Fig. 197.



(297) Mascheroni's Theorem. *The surface of any polygon is equal to half the sum of the products of its sides (omitting any one side) taken two and two, into the sines of the angles which those sides make with each other.*

Fig. 198.



Thus, take any polygon, such as the five-sided one in the figure. Express the angle which the directions of any two sides, as AB, CD, make with each other, thus (AB $\wedge$ CD). Then will the content of that polygon be, as below;

$$= \tfrac{1}{2}\,[\text{AB}\,.\,\text{BC}\,.\,\sin(\text{AB} \wedge \text{BC}) + \text{AB}\,.\,\text{CD}\,.\,\sin(\text{AB} \wedge \text{CD})$$
$$+ \text{AB}\,.\,\text{DE}\,.\,\sin(\text{AB} \wedge \text{DE}) + \text{BC}\,.\,\text{CD}\,.\,\sin(\text{BC} \wedge \text{CD})$$
$$+ \text{BC}\,.\,\text{DE}\,.\,\sin(\text{BC} \wedge \text{DE}) + \text{CD}\,.\,\text{DE}\,.\,\sin(\text{CD} \wedge \text{DE})]$$

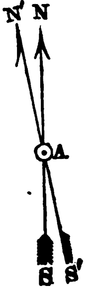
The demonstration consists merely in dividing the polygon into triangles by lines drawn from any angle, (as A); then expressing the area of each triangle by half the product of its base and the perpendicular let fall upon it from the above named angle; and finally separating the perpendicular into parts which can each be expressed by the product of some one side into the sine of the angle made by it with another side. The sum of these triangles equals the polygon.

The expressions are simplified by dividing the proposed polygon into two parts by a diagonal, and computing the area of each part separately, making the diagonal the side omitted.*

CHAPTER VII.

THE VARIATION OF THE MAGNETIC NEEDLE.

Fig. 199.



(298) Definitions. The *Magnetic Meridian* is the direction indicated by the Magnetic Needle. The *True Meridian* is a true North and South line, which, if produced, would pass through the poles of the earth. The *Variation*, or *Declination*, of the needle is the angle which one of these lines makes with the other.†

In the figure, if NS represent the direction of the True Meridian, and N′S′ the direction of the Magnetic Meridian at any place, then is the angle NAN′ the *Variation of the Needle* at that place.

(299) Direction of Needle. The directions of these two meridians do not generally coincide, but the needle in most places points to the East or to the West of the true North, more or less

* The original Theorem is usually accredited to Lhuillier, of Geneva, who published it in 1789. But Mascheroni, the ingenious author of the "Geometry of the Compasses," had published it at Pavia, two years previously. The method is well developed in Prof. Whitlock's "Elements of Geometry."

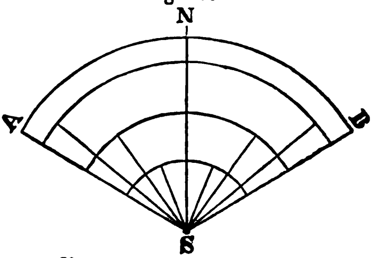
† "Declination" is the more correct term, and "Variation" should be reserved for the *change* in the Declination which will be considered in the next chapter; but custom has established the use of Variation in the sense of Declination.

according to the locality. Observations of the amount and the direction of this variation have been made in nearly all parts of the world. In the United States the Variation in the Eastern States is Westerly, and in the Western States is Easterly, as will be given in detail, after the methods for determining the True Meridian, and consequently the Variation, at any place, have been explained.

TO DETERMINE THE TRUE MERIDIAN.

(200) By equal shadows of the Sun. On the South side of any level surface, erect an upright staff, shown, in horizontal projection, at S. Two or three hours before noon, mark the extremity, A, of its shadow. Describe an arc of a circle with S, the foot of the staff, for centre, and SA, the distance to the extremity of the shadow, for radius. About as many hours after noon as it had been before noon when the first mark was made, watch for the moment when the end of the shadow touches the arc at another point, B. Bisect the arc AB at N. Draw SN, and it will be the true meridian, or North and South line required.

Fig. 200.



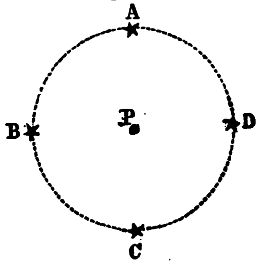
For greater accuracy, describe several arcs before hand, mark the points in which each of them is touched by the shadow, bisect each, and adopt the average of all. The shadow will be better defined, if a piece of tin with a hole through it be placed at the top of the staff, as a bright spot will thus be substituted for the less definite shadow. Nor need the staff be vertical, if from its summit a plumb-line be dropped to the ground, and the point which this strikes be adopted as the centre of the arcs.

This method is a very good approximation, though perfectly correct only at the time of the solstices; about June 21st and December 22d. It was employed by the Romans in laying out cities.

To get the Variation, set the compass at one end of the True Meridian line thus obtained, sight to the other end of it, and take the Bearing as of any ordinary line. The number of degrees in the reading will be the desired variation of the needle.

(301) By the North Star, when in the Meridian. The North Star, or Pole Star, (called by astronomers *Alpha Ursæ Minoris*, or *Polaris*), is not situated precisely at the North Pole of the heavens. If it were, the Meridian could be at once determined by sighting to it, or placing the eye at some distance behind a plumb-line so that this line should hide the star. But the North Star is about $1\frac{1}{2}°$ from the Pole. Twice in 24 hours, however, (more precisely 23h. 56m.), it is in the Meridian, being then exactly above or below the Pole, as at A and C in the figure. To know when it is so, is rendered easy by the aid of another star, easily identified, which at these times is almost exactly above or below the North Star, i. e. situated in the same vertical plane. If then we watch for the moment at which a suspended plumb-line will cover both these stars, they will then be in the Meridian.

Fig. 201.



The other star is in the well known constellation of the Great Bear, called also the Plough, or the Dipper, or Charles's Wain.

Fig. 202.

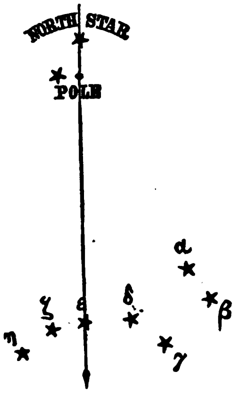
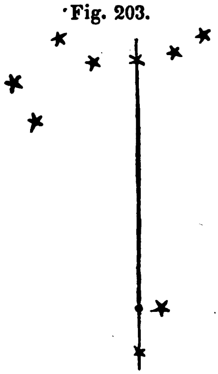


Fig. 203.



Two of its five bright stars (the right-hand ones in Fig. 202) are known as the "Pointers," from their pointing near to the North

Star, thus assisting in finding it. The star in the tail or handle, nearest to the four which form a quadrilateral, is the star which comes to the Meridian at the same time with the North Star, twice in 24 hours, as in Fig. 202 or 203. It is known as *Alioth*, or *Epsilon Ursæ Majoris*.*

To determine the Meridian by this method, suspend a long plumb-line from some elevated point, such as a stick projecting from the highest window of a house suitably situated. The plumb-bob may pass into a pail of water to lessen its vibrations. South of this set up the compass, at such a distance from the plumb-line that neither of the stars will be seen above its highest point, i. e. in Latitudes of 40° or 50° not quite as far from the plumb-line as it is long. Or, instead of a compass, place a board on two stakes, so as to form a sort of bench, running East and West, and on it place one of the compass-sights, or anything having a small hole in it to look through. As the time approaches for the North Star to be on the Meridian (as taken from the table given below) place the compass, or the sight, so that, looking through it, the plumb-line shall seem to cover or hide the North Star. As the star moves one way, move the eye and sight the other way, so as to constantly keep the star behind the plumb-line. At last *Alioth*, too, will be covered by the plumb-line. At that moment the eye and the plumb-line are (approximately) in the Meridian. Fasten down the sight on the board till morning, or with the compass take the bearing at once, and the reading is the variation.†

Instead of one plumb-line and a sight, two plumb-lines may be suspended at the end of a horizontal rod, turning on the top of a pole.

The line thus obtained points to the East of the true line when the North Star is above Alioth, and vice versa. The North Star is exactly in the Meridian about 17 minutes after it has been in the same vertical plane with Alioth, and may be sighted to after that interval of time, with perfect accuracy.

* The North Pole is very nearly at the intersection of the line from Polaris to Alioth, and a perpendicular to this line from the small star seen to the left of it in Fig. 202.

† If a Transit or Theodolite be used, the cross-hairs must be illuminated by throwing the light of a lamp into the telescope by its reflection from white paper.

Another bright star, which is on the opposite side of the Pole, and is known to astronomers as *Gamma Cassiopeiæ*, also comes on the Meridian nearly at the same time as the North Star, and will thus assist in determining its direction.

(302) The time at which the North Star passes the Meridian above the Pole, for every 10th day in the year, is given in the following Table, in common clock time.* The upper transit is the most convenient, since at the other transit Alioth is too high to be conveniently observed.

Times of North Star passing the Meridian.

MONTH.	1st DAY.	11th DAY.	21st DAY.
	H. M.	H. M.	H. M.
January,	6 21 P. M.	5 41 P. M.	5 02 P. M.
February,	4 18 P. M.	3 39 P. M.	3 00 P. M.
March,	2 28 P. M.	1 49 P. M.	1 09 P. M.
April,	0 26 P. M.	11 47 A. M.	11 08 A. M.
May,	10 28 A. M.	9 49 A. M.	9 10 A. M.
June,	8 27 A. M.	7 48 A. M.	7 08 A. M.
July,	6 29 A. M.	5 50 A. M.	5 11 A. M.
August,	4 28 A. M.	3 49 A. M.	3 09 A. M.
September,	2 26 A. M.	1 47 A. M.	1 07 A. M.
October,	0 28 A. M.	11 45 P. M.	11 06 P. M.
November,	10 22 P. M.	9 43 P. M.	9 04 P. M.
December,	8 24 P. M.	7 45 P. M.	7 06 P. M.

* To calculate the time of the North Star passing the Meridian at its upper culmination: Find in the "American Almanac," (Boston), or the "Astronomical Ephemeris," (Washington), or the "Nautical Almanac," (London), or by interpolation from the data at the end of this note, the right ascension of the star, and from it (increased by twenty-four hours if necessary to render the subtraction possible) subtract the Right ascension of the Sun at mean noon, or the sidereal time at mean noon, for the given day, as found in the "Ephemeris of the Sun," in the same Almanacs. From the remainder subtract the acceleration of sidereal on mean time corresponding to this remainder, (3m. 56s. for 24 hours), and the new remainder is the required mean solar time of the upper passage of the star across the Meridian, in "Astronomical" reckoning, the astronomical day beginning at noon of the common civil day of the same date.

The right ascension of the North Star for Jan. 1, 1850, is 1h. 05m. 01.4s.; for 1860, 1h. 08m. 02.8s.; for 1870, 1h. 11m. 16.9s.; for 1880, 1h. 14m. 45.1s.; for 1890, 1h. 18m. 29.2s.; for 1900, 1h. 22m. 31s.

To find the time of the star's passage of the Meridian for other days than those given in the Table, take from it the time for the day most nearly preceding that desired, and subtract from this time 4 minutes for each day from the date of the day in the Table to that of the desired day; or, more accurately, interpolate, by saying: *As* the number of days between those given in the Table *is to* the number of days from the next preceding day in the Table to the desired day, *so is* the difference between the times given in the Table for the days next preceding and following the desired day *to* the time to be subtracted from that of the next preceding day. The first term of the preceding proportion is always *ten*, except at the end of months having more or less than 30 days. For example, let the time of the North Star's passing the Meridian on July 26th be required. From July 21st to August 1st being 11 days, we have this proportion: 11 days : 5 days :: 43 minutes : $19\frac{6}{11}$ minutes. Taking this from 5h. 11m. A. M., we get 4h. $51\frac{1}{2}$m. A. M. for the time of passage required.

The North Star passes the Meridian later every year. In 1860, it will pass the Meridian about two minutes later than in 1854; in 1870, five minutes, in 1880, eight minutes, in 1890, twelve minutes, and in 1900, sixteen minutes, later than in 1854: the year for which the preceding table has been calculated.

The times at which the North Star passes the Meridian *below* the Pole, in its lower Transit, can be found by adding 11h. 58m. to the time of the upper Transit, or by subtracting that interval from it.*

(303) By the North Star at its extreme elongation. When the North Star is at its greatest *apparent* angular distance East or West of the Pole, as at B or D in Fig. 201, it is said to be at its extreme Eastern, or extreme Western, Elongation. If it be observed at either of these times, the direction of the Meridian can be easily

* The North Star, which is now about 1° 28′ from the Pole, was 12° distant from it when its place was first recorded. Its distance is now diminishing at the rate of about a third of a minute in a year, and will continue to do so till it approaches to within half a degree, when it will again recede. The brightest star in the Northern hemisphere, *Alpha Lyræ*, will be the Pole Star in about 12,000 years, being then within about 5° of the Pole, though now more than 51° distant from it

obtained from the observation. The great advantage of this method over the preceding is that then the star's motion apparently ceases for a short time.

(304) The following Table gives the

TIMES OF EXTREME ELONGATIONS OF THE NORTH STAR.*

MONTH.	1ST DAY.		11TH DAY.		21ST DAY.	
	EASTERN.	WESTERN.	EASTERN.	WESTERN.	EASTERN.	WESTERN.
	H. M.	H. M.	H. M.	H. M.	H. M.	H. M.
Jan'y,	0 27 P.M.	0 19 A.M.	11 47 A.M.	11 35 P.M.	11 08 A.M.	10 56 P.M.
Feb'y,	10 24 A.M.	10 13 P.M.	9 45 A.M.	9 33 P.M.	9 06 A.M.	8 54 P.M.
March,	8 34 A.M.	8 22 P.M.	7 55 A.M.	7 43 P.M.	7 15 A.M.	7 04 P.M.
April,	6 32 A.M.	6 20 P.M.	5 53 A.M.	5 41 P.M.	5 14 A.M.	5 02 P.M.
May,	4 34 A.M.	4 22 P.M.	3 55 A.M.	3 43 P.M.	3 16 A.M.	3 04 P.M.
June,	2 33 A.M.	2 21 P.M.	1 53 A.M.	1 42 P.M.	1 14 A.M.	1 02 P.M.
July,	0 35 A.M.	0 23 P.M.	11 52 P.M.	11 44 A.M.	11 13 P.M.	11 05 A.M.
August,	10 30 P.M.	10 22 A.M.	9 51 P.M.	9 43 A.M.	9 11 P.M.	9 03 A.M.
Sept'r,	8 28 P.M.	8 20 A.M.	7 49 P.M.	7 41 A.M.	7 09 P.M.	7 01 A.M.
Oct'r,	6 30 P.M.	6 22 A.M.	5 51 P.M.	5 43 A.M.	5 12 P.M.	5 04 A.M.
Nov'r,	4 28 P.M.	4 21 A.M.	3 49 P.M.	3 41 A.M.	3 10 P.M.	3 02 A.M.
Dec'r,	2 30 P.M.	2 22 A.M.	1 51 P.M.	1 43 A.M.	1 12 P.M.	1 04 A.M.

The Eastern Elongations from October to March, and the Western Elongations from April to September, occurring in the day time, they will generally not be visible except with the aid of a powerful telescope.

* To calculate the times of the greatest elongation of the North Star: Find in one of the Almanacs before referred to, or from the data below, its Polar distance at the given time. Add the logarithm of its tangent to the logarithm of the tangent of the Latitude of the place, and the sum will be the logarithm of the cosine of the Hour angle before or after the culmination. Reduce the space to time; correct for sidereal acceleration (3m. 56s. for 24 hours) and subtract the result from the time of the star's passing the meridian on that day, to get the time of the Eastern elongation, or add it to get the Western.

The Polar distance of the North Star, for Jan. 1, 1850, is 1° 29′ 25″; for 1860, 1° 26′ 12″.7; for 1870, 1° 23° 01″; for 1880, 1° 19′ 50″.4; for 1890, 1° 16′ 40″.7; for 1900, 1° 13′ 32″.2.

The preceding Table was calculated for Latitude 40°. The Time at which the Elongations occur vary slightly for other Latitudes. In Latitude 50°, the Eastern Elongations occur about 2 minutes later and the Western Elongations about 2 minutes earlier than the times in the Table. In Latitude 26°, precisely the reverse takes place.

The Times of Elongation are continually, though slowly, becoming later. The preceding Table was calculated for July 1st, 1854. In 1860, the times will be nearly 2 minutes later; and in 1900, the Eastern Elongations will be about 15 minutes, and the Western Elongations 17 minutes later than in 1854.

(305) Observations. Knowing from the preceding Table the hour and minute of the extreme Elongation on any day, a little before that time suspend a plumb-line, precisely as in Art. (**301**), and place yourself south of it as there directed. As the North Star moves one way, move your eye the other, so that the plumb-line shall continually seem to cover the star. At last the star will appear to stop moving for a time, and then begin to move backwards. Fix the sight on the board (or the compass, &c.) in the position in which it was when the star ceased moving; for the star was then at its extreme apparent Elongation, East or West, as the case may be.

(306) Azimuths. The angle which the line from the eye to the plumb-line, makes with the True Meridian (i. e. the angle between the meridian plane and the vertical plane passing through the eye and the star) is called the *Azimuth* of the Star. It is given in the following Table for different Latitudes, and for a number of years to come. For the intermediate Latitudes, it can be obtained by a simple proportion, similar to that explained in detail in Art. (**302**).*

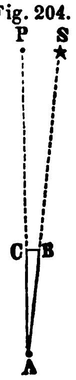
* To calculate this Azimuth: From the logarithm of the sine of the Polar distance of the star, subtract the logarithm of the cosine of the Latitude of the place; the remainder will be the logarithm of the sine of the angle required. The Polar distance can be obtained as directed in the last note.

AZIMUTHS OF THE NORTH STAR.

Latitudes.	1854	1855	1856	1857	1858	1859	1860	1870
50°	2° 16¾′	2° 16¼′	2° 16′	2° 15½′	2° 15′	2° 14½′	2° 14¼′	2° 09¼′
49°	2° 14′	2° 13½′	2° 13¼′	2° 12¾′	2° 12¼′	2° 12′	2° 11½′	2° 06½′
48°	2° 11½′	2° 11′	2° 10½′	2° 10¼′	2° 09¾′	2° 09¼′	2° 09′	2° 04′
47°	2° 09′	2° 08½′	2° 08′	2° 07¾′	2° 07¼′	2° 06¾′	2° 06½′	2° 01¾′
46°	2° 06¾′	2° 06¼′	2° 05¾′	2° 05½′	2° 05′	2° 04½′	2° 04¼′	1° 59½′
45°	2° 04½′	2° 04′	2° 03½′	2° 03¼′	2° 02¾′	2° 02¼′	2° 02′	1° 57½′
44°	2° 02½′	2° 02′	2° 01¾′	2° 01½′	2° 01′	2° 00½′	2° 00′	1° 55½′
43°	2° 00½′	2° 00′	1° 59½′	1° 59′	1° 58¾′	1° 58¼′	1° 58′	1° 53½′
42°	1° 58½′	1° 58′	1° 57½′	1° 57¼′	1° 56¾′	1° 56½′	1° 56′	1° 51¾′
41°	1° 56¾′	1° 56¼′	1° 55¾′	1° 55½′	1° 55′	1° 54½′	1° 54¼′	1° 50′
40°	1° 55′	1° 54½′	1° 54′	1° 53¾′	1° 53¼′	1° 53′	1° 52½′	1° 48¼′
39°	1° 53¼′	1° 52¾′	1° 52½′	1° 52′	1° 51¾′	1° 51¼′	1° 51′	1° 46¾′
38°	1° 51¾′	1° 51¼′	1° 51′	1° 50½′	1° 50′	1° 49¾′	1° 49½′	1° 45¼′
37°	1° 50¼′	1° 49¾′	1° 49½′	1° 49′	1° 48¾′	1° 48¼′	1° 48′	1° 44′
36°	1° 48¾′	1° 48¼′	1° 48′	1° 47¾′	1° 47¼′	1° 47′	1° 46½′	1° 42¾′
35°	1° 47½′	1° 47′	1° 46¾′	1° 46¼′	1° 46′	1° 45½′	1° 45¼′	1° 41½′
34°	1° 46¼′	1° 45¾′	1° 45½′	1° 45′	1° 44¾′	1° 44¼′	1° 44′	1° 40¼′
33°	1° 45′	1° 44½′	1° 44¼′	1° 43¾′	1° 43½′	1° 43′	1° 42¾′	1° 39′
32°	1° 44′	1° 43½′	1° 43′	1° 42¾′	1° 42¼′	1° 42′	1° 41½′	1° 38′
31°	1° 42¾′	1° 42¼′	1° 42′	1° 41½′	1° 41′	1° 40¾′	1° 40½′	1° 37′
30°	1° 41½′	1° 41¼′	1° 41′	1° 40½′	1° 40¼′	1° 40′	1° 39½′	1° 36′

(307) Setting out a Meridian. When two points in the direction of the North Star at its extreme elongation have been obtained, as in Art. (305), the True Meridian can be found thus. Let A and B be the two points. Multiply the natural tangent of the Azimuth given in the Table, by the distance AB. The product will be the length of a line which is to be set off from B, perpendicular to AB, to some point C. A and C will then be points in the True Meridian. This operation may be postponed till morning.

Fig. 204.



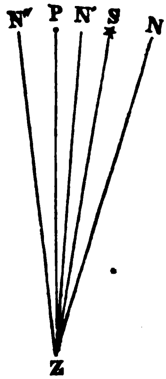
If the directions of both the extreme Eastern and extreme Western elongations be set out, the line lying midway between them will be the True Meridian.

(308) Determining the Variation. The variation would of course be given by taking the Bearing of the Meridian thus obtained, but it can also be determined by taking the Bearing of the star at the time of the extreme elongation, and applying the following rules.

When the Azimuth of the star and its magnetic bearing are one East and the other West, the sum of the two is the Magnetic Varition, which is of the same name as the Azimuth; i. e. East, if that be East, and West, if it be West.

When the Azimuth of the star and its Magnetic Bearing are both East, or both West, their difference is the Variation, which will be of the same name as the Azimuth and Bearing, if the Azimuth be the greater of the two, or of the contrary name if the Azimuth be the smaller.

Fig. 205.



All these cases are presented together in the figure, in which P is the North Pole; Z the place of the observer; ZP the True Meridian; S the star at its greatest Eastern elongation; and ZN, ZN′, ZN″, various supposed directions of the needle.

Call the Azimuth of the star, i. e. the angle PZS, 2° East.

Suppose the needle to point to N, and the Bearing of the star, i. e. SZN, to be 5° West of Magnetic North. The variation PZN will evidently be 7° East of true North.

Suppose the needle to point to N′, and the bearing of the star, i. e. N′ZS, to be $1\frac{1}{4}$° East of Magnetic North. The Variation will be $\frac{3}{4}$° East of true North, and of the same name as the Azimuth, because that is greater than the bearing.

Suppose the needle to point to N″ and the bearing of the star, i. e. N″ZS, to be 10° East of Magnetic North. The Variation will be 8° West of true North, of the contrary name to the Azimuth, because that is the smaller of the two.*

* Algebraically, always subtract the Bearing from the Azimuth, and give the remainder its proper resulting algebraic sign. It will be the Variation; East if *plus*, and West, if minus. Thus in the first case above, the Variation $= +2° - (-5°) = +7° = 7°$ East. In the second case, the Variation $= +2° - (+1\frac{1}{4}°) = +\frac{3}{4}° = \frac{3}{4}°$ East. In the third case, the Variation $= +2° - (+10°) = -8° = 8°$ West.

If the star was on the other side of the Pole, the rules would apply likewise.

(309) Other Methods. Many other methods of determining the true Meridian are employed; such as by equal altitudes and azimuths of the sun, or of a star; by one azimuth, knowing the time; by observations of circumpolar stars at equal times before and after their culmination, or before and after their greatest elongation, &c.

All these methods however require some degree of astronomical knowledge; and those which have been explained are abundantly sufficient for all the purposes of the ordinary Land-Surveyor.

"Burt's Solar Compass" is an instrument by which, "when adjusted for the Sun's declination, and the Latitude of the place, the azimuth of any line from the true North and South can be read off, and the difference between it and the Bearing by the compass will then be the variation."

(310) Magnetic variation in the United States. The variation in any part of the United States, east of the Territories, can be approximately obtained by mere inspection of the map at the beginning of this volume.* Through all the places at which the needle in 1840,† pointed to the true North, a line is drawn on the map, and called the *Line of no Variation.* It will be seen to be nearly straight, and to pass in a N.N.W. direction from a little West of Cape Hatteras, N. C., through the middle of Virginia, about midway between Cleveland, (Ohio), and Erie, (Pa.), and through the middle of Lake Erie and Lake Huron. If followed South-Easterly it would be found to touch the most Easterly point of South America. It is now slowly moving Westward.

At all places situated to the East of this line (including the New-England States, New-York, New-Jersey, Delaware, Maryland, nearly all of Pennsylvania, and the Eastern half of Virginia and North Carolina) the Variation is Westerly, i. e. the North end of the needle points to the West of the true North. At all places

* Copied (by permission) from one prepared by Prof. Loomis by the reduction of numerous observations, and originally published in Silliman's "American Journal of Science," for Oct. 1840, Vol. xxxix, p. 41.

† A gradual change in the Variation is going on from year to year, as will be explained in the next Chapter.

situated to the West of this line (including the Western and Southern States) the Variation is easterly, i. e. the North end of the needle points to the East of the true North. This variation increases in proportion to the distance of the place on either side of the line of no variation, reaching 21° of Easterly Variation in Oregon, and 18° of Westerly Variation in Maine.

Lines of equal Variation are lines drawn through all the places which have the same variation. On the map they are drawn for each degree. All the places situated on the line marked 1°, East or West, have 1° Variation; those on the 2° line, have 2° Variation, &c. The variation at the intermediate places can be approximately estimated by the eye. These lines all refer to 1840.

The lines of equal Variation, if continued Northward, would all meet in a certain point called the *Magnetic Pole*, and situated in the neighborhood of 96° West Longitude from Greenwich, and 70° of North Latitude. Towards this pole the needle tends to point.

Another Magnetic pole is found in the Southern hemisphere; but the farther development of this subject belongs to a treatise on Natural Philosophy.

The Variation on the Pacific slope of this country has been very imperfectly ascertained. A few leading points are as below.

California;	Point Conception,	Sept. 1850,	13° 49½′ E.
	Point Penos, Monterey,	Feb. 1851,	14° 58′ E.
	Presidio, San Francisco,	Feb. 1852,	15° 27′ E.
	San Diego,	Mar. 1851,	12° 29′ E.
Oregon;	Cape Disappointment,	July, 1851,	20° 45′ E.
	Ewing Harbor,	Nov. 1851,	18° 29′ E.
Wash. Ter'y.	Scarboro' Harbor,	Aug. 1852,	21° 30′ E.

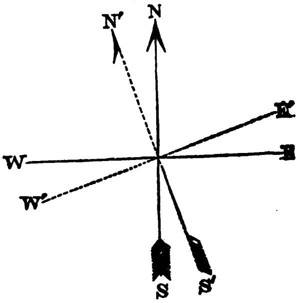
(311) To correct Magnetic Bearings. The Variation at any place and time being known, the Magnetic Bearings taken there and then, may be reduced to their true Bearings, by these Rules.

RULE 1. *When the Variation is West*, as it is in the North-Eastern States, the true Bearing will be the *sum* of the Variation and a Bearing which is North and West, or South and East; and the *difference* of the Variation and a Bearing which is North and East, or South and West. To apply this to the cardinal points, a

North Bearing must be called N. 0° West, an East Bearing N. 90° E., a South Bearing S. 0° E., and a West Bearing S. 90° W.; counting around from N′ to N, in the figure, and so onward, "with the Sun."

The reasons for these corrections are apparent from the Figure, in which the dotted lines and the accented letters represent the direction of the needle, and the full lines and the unaccented letters represent the true North and South and East and West lines.

Fig. 206.

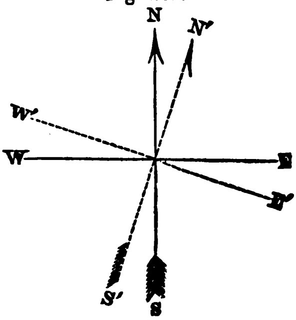


When the sum of the Variation and the Bearing is directed to be taken, and comes to more than 90°, the supplement of the sum is to be taken, and the first letter changed. When the difference is directed to be taken, and the Variation is greater than the Bearing, the last letter must be changed. A diagram of the case will remove all doubts. Examples of all these cases are given below for a Variation of 8° West.

MAGNETIC BEARING.	TRUE BEARING.	MAGNETIC BEARING.	TRUE BEARING.
North.	N. 8° W.	South.	S. 8° E.
N. 1° E.	N. 7° W.	S. 2° W.	S. 6° E.
N. 40° E.	N. 32° E.	S. 60° W.	S. 52° W.
East.	N. 82° E.	West.	S. 82° W.
S. 50° E.	S. 58° E.	N. 70° W.	N. 78° W.
S. 89° E.	N. 83° E.	N. 83° W.	S. 89° W.

RULE 2. *When the Variation is East*, as in the Western and Southern States, the preceding directions must be exactly reversed; i. e. the true Bearing will be the *difference* of the Variation and a Bearing which is North and West, or South and East; and the *sum* of the Variation and a Bearing which is North and East, or South and West. A North Bearing

Fig. 207.



must be called N. 0° E., a West Bearing N. 90° W., a South Bearing S. 0° W., and an East Bearing S. 90° E., counting from N′ to N, and so onward, "against the sun." The reasons for these rules are seen in the Figure. Examples are given below, for a Variation of 5° E.

MAGNETIC BEARING.	TRUE BEARING.	MAGNETIC BEARING.	TRUE BEARING.
North.	N. 5° E.	South.	S. 5° W.
N. 40° E.	N. 45° E.	S. 60° W.	S. 65° W.
N. 89° E.	S. 86° E.	S. 87° W.	N. 88° W.
East.	S. 85° E.	West.	N. 85° W.
S. 1° E.	S. 4° W.	N. 70° W.	N. 65° W.
S. 50° E.	S. 45° E.	N. 2° W.	N. 3° E.

(312) To survey a line with true Bearings. The compass may be set, or adjusted, by means of the Vernier, (noticed in Arts. (**229**) and (**237**), and shown in Fig. 148, page 126) according to the Variation in any place, so that the Bearings of any lines then taken with it will be their true Bearings. To effect this, turn aside the compass plate, by means of the Tangent Screw which moves the Vernier, a number of degrees equal to the Variation, moving the S. end of the Compass-box to the *right*, (the North end being supposed to go ahead) if the Variation be Westerly, and *vice versa;* for that moves the North end of the Compass-box in the contrary direction, and thus makes a line which before was N. by the needle, now read, as it should truly, North, so many degrees, West if the Variation was West; and similarly in the reverse case.

CHAPTER VIII.

CHANGES IN THE VARIATION.

(313) The *Changes* in the Variation are of more practical importance than its absolute amount. They are of four kinds: Irregular, Diurnal, Annual and Secular.

(314) Irregular changes. The needle is subject to sudden and violent changes, which have no known law. They are sometimes coincident with a thunder storm, or an Aurora Borealis, (during which, changes of nearly 1° in one minute, $2\frac{1}{2}$° in eight minutes, and 10° in one night, have been observed), but often have no apparent cause, except an otherwise invisible "Magnetic Storm."

(315) The Diurnal change. On continuing observations of the direction of the needle throughout an entire day, it will be found, in the Northern Hemisphere, that the North end of the needle moves Westward from about 8 A. M. till about 2 P. M. over an arc of from 10′ to 15′, and then gradually returns to its former position.* In the Southern Hemisphere, the direction of this motion is reversed. The period of this change being a day, it is called the *Diurnal Variation.* Its effect on the permanent Variation is necessarily to cause it, in places where it is West, to attain its maximum at about 2 P. M., and its minimum at about 8 A. M.; and the reverse where the Variation is East.

This Diurnal change adds a new element to the inaccuracies of the compass; since the Bearings of any line taken on the same day, at a few hours interval, might vary a quarter of a degree, which would cause a deviation of the end of the line, amounting to nearly half a link at the end of a chain, and to 35 links, or 23 feet, at the end of a mile. The hour of the day at which any important Bearing is taken should therefore be noted.

* A similar but smaller [illegible] takes place during the night.

(316) The Annual change. If the observations be continued throughout an entire year, it will be found that the Diurnal changes vary with the seasons, being about twice as great in Summer as in Winter. The period of this change being a year, it is called the Annual Variation.

(317) The Secular change. When accurate observations on the Variation of the needle in the same place are continued for several years, it is found that there is a continual and tolerably regular increase or decrease of the Variation, continuing to proceed in the same direction for so long a period, that it may be called the *Secular* change of Variation.*

The most ancient observations are those taken in Paris. In the year 1541 the needle pointed 7° East of North; in 1580 the Variation had increased to 11½° East, being its maximum; the needle then began to move Westward, and in 1666, it had returned to the Meridian; the Variation then became West, and continued to increase till in 1814 it attained its maximum, being 22° 34′ West of North. It is now decreasing, and in 1853 was 20° 17′ W. In London, the Variation in 1576 was 11° 15′ E.; in 1662, 0°; in 1700, 9° 40′ W.; in 1778, 22° 11′ W.; in 1815, 24° 27′ W.; and in 1843, 23° 8′ W.

In this country the north end of the needle was moving Eastward at the earliest recorded observations, and continued to do so till about the year 1810 (variously recorded as from 1793 to 1819), when it began to move Westward which it has ever since continued to do. Thus, in Boston, from 1708 to 1807 the Variation changed from 9° W. to 6° 5′ W., and from 1807 to 1840, it changed from 6° 5′ W. to 9° 18′ W.

Valuable Tables of the Secular changes of the Variation in various parts of the United States have been published by Prof. Loomis in Silliman's "American Journal of Science," Vol. 34, July, 1838, p. 301; Vol. 39, Oct. 1840, p. 42; and Vol. 43, Oct. 1842, p. 107. An abstract of the most reliable of them is here given. Troy and Schenectady are from other sources.

* If the term "Declination of the Needle" could be restored to its proper use, this "Change of Variation" would be properly called the "Variation of the Declination."

PLACE.	LATITUDE.	LONGITUDES.	DATES.	ANNUAL MOTION.
Burlington, Vt.	44° 27′	73° 10′	1811...1834	4′.4
Chesterfield, N. H.	42° 53′	72° 20′	1820...1836	6′.4
Deerfield, Mass.	42° 34′	72° 29′	1811...1837	5′.7
Cambridge, Mass.	42° 22′	71° 7′	1810...1840	3′.4
New-Haven, Conn.	41° 18′	72° 58′	1819...1840	4′.6
Keeseville, N. Y.	44° 28′	73° 32′	1825...1838	5′.4
Albany, N. Y.	42° 39′	73° 45′	1818...1842	3′.6
"	"	"	1842...1854	4′.9
Troy, N. Y.	42° 44′	73° 40′	1821...1837	6′.2
Schenectady, N. Y.	42° 49′	73° 55′	1829...1841	7′.2
"	"	"	1841...1854	6′.0
New-York City.	40° 43′	74° 01′	1824...1837	3′.7
Philadelphia.	39° 57′	75° 11′	1813...1837	3′.6
Milledgeville, Ga.	33° 7′	83° 20′	1805...1835	1′.7
Mobile, Ala.	30° 40′	88° 11′	1809...1835	2′.2
Cleveland, O.	41° 30′	81° 46′	1825...1838	4′.5
Marietta, O.	39° 25′	81° 26′	1810...1838	2′.4
Cincinnati, O.	39° 6′	84° 27′	1825...1840	2′.0
Detroit, Mich.	42° 24′	82° 58′	1822...1840	4′.3
Alton, Ill.	38° 52′	90° 12′	1835...1840	3′.0

From these and other observations it appears that at present the lines of equal variation are moving Westward, producing an annual change of variation (increasing the Westerly and lessening the Easterly) which is different in different parts of the country, and is about five or six minutes in the North-Eastern States, three or four minutes in the Middle States, and two minutes in the Southern States.

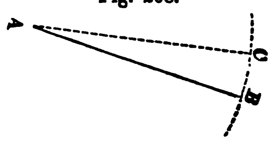
(318) Determination of the change, by Interpolation. To determine the change at any place and for any interval not found in the recorded observations, an approximation, sufficient for most purposes of the surveyor, may be obtained by interpolation (by a simple proportion) between the places given in the Tables, assuming the movements to have been uniform between the given dates; and also assuming the change at any place not found in the Tables, to have been intermediate between those of the lines of equal variation, which pass through the places of recorded observations on each side of it, and to have been in the ratio of its respective distances from those two lines; for example, taking their arithmetical mean, if the required place is midway between them; if it be twice as near one as the other, dividing the sum of twice the change of the nearest line, and once the change of the other, by three; and so in other cases; i. e. giving the change at each place, a "weight" inversely as its distance from the place at which the change is to be found.

(319) Determination of the change, by old lines. When the former Bearing of any old line, such as a farm-fence, &c. is recorded, the change in the Variation from the date of the original observation to the present time can be at once found by setting the compass at one end of the line and sighting to the other. The difference of the two Bearings is the required change.

If one end of the old line cannot be seen from the other, as is often the case when the line is fixed only by a "corner" at each end of it, proceed thus. Run a line from one corner with the old Bearing and with its distance. Measure the distance from the end of this line to the other corner, to which it will be opposite. Multiply this distance by 57.3, and divide by the length of the line. The quotient will be the change of variation in degrees.*

For example, a line 63 chains long, in 1827 had a Bearing of North 1° East. In 1847 a trial line was run from one end of the former line with the same Bearing and distance, and its other end was found to be 125 links to the West of the true corner. The change of Variation was therefore $\frac{1.25 \times 57.3}{63} = 1^\circ.137 = 1^\circ\ 8'$ Westerly.

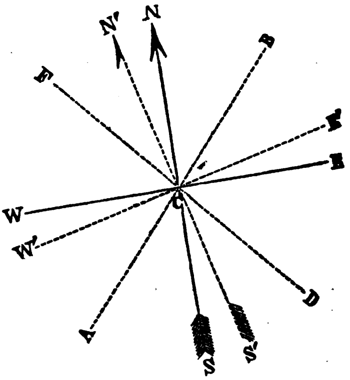
Fig. 208.



* Let AB be the original line; AC the trial line, and BC the distance between their extremities. AB and AC may be regarded as radii of a circle and BC as a chord of the arc which subtends their angle. Assuming the chord and arc to coincide (which they will, nearly, for small angles) we have this proportion; Whole circumference : arc BC :: 360° : BAC : or, 2 × AC × 3.1416 : BC :: 360° : BAC, whence BAC $= \frac{BC}{AB} \times 57.3$; or more precisely 57.29578.

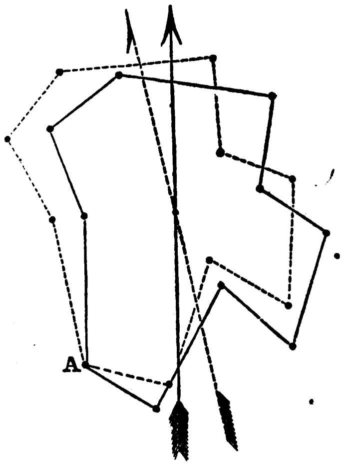
(320) Effects of the Secular change. These are exceedingly important in the re-survey of farms by the Bearings recorded in old deeds. Let SN denote the direction of the needle at the time of the original Survey, and S′N′ its direction at the time of the re-survey, a number of years later. Suppose the change to have been 3°, the needle pointing so much farther to the west of North. The line SN, which before was due North and South by the needle will now bear N. 3° E. and S. 3° W; the line AB, which before was N. 40° E. will now bear N. 43° E; the line DF which before was N. 40° W. will now bear N. 37° W; and the line WE, which before was due East and West, will now bear S. 87° E. and N. 87° W. Any line is similarly changed. The proof of this is apparent on inspecting the figure.

Fig. 209.



Suppose now that a surveyor, ignorant or neglectful of this change, should attempt to run out a farm by the old Bearings of the deed, none of the old fences or corners remaining. The full lines in the figure represent the original bounds of the farm, and the dotted lines those of the *new* piece of land which, starting from A, he would unwittingly run out. It would be of the same size and the same shape as the true one, but it would be in the wrong place. None of its lines would agree with the true ones, and in some places it would encroach on one neighbor, and in other places would leave a gore which belongs to it, between itself and another neighbor. Yet this is often done, and is the source of a great part of the litigation among farmers respecting their "lines."

Fig. 210.



(321) To run out old lines. To succeed in retracing old lines, proper allowance must be made for the change in the variation since the date of the original survey. That date must first be accurately ascertained; for the survey may be much older than the deed, into which its bearings may have been copied from an older one. The amount and direction of the change is then to be ascertained by the methods of Arts. (**318**) or (**319**). The bearings may then be corrected by the following RULES.

When the North end of the needle has been moving Westerly, (as it has for about forty years), the present Bearings will be the *sums* of the change and the old Bearings which were North-Easterly or South-Westerly, and the *differences* of the change and the old Bearings which were North-Westerly or South-Easterly.

If the change have been *Easterly*, reverse the preceding rules, subtracting where it is directed to add, and adding where it is directed to subtract.

Run out the lines with the Bearings thus corrected.

It will be noticed that the process is precisely the reverse of that in Art. (**311**). The rules there given in more detail, may therefore be used; RULE 1, "when the Variation is West," being employed when the *change* has been a movement of the N. end of the needle to the East; and RULE 2, "when the Variation is East," being employed when the N. end of the needle has been moving to the West.

If the compass has a Vernier, it can be set for the change, once for all, precisely as directed in Art. (**312**), and then the courses can be run out as given in the deed, the correction being made by the instrument.

(322) Example. The following is a remarkable case which recently came before the Supreme Court of New-York. The North line of a large Estate was fixed by a royal grant, dated in 1704, as a due East and West line. It was run out in 1715, by a surveyor, whom we will call Mr. A. It was again surveyed in 1765, by Mr. B. who ran a course N. 87° 30′ E. It was run out for a third time in 1789, by Mr. C. who adopted the course N. 86° 18′ E. In 1845 it was surveyed for the fourth time by

Mr. D. with a course of N. 88° 30′ E. He found old "corners," and "blazes" of a former survey, on his line. They are also found on another line, South of his. Which of the preceding courses were correct, and where does the true line lie?

The question was investigated as follows. There were no old records of variation at the precise locality, but it lies between the lines of equal variation which pass through New-York and Boston, its distance from the Boston line being about twice its distance from the New-York line. The records of those two cities (referred to in Art. (**317**)) could therefore be used in the manner explained in Art. (**318**). For the later dates, observations at New-Haven could serve as a check. Combining all these, the author inferred the variation at the desired place to have been as follows:

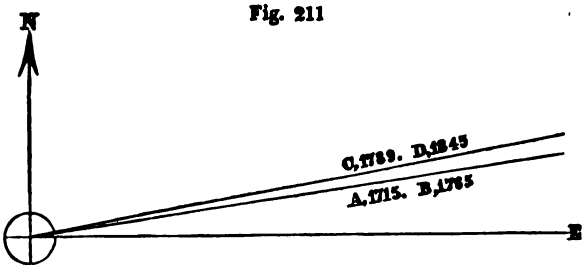
In 1715,	Variation	8° 02′	West.	
In 1765,	"	5° 32′	"	Decrease since 1715, 2° 30′.
In 1789,	"	5° 05′	"	Decrease since 1765, 0° 27′.
In 1845,	"	7° 23′	"	Increase since 1789, 2° 18′.

We are now prepared to examine the correctness of the allowances made by the old surveyors.

The course run by Mr. B. in 1765, N. 87° 30′ E. made an allowance of 2° 30′ as the decrease of variation, agreeing precisely with our calculation. The course of Mr. C. in 1789, N. 86° 18′ E., allowed a change of 1° 12′, which was wrong by our calculation, which gives only about 27′, and was deduced from three different records. Mr. D. in 1845, ran a course of N. 88° 30′ E, calling the increase of variation since 1789, 2° 12′. Our estimate was 2° 18′, the difference being comparatively small. Our conclusion then is this: the second surveyor retraced correctly the line of the first: the third surveyor ran out a *new* and incorrect line: and the fourth surveyor correctly retraced the line of the third, and found his marks, but this line was wrong originally and therefore wrong now. All the surveyors ran their lines on the supposition that the original "due East and West line" meant East and West as the needle pointed at that time.

The preponderance of the testimony as to old land marks agreed with the results of the above reasoning, and the decision of the court was in accordance therewith.

Fig. 211



In the above figure the horizontal and vertical lines represent true East and North lines; and the two upper lines running from left to right represent the two lines set out by the surveyors and in the years, there named.

(323) Remedy for the evils of the Secular change. The only complete remedy for the disputes, and the uncertainty of bounds, resulting from the continued change in the variation, is this. Let a Meridian, i. e. a true North and South line, be established in every town or county, by the authority of the State; monuments, such as stones set deep in the ground, being placed at each end of it. Let every surveyor be obliged by law to test his compass by this line, at least once in each year. This he could do as easily as in taking the Bearing of a fence, by setting his instrument on one monument, and sighting to a staff held on the other. Let the variation thus ascertained be inserted in the notes of the survey and recorded in the deed. Another surveyor, years or centuries afterwards, could test his compass by taking the Bearing of the same monuments, and the difference between this and the former Bearing would be the change of variation. He could thus determine with entire certainty the proper allowance to be made (as in Art. (**321**)) in order to retrace the original line, no matter how much, or how irregularly, the variation may have changed, or how badly adjusted was the compass of the original survey. Any permanent line employed in the same manner as the meridian line, would answer the same purpose, though less conveniently, and every surveyor should have such a line at least, for his own use.*

* This remedy seems to have been first suggested by Rittenhouse. It has since been recommended by T. Sopwith, in 1822; by E. F. Johnson, in 1831, and by W. Roberts, of Troy, in 1839. The errors of re-surveys, in which the change is neglected, were noticed in the "Philosophical Transactions," as long ago as 1679.

PART IV.

TRANSIT AND THEODOLITE SURVEYING:

By the Third Method.

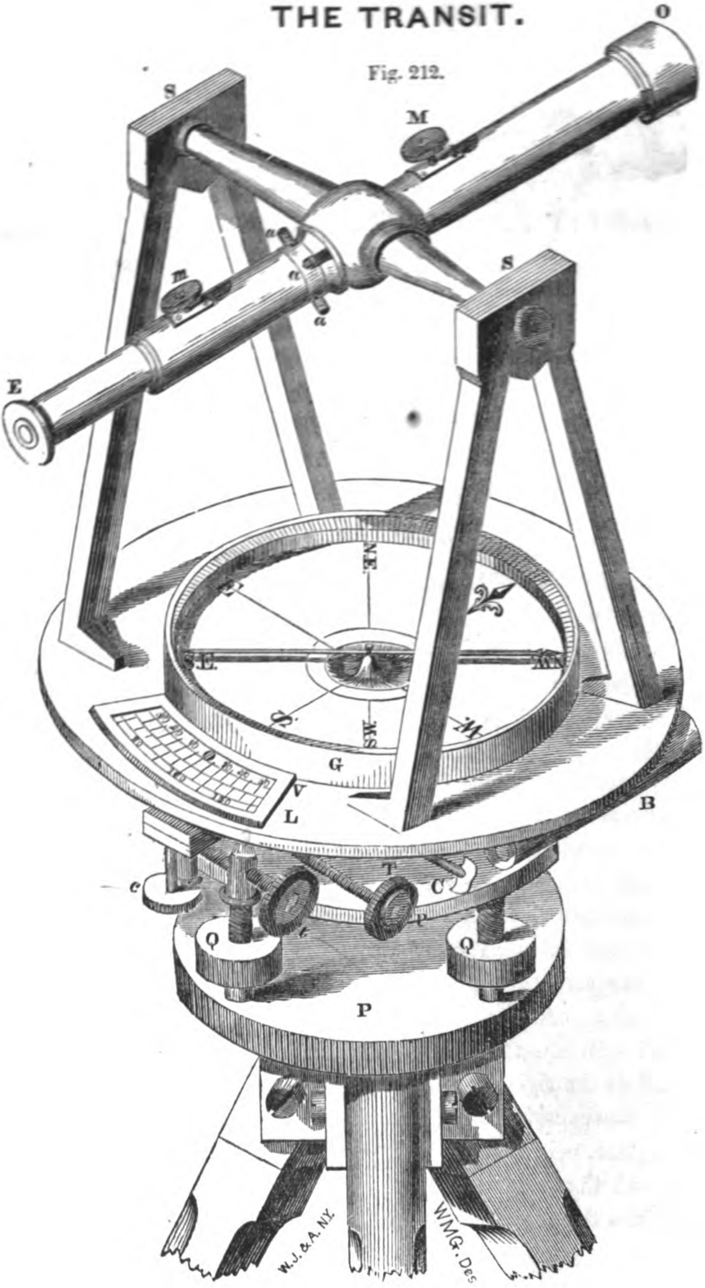
CHAPTER I.

THE INSTRUMENTS.

(324) THE TRANSIT and THE THEODOLITE (figures of which are given on the next two pages) are *Goniometers*, or Angle-Measurers. Each consists, essentially, of a circular plate of metal, supported in such a manner as to be horizontal, and divided on its outer circumference into degrees, and parts of degrees. Through the centre of this plate passes an upright axis, and on it is fixed a second circular plate, which nearly touches the first plate, and can turn freely around to the right and to the left. This second plate carries a Telescope, which rests on upright standards firmly fixed to the plate, and which can be pointed upwards and downwards. By the combination of this motion and that of the second plate around its axis, the Telescope can be directed to any object. The second plate has some mark on its edge, such as an arrow-head, which serves as a pointer or index for the divided circle, like the hand of a clock. When the Telescope is directed to one object, and then turned to the right or to the left, to some other object, this index which moves with it, and passes around the divided edge of the other plate, points out the arc passed over by this change of direction, and thus measures the angle made by the lines imagined to pass from the centre of the instrument to the two objects.

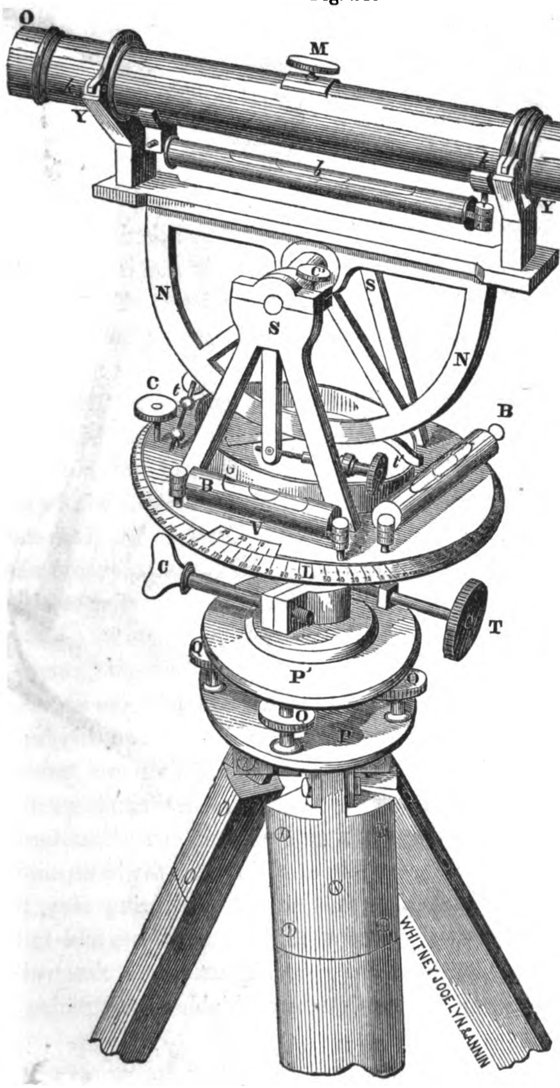
THE TRANSIT.

Fig. 212.



THE THEODOLITE.

Fig. 213



(325) Distinction. The preceding description applies to both the Transit and the Theodolite. But an essential difference between them is, that in the Transit the Telescope can turn completely over, so as to look both forward and backward, while in the Theodolite it cannot do so. Hence the name of the Transit.*

This capability of reversal enables a straight line to be prolonged from one end of it, or to be ranged out in both directions from any one point. The Telescope of the Theodolite can indeed be taken out of the Y shaped supports in which it rests, and be replaced end for end, but this operation is an imperfect substitute for the revolution of the Telescope of the Transit. So also is the turning half way around of the upper plate which carries the Telescope.

The Theodolite has a level attached to its Telescope, and a vertical circle for measuring vertical angles. The Transit does not usually have these, though they are sometimes added to it. The instrument may then be named a Transit-Theodolite. It then corresponds to the altitude and azimuth instrument of Astronomy. As the greater part of the points to be explained are common to both the Transit and the Theodolite, the descriptions to be given may be regarded as applicable to either of the instruments, except when the contrary is expressly stated, and some point peculiar to either is noticed.

(326) The great value of these instruments, and the accuracy of their measurements of angles are due chiefly to two things; to the *Telescope*, by which great precision in sighting to a point is obtained; and to the *Vernier Scale*, which enables minute portions of any arc to be read with ease and correctness. The former assists the eye in directing the line of sight, and the latter aids it in reading off the results. Arrangements for giving slow and steady motion to the movable parts of the instruments add to the value of the above. A contrivance for *Repeating* the observation of angles still farther lessens the unavoidable inaccuracies of these observations.

* It is sometimes called the "Engineers' Transit, or "Railroad Transit," to distinguish it from the Astronomical Transit-instrument.

The inaccurate division of the limb of the instrument is also averaged and thus diminished by the last arrangement. Its want of true "centring," is remedied by reading off on opposite sides of the circle.

Imperfections in the parallelism and perpendicularity of the parts of the instrument in which those qualities are required, are corrected by various "adjustments," made by the various screws whose heads appear in the engravings.

The arrangements for attaining all these objects render necessary the numerous parts and apparent complication of the instrument. But this complication disappears when each part is examined in turn, and its uses and relations to the rest are distinctly indicated. This we now propose to do, after explaining the engravings.

(327) In the figures of the instruments, given on pages 212 and 213, the same letters refer to both figures, so far as the parts are common to both.* L is the limb or divided circle. V is the index, or "Vernier," which moves around it. In the Transit, only a small portion of the divided limb is seen, the upper circle (which in it is the movable one) covering it completely, so that only a short piece of the arc is visible through an opening in the upper plate. S, S, are standards, fastened to the upper plate and supporting the telescope, EO. G is a compass-box, also fastened to the upper plate. *c* is a clamp-screw, which presses together the two plates, and prevents one from moving over the other. *t* is a tangent-screw, or slow-motion screw, which gives a slow and gentle motion to one plate over the other. C is a clamp-screw which fastens the lower plate to the body of the instrument, and thus prevents it from moving on its own axis. T is the tangent-screw to give this part a slow-motion. P and P′ are parallel plates through which pass four screws, Q, Q, Q, Q, by which the circular plate L is made level,

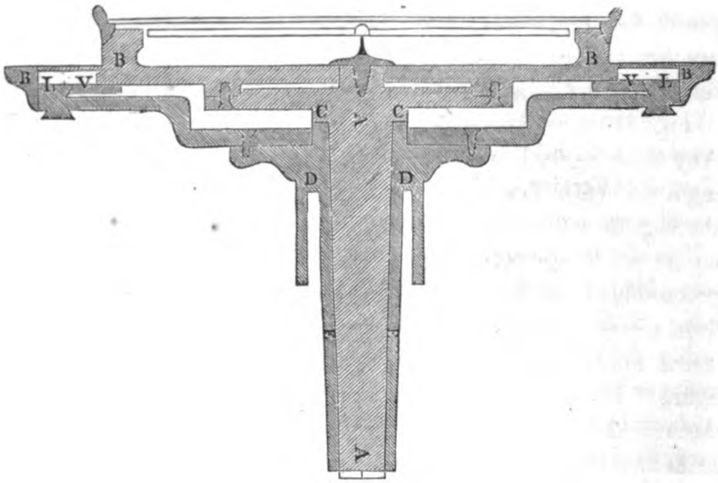
* The arrangements of these instruments are differently made by almost every maker; but any form of them being thoroughly understood, any new one will cause no difficulty. The figure of the Transit was drawn from one made by W. & L. E. Gurley, of Troy, N. Y. to the latter of whom the Author is indebted for some valuable information respecting the details of the instrument. The Theodolite is of the favorite English form.

as determined by the bubbles in the small spirit levels, B, B, of which there are two at right angles to each other.

In the figure of the *Theodolite*, the large level *b*, and the semicircle NN are for the purposes of Levelling, and of measuring Vertical angles. They will therefore not be described in this place.

(328) As the value of either of these instruments depends greatly on the accurate fitting and bearings of the two concentric vertical axes, and as their connection ought to be thoroughly understood, a vertical section through the body of the instrument is given in Fig. 214, to half the real size. The tapering spindle or

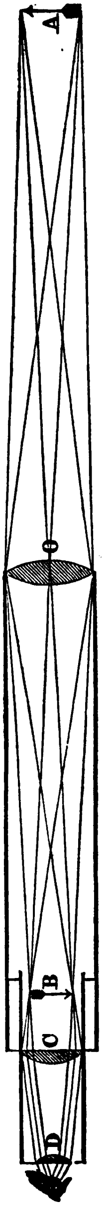
Fig. 214.



inverted frustum of a cone, marked AA, supports the upper plate BB, which carries the index, or Verniers, V, V, and the Telescope. The whole bearing of this plate is at C, C, on the top of the hollow inverted cone EE, in which the spindle turns freely, but steadily. This interior position of the bearings preserves them from dust and injury. This hollow cone carries the lower or graduated plate, and it can itself turn around on the bearings D, D, carrying with it the lower circle, and also the upper one and all above it.

The Vernier scales V, V, are attached to the upper plate, but lie in the same plane as the divisions L, L, of the lower plate, (so that the two can be viewed together, without parallax,) and are covered with glass, to exclude dust and moisture. In the figure the hatchings are drawn in different directions on the parts which move with the Vernier, and on those which move only with the limb.

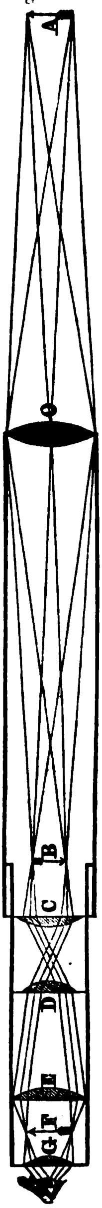
Fig. 215.



(329) The Telescope. This is a combination of lenses, placed in a tube, and so arranged, in accordance with the laws of optical science, that an image of any object to which the Telescope may be directed, is formed within the tube, (by the rays of light coming from the object and bent in passing through the object-glass) and there magnified by an Eye-glass, or Eye-piece, composed of several lenses. The arrangement of these lenses are very various. Those two combinations, which are preferred for surveying instruments, will be here explained.

Fig. 215 represents a Telescope which inverts objects. Any object is rendered visible by every point of it sending forth rays of light in every direction. In this figure, the highest and lowest points of the object, which here is an arrow, A, are alone considered. Those of the rays proceeding from them, which meet the object-glass, O, form a cone. The centre line of each cone, and its extreme upper and lower lines are alone shown in the figure. It will be seen that these rays, after passing through the object-glass, are refracted, or bent, by it, so as to cross one another, and thus to form at B an inverted image of the object. This would be rendered visible, if a piece of ground glass, or other semi-transparent substance, was placed at the point B, which is called the *focus* of the object-glass. The rays which form this image continue onward and pass through the two lenses C and D, which act like one magnifying glass, so that the rays, after being refracted by them, enter the eye at such angles as to form there a magnified and inverted image of the object. This combination of the two plano-convex lenses, C and D, is known as "Ramsden's Eye-piece."

Fig. 216.



This Telescope, inverting objects, shows them upside down, and the right side on the left. They can be shown erect by adding one or two more lenses as in the marginal figure. But as these lenses absorb light and lessen the distinctness of vision, the former arrangement is preferable for the glasses of a Transit or a Theodolite. A little practice makes it equally convenient for the observer, who soon becomes accustomed to seeing his flagmen standing on their heads, and soon learns to motion them to the right when he wishes them to go to the left, and *viçe versa.*

Figure 216 represents a Telescope which shows objects erect. Its eye-piece has four lenses. The eye-piece of the common terrestrial Telescope, or spy-glass, has three. Many other combinations may be used, all intended to show the object achromatically, or free from false coloring, but the one here shown is that most generally preferred at the present day. It will be seen that an inverted image of the object A, is formed at B, as before, but that the rays continuing onward are so refracted in passing through the lens C as to again cross, and thus, after farther refraction by the lenses D and E, to form, at F, an erect image, which is magnified by the lens G.

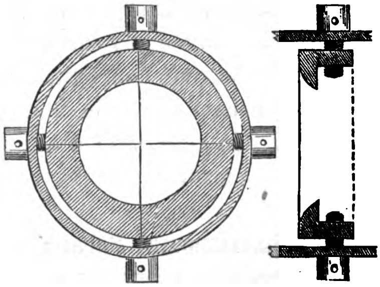
In both these figures, the limits of the page render it necessary to draw the angles of the rays very much out of proportion.

(330) Cross-hairs. Since a considerable field of view is seen in looking through the Telescope, it is necessary to provide means for directing the line of sight to the precise point which is to be observed. This could be effected by placing a very fine point, such as that of a needle, within the Telescope, at some place where it could be distinctly seen. In practice this fine point is obtained by the intersection of two very fine lines, placed in the common *focus* of the object-glass and of the eye-piece. These lines are called the *cross-hairs*, or *cross-wires*. Their intersection can be seen through the eye-piece, at the same time, and apparently at the same place, as the image of the distant object. The magnifying powers of the eye-piece will then detect the slightest deviation from perfect coincidence. "This application of the Telescope may be considered as completely annihilating that part of the error of observation which might otherwise arise from an erroneous estimation of the direction in which an object lies from the observer's eye, or from the centre of the instrument. It is, in fact, the grand source of all the precision of modern Astronomy, without which all other refinements in instrumental workmanship would be thrown away." What Sir John Herschel here says of its utility to Astronomy, is equally applicable to Surveying.

The imaginary line which passes through the intersection of the cross-hairs and the optical centre of the object-glass, is called the *line of collimation* of the Telescope.*

The cross-hairs are attached to a ring, or short thick tube of brass, placed within the Telescope tube, through holes in which pass loosely four screws, (their heads being seen at *a*, *a*, *a*, in Figs. 212 and 213), whose threads enter and take hold of the ring, behind or in front of the cross-hairs, as shown (in front view and in section) in the two figures in the margin. Their movements will be explained in Chapter III.

Fig. 217.



Usually, one cross-hair is horizontal, and the other vertical, as in Fig. 217, but sometimes they are arranged as in Fig. 218, which is thought to enable the object to be bisected with more precision. A horizontal hair is sometimes added.

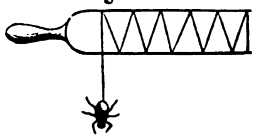
Fig. 218.

The cross-hairs are best made of platinum wire, drawn out very fine by being previously enclosed

* From the Latin word *Collimo*, or *Collineo*, meaning to direct one thing towards another in a straight line, or to aim at. The *line of aim* would express the meaning.

in a larger wire of silver, and the silver then removed by nitric acid. Silk threads from a cocoon are sometimes used. Spiders' threads are, however, the most usual. If a cross-hair is broken, the ring must be taken out by removing two opposite screws, and inserting a wire with a screw cut on its end, or a stick of suitable size, into one of the holes thus left open in the ring, it being turned sideways for that purpose, and then removing the other screws. The spider's threads are then stretched across the notches seen in the end of the ring, and are fastened by gum, or varnish, or bees-wax. The operation is a very delicate one. The following plan has been employed. A piece of wire is bent, as in the figure, so as to leave an opening a little wider than the ring of the cross-hairs. A cobweb is chosen, at the end of which a spider is hanging, and it is wound around the bent wire, as in the figure, the weight of the insect keeping it tight and stretching it ready for use, each part being made fast by gum, &c. When a cross-hair is wanted, one of these is laid across the ring and there attached. Another method is to draw the thread out of the spider, persuading him to spin, if he sulks, by tossing him from hand to hand. A stock of such threads must be obtained in warm weather for the winter's wants. A piece of thin glass, with a horizontal and a vertical line etched on it, may be made a substitute.

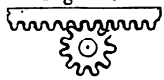
Fig. 219.



(331) Instrumental Parallax. This is an apparent movement of the cross-hairs about the object to which the line of sight is directed, taking place on any slight movement of the eye of the observer. It is caused by the image and the cross-hairs not being precisely in the common focus, or point of distinct vision of the eye-piece and the object glass. To correct it, move the eye-piece out or in till the cross-hairs are seen clearly and sharply defined against any white object. Then move the object glass in or out till the object is also distinctly seen. The cross-hairs will then seem to be fixed to the object, and no movement of the eye will cause them to appear to change their place.

(332) The milled-headed screw seen at M, passing into the telescope has a pinion at its other end entering a toothed rack, and is used to move the object glass, O, out and in, according as the object looked at is nearer or farther than the one last observed. Short distances require a long tube: long distances a short tube.

Fig. 220.



The eye-piece, E, is usually moved in and out by hand, but a similar arrangement to the preceding is a great improvement. This movement is necessary in order to obtain a distinct view of the cross-hairs. Short-sighted persons require the eye-piece to be pushed farther in than persons of ordinary sight, and old or long-sighted persons to have it drawn further out.

(333) Supports. The Telescope of the *Transit* is supported by a hollow axis at right angles to it, which itself rests at each end, on two upright pieces, or standards, spreading at their bases so as to increase their stability. In the *Theodolite*, the telescope rests at each end in forked supports, called Ys, from their shape. These Ys are themselves supported by a cross-bar, which is carried by an axis at right angles to it and to the telescope. This axis rests on standards similar to those of the Transit. The Telescope of the *Theodolite* can be taken out of the Ys, and turned "end for end." This is not usual in the Transit. Either of the above arrangements enables the Telescope to be raised or depressed so as to suit the height of the object to which it is directed. A telescope so disposed is called a "plunging telescope."

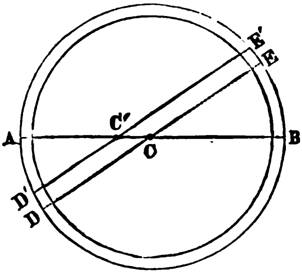
In some instruments there is an arrangement for raising or lowering one end of the axis. This is sometimes required for reasons to be given in connection with "Adjustments."

(334) The Indexes. The supports, or standards, of the telescope just described are attached to the upper, or index-carrying circle.* This, as has been stated, can turn freely on the lower or graduated circle, by means of its conical axis moving in the hollow conical axis of the latter circle. This upper circle carries the index, V,

* In some instruments this circle is the under one. In our figures it is the upper one, and we will therefore always speak of it as such.

which is an arrow-head or other mark on its edge, or the zero-point of a Vernier scale. There are usually two of these, situated exactly opposite to each other, or at the extremities of a diameter of the upper circle, so that the readings on the graduated circle pointed out by them differ, if both are correct, exactly 180°. The object of this arrangement is to correct any error of *eccentricity*, arising from the centre of the axis which carries the upper circle, (and with which it and its index pointers turn), not being precisely in the centre of the graduated circle. In the figure, let C be the true centre of the graduated circle, but C′ the centre on which the plate carrying the indexes turns. Let AC′B represent the direction of a sight taken to one object, and D′C′E′ the direction when turned to a second object. The angle subtended by the two objects at the centre of the instrument is required.

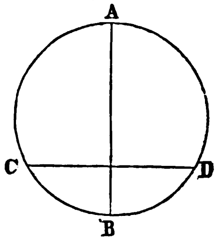
Fig. 221.



Let DE be a line passing through C, and parallel to D′E′. The angle ACD equals the required angle, which is therefore truly measured by the arc AD or BE. But if the arc shown by the index is read, it will be AD′ on one side, and BE′ on the other; the first being too small by the arc DD′ and the other too large by the equal arc EE′. If however the half-sum of the two arcs AD′ and BE′ be taken, it will equal the true arc, and therefore correctly measure the angle. Thus if AD′ was 19°, and BE′ 21°, their half sum, 20°, would be the correct angle.

Three indexes, 120° apart, are sometimes used. They have the advantage of *averaging* the unavoidable inaccuracies and inequalities of graduation on different parts of the limb, and thus diminishing their effect on the resulting angle.

Fig. 222.



Four were used on the large Theodolite of the English Ordnance Survey, two, A and B, opposite to each other, and two, C and D, 120° from A and from each other. The half-sum or arithmetical mean, of A and B was taken, then the mean of A, C, and D, and then the mean of these two means. But this was wrong, for it gave too great value to the reading of A, and also to B, though in a less degree; since the share of each Vernier in the final mean was as follows: A = 5, B = 3, C = 2, D = 2. This results from the expression for that mean, $= \frac{1}{2}\left(\frac{A + B}{2} + \frac{A + C + D}{3}\right) = \frac{1}{12}$ (5 A + 3 B + 2 C + 2 D).

(335) The graduated circle. This is divided into three hundred and sixty equal parts, or Degrees, and each of these is subdivided into two or three parts or more, according to the size of the instrument. In the first case, the smallest division on the circle will of course be 30′; in the second case 20′. More precise reading, to single minutes or even less, is effected by means of the Vernier of the index, all the varieties of which will be fully explained in the next chapter. The numbers run from 0° around to 360°, which number is necessarily at the same point as the 0, or *zero-point.** Each tenth degree is usually numbered, each fifth degree is distinguished by a longer line of division, and each degree-division line is longer than those of the sub-divisions. A magnifying glass is needed for reading the divisions with ease. In the Theodolite engraving this is shown at *m*. It should be attached to each Vernier.

(336) Movements. When the line of sight of the telescope is directed to a distant well-defined point, the unaided hand of the observer cannot move it with sufficient delicacy and precision to make the intersection of the cross hairs exactly cover or "bisect" that point. To effect this, a clamp, and a Tangent, or slow-motion, screw are required. This arrangement, as applied to the movement of the upper, or Vernier plate, consists of a short piece of brass, D, which is attached to the Vernier plate, and through which passes a long and fine-threaded "Tangent-screw," *t*. The other end of this screw enters into and carries the *clamp*. This consists of two pieces of brass, which, by turning the clamp-screw *c*, which passes through them on the outside, can be made to take

* In some instruments there is another concentric circle on which the degrees are also numbered from 0° to 90° as on the compass circle.

hold of and pinch tightly the edge of the lower circle, which lies between them on the inside. The upper circle is now prevented from moving on the lower one; for, the tangent-screw, passing through hollow screws in both the clamp and the piece D, keeps them at a fixed distance apart, so that they cannot move to or from one another, nor consequently the two circles to which they are respectively made fast. But when this tangent-screw is turned by its milled-head, it gives the clamp and with it the upper plate a smooth and slow motion, backward or forward, whence it is called the "Slow motion screw," as well as "Tangent-screw," from the direction in which it acts. It is always placed at the south end of the compass-box.

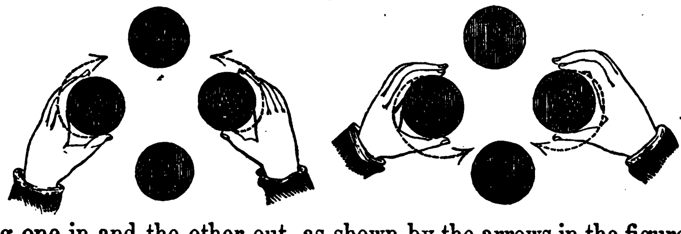
A little different arrangement is employed to give a similar motion to the lower circle (which we have hitherto regarded as immovable) on the body of the instrument. Its axis is embraced by a brass ring, into which enters another tangent-screw, which also passes through a piece fastened to the plate P. The clamp-screw, C, causes the ring to pinch and hold immovably the axis of the lower circle, while a turn of the Tangent-screw, T, will slowly move the clamp ring itself, and therefore with it the lower circle. When the clamp is loosened, the lower circle, and with it every thing above it, has a perfectly free motion. A recent improvement is the employment for this purpose of *two* tangent screws, pressing against opposite sides of a piece projecting from the clamp-ring. One is tightened as the other is loosened, and a very steady motion is thus obtained.

(337) Levels. Since the object of the instrument is to measure *horizontal* angles, the circular plate on which they are measured must itself be made horizontal. Whether it is so or not is known by means of two small levels placed on the plate at right angles to each other. Each consists of a glass tube, slightly curved upward in its middle and so nearly filled with alcohol, that only a small bubble of air is left in the tube. This always rises to the highest part of the tubes. They are so "adjusted" (as will be explained in chapter III) that when this bubble of air is in the middle of the tubes, or its ends equidistant from the central mark, the plate on which they are fastened shall be level, which way soever it may be turned.

The levels are represented in the figure of the Transit, on page 212, as being under the plate. They are sometimes placed above it. In that case, the Verniers are moved to one side, between the feet of the standards, and one of the levels is fixed between the standards above one of the Verniers, and the other on the plate at the south end of the compass-box.

(338) Parallel Plates. To raise or lower either side of the circle, so as to bring the bubbles into the centres of the tubes, requires more gentle and steady movements than the unaided hands can give, and is attained by the Parallel Plates P, P′, (so called because they are never parallel except by accident), and their four screws Q, Q, Q, Q, which hold the plates firmly apart, and, by being turned in or out, raise or lower one side or the other of the upper plate P′, and thereby of the graduated circle. The two plates are held together by a ball and socket joint. To level the instrument, loosen the lower clamp and turn the circle till each level is parallel to the vertical plane passing through a pair of opposite screws. Then take hold of two opposite screws and turn them simultaneously and equally, but in contrary directions, screw-

Fig. 223.



ing one in and the other out, as shown by the arrows in the figures. A rule easily remembered is that both thumbs must turn in, or both out. The movements represented in the first of these figures would raise the left-hand side of the circle and lower the right-hand side. The movements of the second figure would produce the reverse effect. Care is needed to turn the opposite screws equally, so that they shall not become so loose that the instrument will rock, or so tight as to be cramped. When this last occurs, one of the other pair should be loosened.

Sometimes one of each pair of the screws is replaced by a strong spring against which the remaining screws act.

The French and German instruments are usually supported by only three screws. In such cases, one level is brought parallel to one pair of screws and levelled by them, and the other level has its bubble brought to its centre by the third screw. If there is only one level on the instrument, it is first brought parallel to one pair of screws and levelled, and is then turned one quarter around so as to be perpendicular to them and over the third screw, and the operation is repeated.

(339) Watch Telescope. A second Telescope is sometimes attached to the lower part of the instrument. When a number of angles are to be observed from any one station, direct the upper and principal Telescope to the first object, and then direct the lower one to any other well-defined point. Then make all the desired observations with the upper Telescope, and when they are finished, look again through the lower one, to see that it and therefore the divided circle has not been moved by the movements of the Vernier plate. The French call this the *Witness Telescope*, (*Lunette temoin*).

(340) The Compass. Upon the upper plate is fixed a compass. Its use has been fully explained in Part III. It is little used in connection with the Transit or Theodolite, which are so incomparably more accurate, except as a "check," or rough test of the accuracy of the angles taken, which should about equal the difference of the magnetic bearings. Its use will be farther noticed in Chapter IV, on "Field Work."

The name of "Surveyor's Transit" has been given to a modification of the "Engineer's Transit," enabling the compass to be set so as to run lines with an allowance for the magnetic variation, as was explained in Art. (**312**). This cannot be done in the instrument which has been described.

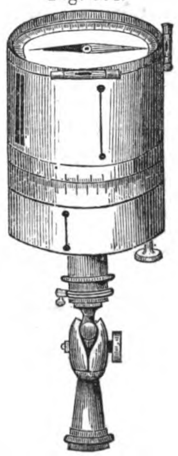
(341) Theodolites. The distinctive peculiarities of the ordinary Theodolite have been explained in connection with those of the Transit. Some modifications will now be described.

Two Telescopes, one exactly under the other, so that their lines of collimation are in the same vertical plane, have been attached to a Theodolite, for the purpose of ranging a straight line forward and backward, as a substitute for the much more elegant operation of reversion as in the Transit.

A remarkable arrangement, made by Gurley, of Troy, had an object-glass, an eye-glass, and a pair of cross-hairs *at each end* of the instrument, the eye glass being set in a skeleton frame so as to obstruct the light as little as possible. The focus of each glass was so adjusted that the Telescope could be looked through in either direction.

Two Telescopes, mounted one above the other, and one attached to the graduated circle and the other to the Vernier plate, have also been used. With these the angle subtended at the instrument, can be measured at once, each object having a Telescope directed to it at the same moment, thus avoiding any danger of slipping, and rendering unnecessary a Watch Telescope, for which one of these Telescopes serves as a more perfect substitute.

Fig. 224.



(342) Goniasmometre. A very compact instrument to which the above name has been given in France, where it is much used, is shown in the figure. The upper half of the cylinder is movable on its lower half. The observations may be taken through the slits, as in the Surveyor's Cross, or a Telescope may be added to it. Readings may be taken both from the compass, and from the divided edge of the lower half of the cylinder, by means of a Vernier on the upper half.*

* The proper care of instruments must not be overlooked. If varnished, they should be wiped gently with fine and clean linen. If polished with oil, they should be rubbed more strongly. The parts neither varnished nor oiled, should be cleaned with Spanish white and alcohol. Varnished wood, when spotted, should be wiped with very soft linen, moistened with a little olive oil or alcohol. Unpainted wood is cleaned with sand-paper. Apply olive oil where steel rubs against brass; and wax softened by tallow where brass rubs against brass.—Clean the glasses with kid or buck skin. Wash them, if dirtied, with alcohol.

CHAPTER II.

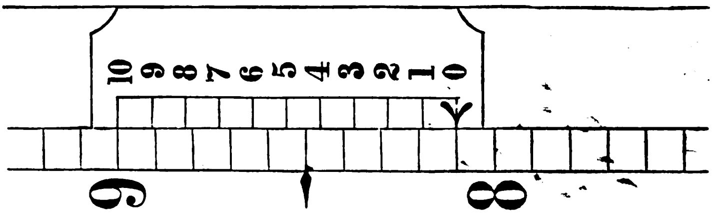
VERNIERS.

(343) Definition. A Vernier is a contrivance for measuring smaller portions of space than those into which a line is actually divided. It consists of a second line or scale, movable by the side of the first, and divided into equal parts, which are a very little shorter or longer than the parts into which the first line is divided. This small difference is the space which we are thus enabled to measure.*

The Vernier scale is usually constructed by taking a length equal to any number of parts on the divided line, and then dividing this length into a number of equal parts, one more or one less than the number into which the same length on the original line is divided.

(344) Illustration. The figure represents (to twice the real size) a scale of inches divided into tenths, with a Vernier scale beside it, by which hundredths of an inch can be measured. The

Fig. 225.

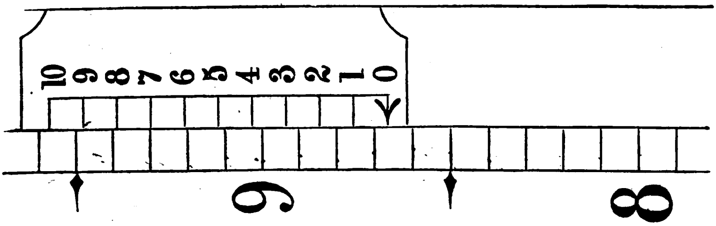


Vernier is made by setting off on it 9 tenths of an inch, and dividing that length into 10 equal parts. Each space on the Vernier is therefore equal to a tenth of nine-tenths of an inch, or to nine-hundredths of an inch, and is consequently one-hundredth of an inch shorter than one of the divisions of the original scale. The

* The Vernier is so named from its inventor, in 1631. The name "Nonius," often improperly given to it, belongs to an entirely different contrivance for a similar object.

first space of the Vernier will therefore fall short of, or be overlapped by, the first space on the scale by this one-hundredth of an inch; the second space of the Vernier will fall short by two-hundredths of an inch; and so on. If then the Vernier be moved up by the side of the original scale, so that the line marked 1 coincides, or forms one straight line, with the line of the scale which was just above it, we know that the Vernier has been moved one-hundredth of an inch. If the line marked 2 comes to coincide with a line of the scale, the Vernier has moved up two-hundredths of an inch; and so for other numbers. If the position of the

Fig. 226.



Vernier be as in this figure, the line marked 7 on the Vernier corresponding with some line on the scale, the zero line of the Vernier is 7 hundredths of an inch above the division of the scale next below this zero line. If this division be, as in the figure, 8 inches and 6 tenths, the reading will be 8.67 inches.*

A Vernier like this is used on some levelling rods, being engraved on the sides of the opening in the part of the target above its middle line. The rod being divided into hundredths of a foot, this Vernier reads to thousandths of a foot. It is also used on some French Mountain Barometers, which are divided to hundredths of a *metre*, and thus read to thousandths of that unit.

(345) General rules. *To find what any Vernier reads to,* i. e. to determine how small a distance it can measure, observe how many parts on the original line are equal to the same number increased or diminished by one on the Vernier, and divide the

* The student will do well to draw such a scale and Vernier on two slips of thick paper, and move one beside the other till he can read them in any possible position; and so with the following Verniers.

length of a part on the original line by this last number. It will give the required distance.*

To read any Vernier, firstly, look at the zero line of the Vernier, (which is sometimes marked by an arrow-head), and if it coincides with any division of the scale, that will be the correct reading, and the Vernier divisions are not needed. But if, as usually happens, the zero line of the Vernier comes between any two divisions of the scale, note the nearest next less division on the scale, and then look along the Vernier till you come to some line on it which exactly coincides, or forms a straight line, with some line (no matter which) on the fixed scale. The number of this line *on the Vernier* (the 7th in the last figure) tells that so many of the sub-divisions which the Vernier indicates, are to be added to the reading of the entire divisions on the scale.

When several lines on the Vernier appear to coincide equally with lines of the scale, take the middle line.

When no line coincides, but one line on the Vernier is on one side of a line on the scale, and the next line on the Vernier is as far on the other side of it, the true reading is midway between those indicated by these two lines.

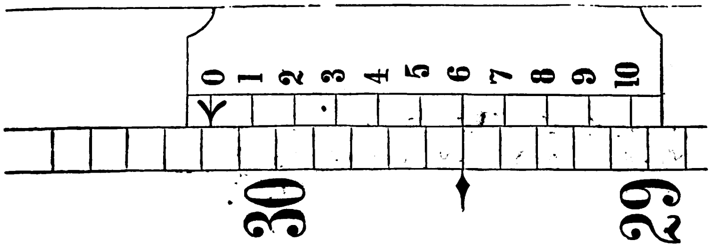
(346) Retrograde Verniers. The spaces of the Vernier in modern instruments, are usually each shorter than those on the scale, a certain number of parts on the scale being divided into a larger number of parts on the Vernier.† In the contrary case,‡ there is the inconvenience of being obliged to number the lines of the Vernier and to count their coincidences with the lines of the scale, in a retrograde or contrary direction to that in which the numbers on the scale run. We will call such arrangements *retrograde* Verniers.

* In Algebraic language, let s equal the length of one part on the original line, and v the unknown length of one part on the Vernier. Let m of the former = $m+1$ of the latter. Then $ms = (m+1)\,v$. $v = \frac{m}{m+1}s$. $s - v = s - \frac{m}{m+1}s = \frac{s}{m+1}$. If $ms = (m-1)\,v$, then $v - s = \frac{s}{m-1}$.

† i. e. Algebraically, $v = \frac{m}{m+1}\,s$, ‡ i. e. When $v = \frac{m}{m-1}s$.

(347) Illustration. In this figure, the scale, as before, represents (to twice the real size) inches divided into tenths, but the Vernier is made by dividing 11 parts of the scale into 10 equal

Fig. 227.

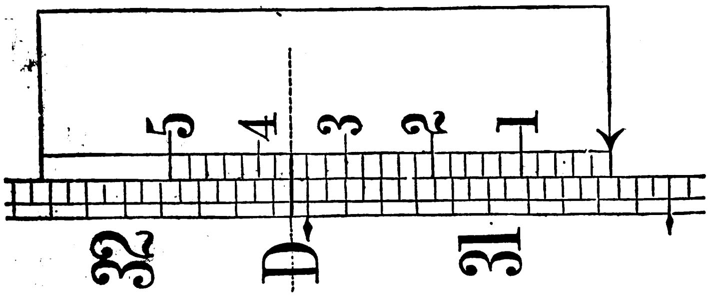


parts, each of which is therefore one-tenth of eleven-tenths of an inch, i. e. eleven-hundredths of an inch, or a tenth and a hundredth. Each space of the Vernier therefore overlaps a space on the scale by one-hundredth of an inch. The manner of reading this Vernier is the same as in the last one, except that the numbers run in a reverse direction. The reading of the figure is 30.16.

This Vernier is the one generally applied to the common Barometer, the zero point of the Vernier being brought to the level of the top of the mercury, whose height it then measures. It is also employed for levelling rods which read downwards from the middle of the target.

(348) The figure below represents (to double size) the usual scale of the English Mountain Barometer.* The scale is first divided into inches. These are subdivided into tenths by the

Fig. 228.



* This figure, and others in this chapter, are from Bree's "Present Practice."

longer lines, and the shorter lines again divide these into half-tenths, or to 5 hundredths. 24 of these smaller parts are set off on the Vernier, and divided into 25 equal parts, each of which is therefore $= \frac{24 \times .05}{25} = .048$ inch, and is shorter than a division of the scale by $.050 - .048 = .002$, or two thousandths of an inch, a twenty-fifth part of a division on the scale, to which minuteness the Vernier can therefore read. The reading in the figure is 30.686, (30.65 by the scale and .036 by the Vernier), the dotted line marked D showing where the coincidence takes place.

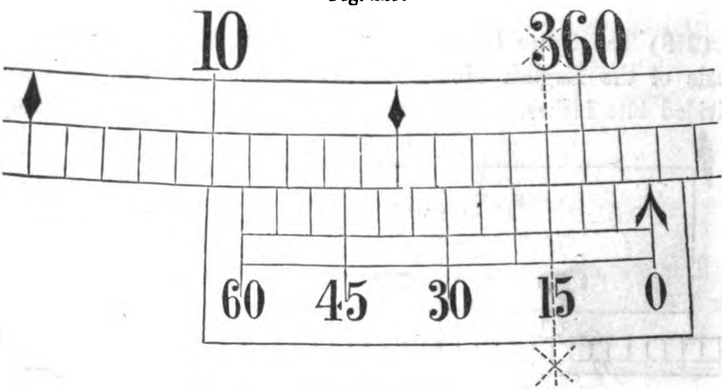
Circle divided into degrees. The following illustrations apply to the measurements of angles, the circle being variously divided. In this article, the circle is supposed to be divided into degrees.

If 6 spaces on the Vernier are found to be equal to 5 on the circle, the Vernier can read to one-sixth of a space on the circle, i. e. to 10′.

If 10 spaces on the Vernier are equal to 9 on the circle, the Vernier can read to one-tenth of a space on the circle, i. e. to 6′.

If 12 spaces on the Vernier are equal to 11 on the circle, the Vernier can read to one-twelfth of a space on the circle, i. e. to 5′.

Fig. 229.



The above figure shows such an arrangement. The index, or zero, of the Vernier is at a point beyond 358°, a certain distance, which the coincidence of the third line of the Vernier (as indicated by the dotted and crossed line) shows to be 15′. The whole reading is therefore 358° 15′.

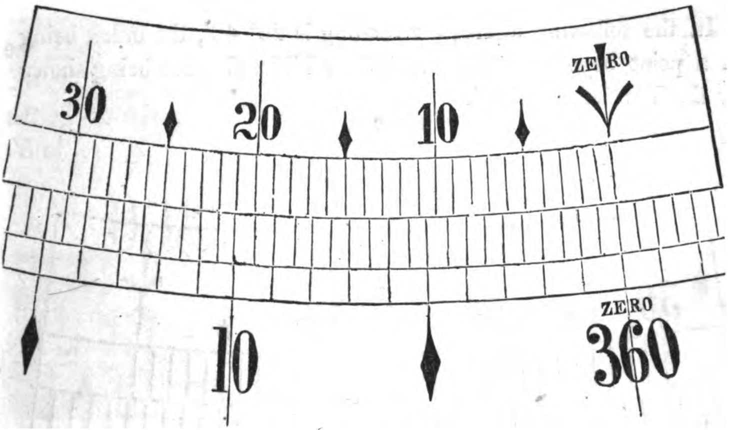
If 20 spaces on the Vernier are equal to 19 on the circle, the Vernier can read to one-twentieth of a division on the circle, i. e. to 3′. English compasses, or "Circumferentors," are sometimes thus arranged.

If 60 spaces on the Vernier are equal to 59 on the circle, the Vernier can read to one-sixtieth of a division on the circle, i. e. to 1′.

(350) Circle divided to 30′. Such a graduation is a very common one. The Vernier may be variously constructed.

Suppose 30 spaces on the Vernier to be equal to 29 on the circle. Each space on the Vernier will be $= \frac{29 \times 30'}{30} = 29'$, and will therefore be less than a space of the circle by 1′, to which the Vernier will then read.

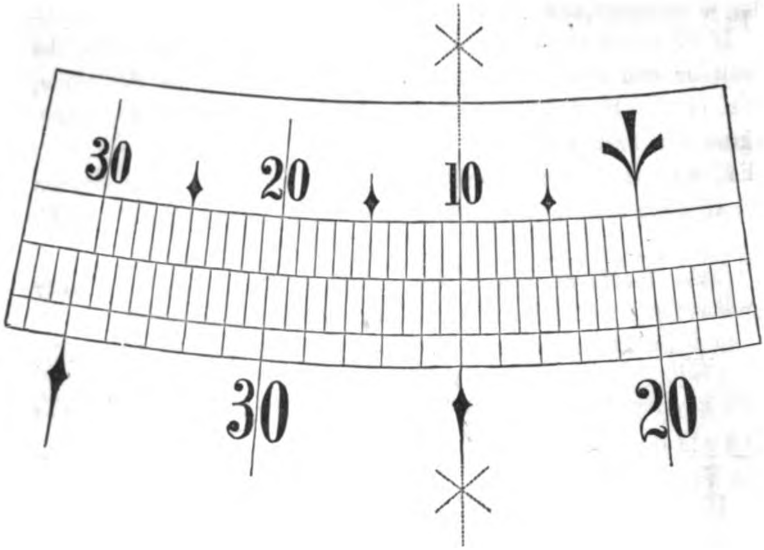
Fig. 230.



The above figure shows this arrangement. The reading is 0°, or 360°.

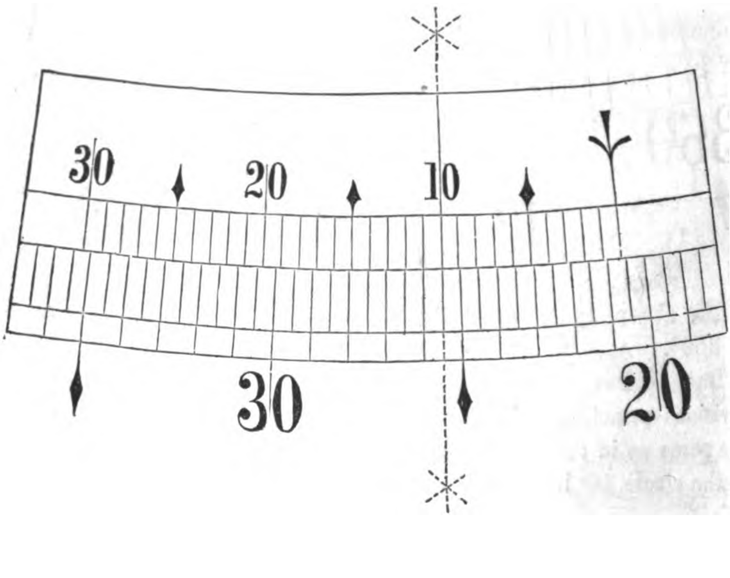
In the following figure, the dotted and crossed line shows what divisions coincide, and the reading is 20° 10′; the Vernier being the same as in the preceding figure, and its zero being at a point of the circle 10′ beyond 20°.

Fig. 231.



In the following figure, the reading is 20° 40′, the index being at a point beyond 20° 30′, and the additional space being shown by the Vernier to be 10′.

Fig. 232.



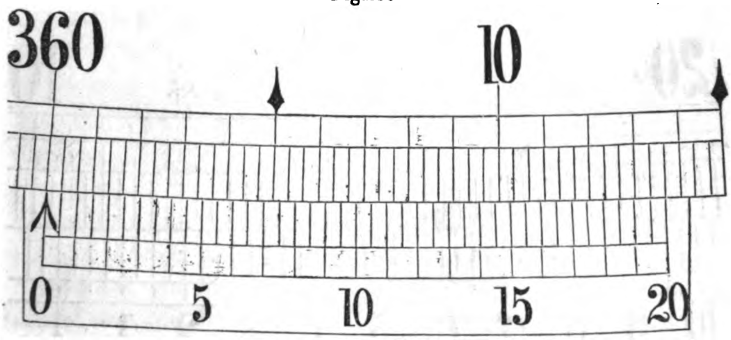
Sometimes 30 spaces on the Vernier are equal to 31 on the circle. Each space on the Vernier will therefore be $= \frac{31 \times 30'}{30} = 31'$, and will be longer than a space on the circle by $1'$, to which it will therefore read, as in the last case, but the Vernier will be "retrograde." This is the Vernier of the compass, Fig. 148. The peculiar manner in which it is there applied is shown in Fig. 239.

If 15 spaces on the Vernier are equal to 16 on the circle, each space on the Vernier will be $= \frac{16 \times 30'}{15} = 32'$, and the Vernier will therefore read to $2'$.

(351) Circle divided to 20'. If 20 spaces on the Vernier are equal to 19 on the circle, each space of the latter will be $= \frac{19 \times 20'}{20} = 19'$, and the Vernier will read to $20' - 19' = 1'$.

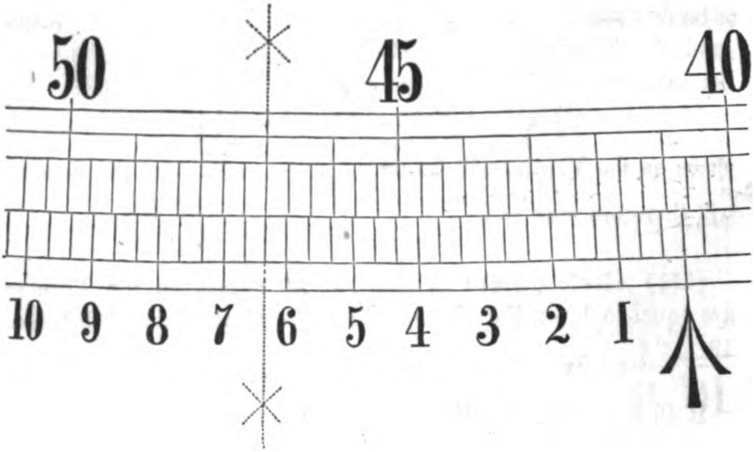
If 40 spaces on the Vernier are equal to 41 on the circle, each space on the Vernier will be $= \frac{41 \times 20''}{40} = 20\frac{1}{2}'$; and the Vernier will therefore read to $20\frac{1}{2}' - 20' = 30''$. It will be retrograde. In the following figure the reading is 360°, or 0°; and it will be seen that the 40 spaces on the Vernier (numbered to whole minutes) are equal to 13° 40′ on the limb, i. e. to 41 spaces, each of 20′.

Fig.233.



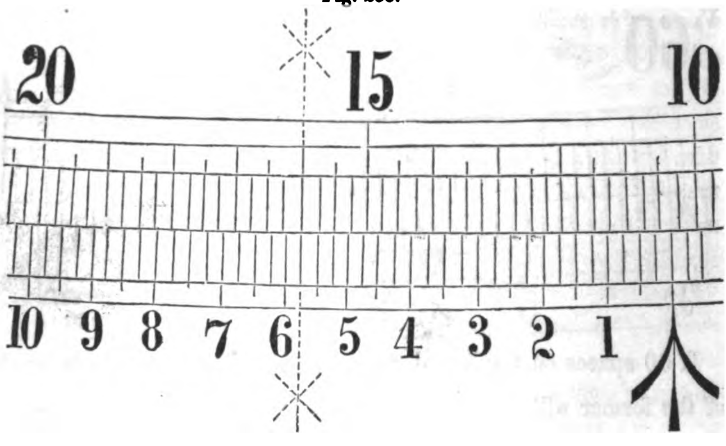
If 60 spaces on the Vernier are equal to 59 on the circle, each of the former will be $= \frac{59 \times 20'}{60} = 19'\ 40''$, and the Vernier will therefore read to 20′ — 19′ 40″ = 20″. The following figure shows such an arrangement. The reading in that position would be 40° 46′ 20″.

Fig. 234.



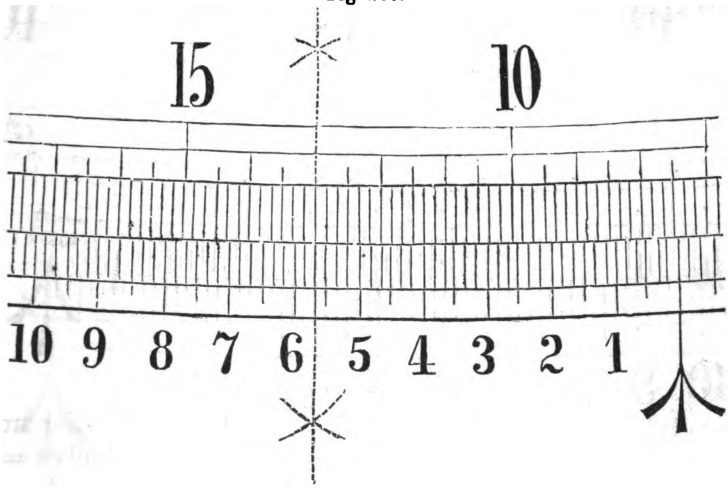
(352) **Circle divided to 15′.** If 60 spaces on the Vernier are equal to 59 on the circle, each space on the Vernier will be $= \frac{59 \times 15'}{60} = 14' \, 45''$, and the Vernier will read to 15″. In the following figure the reading is 10° 20′ 45″, the index pointing to 10° 15′, and something more, which the Vernier shows to be 5′ 45″.

Fig. 235.



(353) Circle divided to 10′. If 60 spaces on the Vernier be equal to 59 on the limb, the Vernier will read to 10″. In the following figure, the reading is 7° 25′ 40″, the reading on the circle being 7° 20′, and the Vernier showing the remaining space to be 5′ 40″.

Fig. 236.



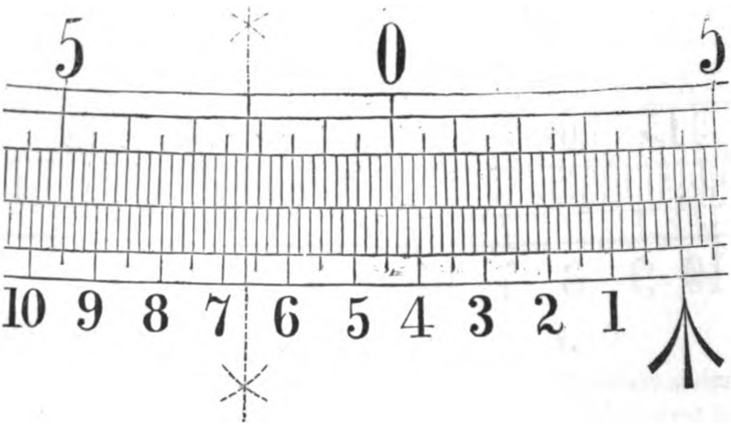
(354) Reading backwards. When an index carrying a Vernier is moved backwards, or in a contrary direction to that in which the numbers on the circle run, if we wish to read the space which it has passed over in this direction from the zero point, the Vernier must be read backwards, (i. e. the highest number be called 0), or its actual reading must be subtracted from the value of the smallest space on the circle. The reason is plain; for, since the Vernier shows how far the index, moving in one direction, has gone past one division line, the distance which it is from the next division line (which it may be supposed to have passed, moving in a contrary direction), will be the difference between the reading and the value of one space.

Thus, in Fig. 229, page 232, the reading is 358° 15′. But, counting backwards from the 360°, or zero point, it is 1° 45′.

Caution on this point is particularly necessary in using small angles of deflection for railroad curves.

(355) Arc of excess. On the sextant and similar instruments, the divisions of the limb are carried onward a short distance beyond the zero point. This portion of the limb is called the "Arc of excess." When the index of the Vernier points to this arc, the reading must be made as explained in the last article. Thus, in the figure, the reading on the arc from the zero of the limb to the

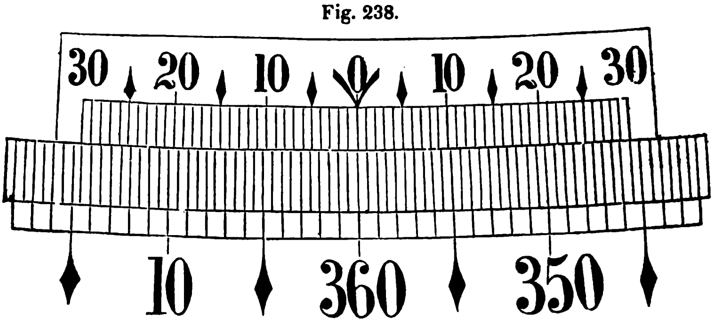
Fig. 237



zero of the Vernier is 4° 20′, and something more, and the reading of the Vernier from 10 towards to the right, where the lines coincide, is 3′ 20″, (or it is 10′ — 6′ 40″ = 3′ 20″), and the entire reading is therefore 4° 23′ 20″.

(356) Double Verniers. To avoid the inconveniences of reading backwards, double Verniers are sometimes used. The figure below shows one applied to a Transit. Each of the Verniers is

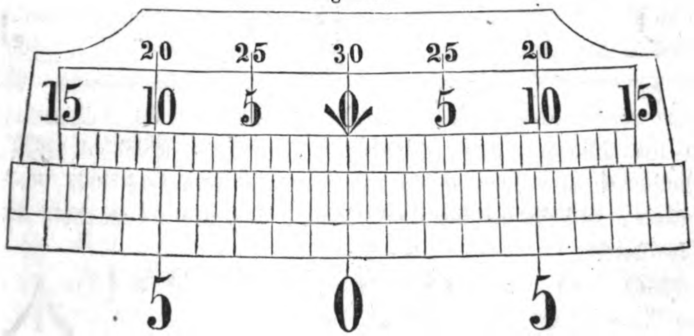
Fig. 238.



like the one described in Art. (**350**), Figs. 230, 231, and 232. When the degrees are counted to the left, or as the numbers run, as is usual, the left-hand Vernier is to be read, as in Art. (**350**); but when the degrees are counted to the right, from the 360° line, the right-hand Vernier is to be used.

(357) Compass-Vernier. Another form of double Vernier, often applied to the compass, is shown in the following figure. The

Fig. 239.



limb is divided to half degrees, and the Vernier reads to minutes, 30 parts on it being equal to 31 on the limb. But the Vernier is only half as long as in the previous case, going only to 15′, the upper figures on one half being a sort of continuation of the lower figures on the other half. Thus in moving the index to the right, read the *lower* figures on the left hand Vernier (it being retrograde) at any coincidence, when the space passed over is less than 15′; but if it be more, read the *upper* figures on the right hand Vernier: and *vice versa.*

CHAPTER III.

ADJUSTMENTS.

(358) The purposes for which the Transit and Theodolite (as well as most surveying and astronomical instruments) are to be used, require and presuppose certain parts and lines of the instrument to be placed in certain directions with respect to others; these respective directions being usually parallel or perpendicular. Such arrangements of their parts are called their *Adjustments*. The same word is also applied to placing these lines in these directions. In the following explanations the operations which determine whether these adjustments are correct, will be called their *Verifications;* and the making them right, if they are not so, their *Rectifications.**

(359) In observations of horizontal angles with the Transit or the Theodolite,† it is required,

1° That the circular plates shall be horizontal in whatever way they may be turned around.

2° That the Telescope, when pointed forward, shall look in precisely the reverse of its direction when pointed backward, i. e. that its two lines of sight (or lines of collimation) forward and backward shall lie in the same plane.

3° That the Telescope in turning upward or downward, shall move in a truly vertical plane, in order that the angle measured between a low object and a high one, may be precisely the same as would be the angle measured between the low object and a point exactly under the high object, and in the same horizontal plane as the low one.

* It has been well said, that "in the present state of science it may be laid down as a maxim, that every instrument should be so contrived, that the observer may easily examine and rectify the principal parts; for, however careful the instrument-maker may be, however perfect the execution thereof, it is not possible that any instrument should long remain accurately fixed in the position in which it came out of the maker's hands."—*Adams*' "*Geometrical and Graphical Essays*," 1791.

† The Theodolite adjustments, which relate only to levelling, or to measuring vertical angles, will not be here discussed.

We shall see that all these adjustments are finally resolvable into these; 1st. Making the vertical axis of the instrument perpendicular to the plane of the levels; 2d. Making the line of collimation perpendicular to its axis; and 3d. Making this axis parallel to the plane of the levels. They are all best tested by the invaluable principle of "Reversion."

We have now, firstly, to examine whether these things *are* so, that is, to "verify" the adjustments; and, secondly, if we find that they are not so, to *make* them so, i. e. to "rectify," or "adjust" them correctly. The above three requirements produce as many corresponding adjustments.

(360) First adjustment. *To cause the circle to be horizontal in every position.**

Verification.—Turn the Vernier plate which carries the levels, till one of them is parallel to one pair of the parallel plate screws. The other will then be parallel to the other pair. Bring each bubble to the middle of its tube, by that pair of screws to which it is parallel. Then turn the vernier plate half way around, i. e. till the index has passed over 180°. If the bubbles remain in the centres of the tubes, they are in adjustment. If either of them runs to one end of the tube, it requires rectification.

Rectification.—The fault which is to be rectified is that the plane of the level (i. e. the plane tangent to the highest point of the level tube) is not perpendicular to the vertical axis, AA in figure 214, on which the plate turns. For, let AB represent this

Fig. 240.

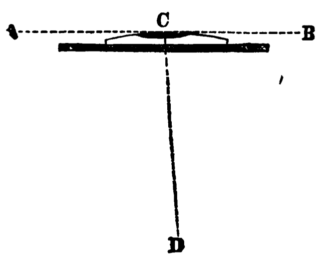
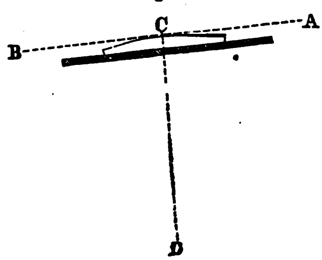


Fig. 241.



plane, seen edgeways, and CD the centre line of the vertical axis,

* This applies equally to the Transit and the Theodolite.

which is here drawn as making an acute angle with this plane on the right hand side. The first figure represents the bubble brought to the centre of the tube. The second figure represents the plate turned half around. The centre line of the axis is supposed to remain unmoved. The acute angle will now be on the left hand side, and the plate will no longer be horizontal. Consequently the bubble will run to the higher end of the tube. The rectification necessary is evidently to raise one end of the tube and lower the other. The real error has been doubled to the eye by the reversion. Half of the motion of the bubble was caused by the tangent plane not being perpendicular to the axis, and half by this axis not being vertical. Therefore raise or lower one end of the level by the screws which fasten it to the plate, till the bubble comes about *half way* back to the centre, and then bring it quite back by turning its pair of parallel plate screws. Then again reverse the vernier plate 180°. The bubble should now remain in the centre. If not, the operation should be repeated. The same must be done with the other level if required. Then the bubbles will remain in the centre during a complete revolution. This proves that the axis of the vernier plate is then vertical; and as it has been fixed by the maker perpendicular to the plate, the latter must then be horizontal.

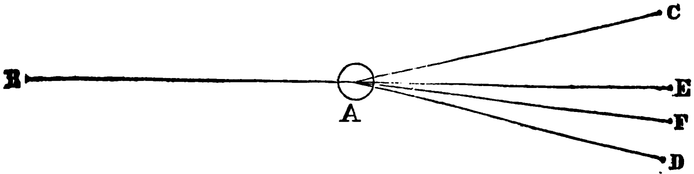
It is also necessary to examine whether the bubbles remain in the centre, when the divided circle is turned round on its axis. If not, the axes of the two plates are not parallel to each other. The defect can be remedied only by the maker; for if the bubbles be altered so as to be right for this reversal, they will be wrong for the vernier plate reversal.

(361) Second adjustment. *To cause the line of collimation to revolve in a plane.**

Verification. Set up the Transit in the middle of a level piece of ground, as at A in the figure. Level it carefully. Set a stake, with a nail driven into its head, or a chain pin, as far from the instrument as it is distinctly visible, as at B. Direct the telescope

* This adjustment is not the same in the Transit and in the Theodolite. That for the Transit will be first given, and that for the Theodolite in the next article.

Fig. 242.



to it, and fix the intersection of the cross-hairs very precisely upon it. Clamp the instrument. Measure from A to B. Then turn over the telescope, and set another stake at an equal distance from the Transit, and also precisely in the line of sight. If the line of collimation has *not* continued in the same plane during its half-revolution, this stake will not be at E, but to one side, as at C. To discover the truth, loosen the clamp and turn the vernier plate half around without touching the telescope. Sight to B, as at first, and again clamp it. Then turn over the telescope, and the line of sight will strike, as at D in the figure, as far to the right of the point, as it did before to its left.

Rectification. The fault which is to be rectified, is that the line of collimation of the telescope is not perpendicular to the horizontal axis on which the telescope revolves. This will be seen by the figures, which represent the position of the lines in each of the four

Fig. 243.

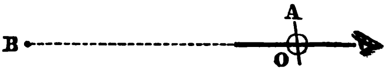


Fig. 244.

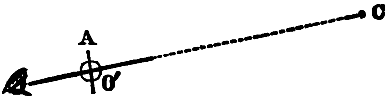


Fig. 245.

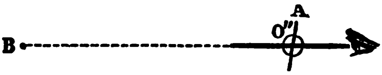
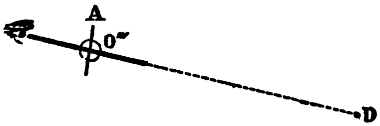


Fig. 246.



observations which have been made. In each of the figures the long thick line represents the telescope, and the short one the axis on which it turns. In Fig. 243 the line of sight is directed to B

In Fig. 244 the telescope has been turned over, and with it the axis, so that the obtuse angle, marked O in the first figure, has taken the place, O′, of the acute angle, and the telescope points to C instead of to E. In Fig. 245 the vernier plate has been turned half around so as to point to B again, and the same obtuse angle has got around to O″. In Fig. 246 the telescope has been turned over, the obtuse angle is at O‴, and the telescope now points to D.

To make the line of collimation perpendicular to the axis, the former must have its direction changed. This is effected by moving the vertical hair the proper distance to one side. As was explained in Art. (330), and represented in Fig. 217, the cross-hairs are on a ring held by four screws. By loosening the left-hand screw and tightening the right-hand one, the ring, and with it the cross-hairs, will be drawn to the right; and *vice versa*. Two holes at right angles to each other pass through the outer heads of the screws. Into these holes a stout steel wire is inserted, and the screws can thus be turned around. Screws so made are called "capstan-headed." One of the other pair of screws may need to be loosened to avoid straining the threads. In some French instruments, one of each pair of screws is replaced by a spring.

To find how much to move this vertical hair, measure from C to D, Fig. 242, page 243. Set a stake at the middle point E, and set another at the point F, midway between D and E. Move the vertical hair till the line of sight strikes F. Then the instrument is adjusted; and if the line of sight be now directed to E, it will strike B, when the telescope is turned over; since the hair is moved half of the doubled error, DE. The operation will generally require to be repeated, not being quite perfected at first.
generally require to be repeated, not being quite perfected at first.

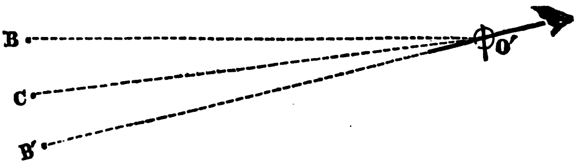
It should be remembered, that if the Telescope used does not invert objects, its eye-piece will do so. Consequently, with such a telescope, if it seems that the vertical hair should be moved to the left, it must be moved to the right, and *vice versa*. An inverting telescope does not invert the cross-hairs.

If the young surveyor has any doubts as to the perfection of his rectification, he may set another stake exactly under the instrument by means of a plumb-line suspended from its centre; and then, in like manner, set his Transit over B or E. He will find that the other two stakes, A and the extreme one, *are* in the same straight line with his instrument.

In some instruments, the horizontal axis of the telescope can be taken out of its supports, and turned over, end for end. In such a case, the line of sight may be directed to any well defined point, and the axis then taken out and turned over. If the line of sight again strikes the same point, this line is perpendicular to the axis. If not, the apparent error is double the real error, as appears from the figures, the obtuse angle O coming to O′, and the desired per-

Fig. 247.

Fig. 248.



pendicular line falling at C midway between B and B′. The rectification may be made as before; or, in some large instruments, in which the telescope is supported on Ys, by moving one of the Ys laterally.

(362) The Theodolite must be treated differently, since its telescope does not reverse. One substitute for this reversal, when it is desired to range out a line forward and backward from one station, is, after sighting in one direction, to take the telescope out of the Ys and turn it end for end, to sight in the reverse direction. This it can be made to do by adjusting its line of collimation as explained in the last article. Another substitute is, after sighting in one direction, and noting the reading, to turn the vernier plate around exactly 180°. But this supposes not only that the graduation is perfectly accurate, but also that the line of collimation is exactly over the centre of the circle. To test this, after sighting to a point, and noting the reading, take the telescope out of the Ys and turn it end for end, and then turn the vernier plate around exactly 180°. If the line of sight again strikes the same point, the latter condition exists. If not, the maker must remedy the defect. This error of eccentricity is similar to that explained with respect to the compass, in the latter part of Art. (226).

(363) Third adjustment. *To cause the line of collimation to revolve in a vertical plane.**

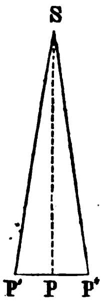
Verification. Suspend a long plumb-line from some high point. Set the instrument near this line, and level it carefully. Direct the telescope to the plumb-line, and see if the intersection of the cross-hairs follows and remains upon this line, when the telescope is turned up and down. If it does, it moves in a vertical plane.

The angle of a new and well-built house will form an imperfect substitute for the plumb-line.

Otherwise; the instrument being set up and levelled as above, place a basin of some reflecting liquid (quicksilver being the best, though molasses, or oil, or even water, will answer, though less perfectly,) so that the top of a steeple, or other point of a high object, can be seen in it through the telescope by reflection. Make the intersection of the cross-hairs cover it. Then turn up the telescope, and if the intersection of the cross-hairs bisects also the object seen directly, the line of sight has moved in a vertical plane. If a star be taken as the object, the star and its reflection will be equivalent (if it be nearly over head) to a plumb-line at least fifty million million miles long.

Otherwise; set the instrument as close as possible to the base of a steeple, or other high object; level it, and direct it to the top of the steeple, or to some other elevated and well defined point. Clamp the plates. Turn down the telescope, and set up a pin in the ground precisely "in line." Then loosen the clamp, turn over the telescope, and turn it half-way around, or so far as to again sight to the high point. Clamp the plates, and again turn down the telescope. If the line of sight again strikes the pin, the telescope has moved in a vertical plane. If not, the apparent error is double the real error. For, let S be the top of the steeple,

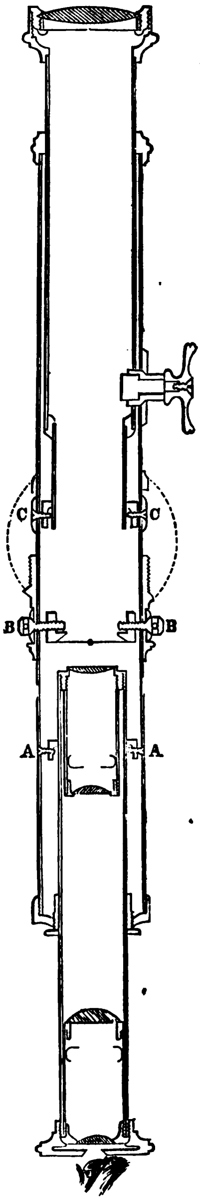
Fig. 249



* This applies to both the Transit and the Theodolite, with the exception of the method of verification by the steeple and pin, which applies only to the Transit.

(Fig. 249) and P′ the pin; then the plane in which the telescope moves, seen edgewise, is SP′; and, after being turned around, the line of sight moves in the plane SP″, as far to one side of the vertical plane SP, as SP′ was on the other side of it.

Fig. 250.



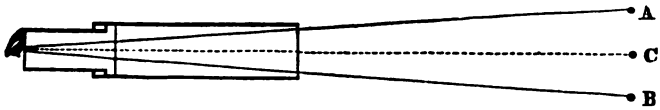
Rectification. Since the second adjustment causes the line of sight to move in a plane perpendicular to the axis on which it turns, it will move in a vertical plane if that axis be horizontal. It may be made so by filing off the feet of the standards which support the higher end of the axis. This will be best done by the maker. In some instruments one end of the axis can be raised or lowered.

(364) Centring eye-piece. In some instruments, such as that of which a longitudinal section is shown in the margin, the inner end of the eye-piece may be moved so that the cross-hairs shall be seen precisely in the centre of its field of view. This is done by means of four screws, arranged in pairs, like those of the cross-hair-ring screws, and capable of moving the eye-piece up and down, and to right or left, by loosening one and tightening the opposite one. Two of them are shown at A, A, in the figure; in which B, B, are two of the cross-hair screws.

(365) Centring object-glass. In some instruments four screws, similarly arranged, two of which are shown at C, C, can move, in any direction, the inner end of the slide which carries the object-glass. The necessity for such an arrangement arises from the impossibility of drawing a tube perfectly straight. Consequently, the line of collimation, when the tube is drawn in, will not coincide with the same line when the tube is pushed out. If adjusted for one position, it will therefore be wrong for the other. These screws, however, can make it right in both positions. They are used as follows.

Sight to some well defined point as far off as it can be distinctly seen. Then revolve the telescope half around in its supports; i. e. turn it upside down.* If the line of collimation was not in the imaginary axis of the rings or collars on which the telescope rests, it will now no longer bisect the object sighted to. Thus, if the horizontal hair was too high, as in Fig. 251. this line of

Fig. 251.



collimation would point at first to A, and after being turned over, it would point to B. The error is doubled by the reversion, and it should point to C, midway between A and B. Make it do so, by unscrewing the upper capstan-headed screw, and screwing in the lower one, till the horizontal hair is brought half way back to the point. Remember that in an erecting telescope, the cross-hairs are reversed, and *vice versa*. Bring it the rest of the way by means of the parallel plate screws. Then revolve it in the Ys back to its original position, and see if the intersection of the cross-hairs now bisects the point, as it should. If not, again revolve, and repeat the operation till it is perfected. If the vertical hair passes to the right or to the left of the point when the telescope is turned half around, it must be adjusted in the same manner by the other pair of cross-hairs screws. One of these adjustments may disturb the other, and they should be repeated alternately. When they are perfected, the intersection of the cross-hairs, when once fixed on a point, will not move from it when the telescope is revolved in its

* In Theodolites, the Telescope is revolved in the Ys. In Transits, the maker, by whom this adjustment is usually performed, revolves the Telescope, in the same manner, before it is fixed in its cross-bar.

supports. This double operation is called *adjusting the line of collimation.**

This line is now adjusted for distant objects. It would be so for near ones also, if the tube were perfectly straight. To test this, sight to some point, as near as is distinctly visible. Then turn the telescope half over. If the intersection does not now bisect the point, bring it half way there by the screws C, C, of Fig. 250, moving only one of the hairs at a time, as before. Then repeat the former adjustment on the distant object. If this is not quite perfect, repeat the operation.

This adjustment, in instruments thus arranged, should precede the first one which we have explained. It is usually performed by the maker, and its screws are not visible in the Transit, being enclosed in the ball seen where the telescope is connected with the cross-bar.†

All the adjustments should be meddled with as little as possible, lest the screws should get loose; and when once made right they should be kept so by careful usage.

* This "adjustment of the line of collimation" has merely brought the intersection of the cross-hairs (which fixes the line of sight) into the line joining the centres of the collars on which the telescope turns in the Ys; but the maker is supposed to have originally fixed the optical axis of the telescope, (i. e. the line joining the optical centres of the glasses), in the same line.

† The adjustment of "Centring the object-glass is the invention of Messrs. Gurley, of Troy.

CHAPTER IV.

THE FIELD-WORK.

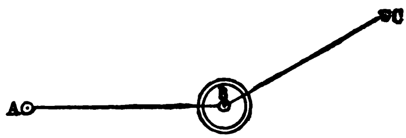


Fig. 252.

(366) To measure a horizontal angle. Set up the instrument so that its centre shall be exactly over the angular point, or in the intersection of the two lines whose difference of direction is to be measured; as at B in the figure. A plumb line must be suspended from under the centre. Dropping a stone is an imperfect substitute for this. Set the instrument so that its *lower* parallel plate may be as nearly horizontal as possible. The levels will serve as guides, if the four parallel-plate screws be first so screwed up or down that equal lengths of them shall be above the upper plate. Then level the instrument carefully, as in Art. (338). Direct the telescope to a rod, stake, or other object, A in the figure, on one of the lines which form the angle. Tighten the clamps, and by the tangent-screw, (see Art. (336)), move the telescope so that the intersection of the cross-hairs shall very precisely bisect this object. Note the reading of the vernier, as explained in the preceding chapter. Then loosen the clamp of the vernier, and direct the telescope on the other line (as to C) precisely as before, and again read. The difference of the two readings will be the desired angle, ABC. Thus, if the first reading had been 40° and the last 190°, the angle would be 150°. If the vernier had passed 360° in turning to the second object, 360° should be added to the last reading before subtracting. Thus, if the first reading had been 300°, and the last reading 90°, the angle would be found by calling the last reading, as it really is, $360° + 90° = 450°$, and then subtracting 300°.

It is best to sight first to the left hand object and then to the right hand one, turning "with the sun," or like the hands of a watch, since the numbering of the degrees usually runs in that direction.

It is convenient, though not necessary, to begin by setting the vernier at zero, by the upper movement (that of the vernier plate on the circle) and then, by means of the lower motion, (that of the whole instrument on its axis), to direct the telescope to the first object. Then fasten the lower clamp, and sight to the second object as before. The reading will then be the angle desired. An objection to this is that the two verniers seldom read alike.*

After one or more angles have been observed from one point, the telescope must be directed back to the first object, and the reading to it noted, so as to make sure that it has not slipped. A watch-telescope (see Art. 339) renders this unnecessary.

The error arising from the instrument not being set precisely over the centre of the station, will be greater the nearer the object sighted to. Thus a difference of one inch would cause an error of only 3″ in the apparent direction of an object a mile distant, but one of nearly 3′ at a distance of a hundred feet.

(367) Reduction of high and low objects. When one of the objects sighted to is higher than the other, the "plunging telescope" of these instruments causes the angle measured to be the true horizontal angle desired; i. e. the same angle as if a point exactly under the high object and on a level with the low object (or *vice versa*) had been sighted to. For, the telescope has been caused to move in a vertical plane by the 3d adjustment of Chapter II, and the angle measured is therefore the angle between the vertical planes which pass through the two objects, and which "project" the two lines of sight on the same horizontal plane.

This constitutes the great practical advantage of these instruments over those which are held in the planes of the two objects observed, such as the sextant, and the "circle" much used by the French.

* The learner will do well to gauge his own precision and that of the instrument (and he may rest assured that his own will be the one chiefly in fault) by measuring, from any station, the angles between successive points all around him, till he gets back to the first point, beginning at different parts of the circle for each angle. The sum of all these angles *should* exactly equal 360°. He will probably find quite a difference from that.

(368) Notation of angles. The angles observed may be noted in various ways. Thus, the observation of the angle ABC, in Fig. 252, may be noted "At B, from A to C, 150°," or better, "At B, between A and C, 150°." In column form, this becomes

Between A	150°	and C.
At	B	

When the vernier had been set at zero before sighting to the first object, and other objects were then sighted to, those objects, the readings to which were less than 180°, will be on the left of the first line, and those to which the readings were more than 180°, will be on its right, looking in the direction in which the survey is proceeding, from A to B, and so on.*

(369) Probable error. When a number of separate observations of an angle have been made, the mean or average of them all, (obtained by dividing the sum of the readings by their number,) is taken as the true reading. The "Probable error" of this mean, is the quantity, (minutes or seconds) which is such that there is an even chance of the real error being more or less than it. Thus, if ten measurements of an angle gave a mean of 35° 18′, and it was an equal wager that the error of this result, too much or too little, was half a minute, then half a minute would be the "Probable error" of this determination. This probable error is equal to the square root of the sum of the squares of the errors (i. e. the differences of each observation from the mean) divided by the number of observations, and multiplied by the decimal 0.674489.

The same result would be obtained by using what is called "*The weight*" of the observation. It is equal to the square of the number of observations divided by twice the sum of the squares of the errors. The "Probable error" is equal to 0.476936 divided by the square root of the weight. These rules are proved by the "Theory of Probabilities."

(370) To repeat an angle. Begin as in Art. (366), and measure the angle as there directed. Then unclamp below, and turn the circle around till the telescope is again directed to the first object, and made to bisect it precisely by the lower tan-

* This is very useful in preventing any ambiguity in the field-notes.

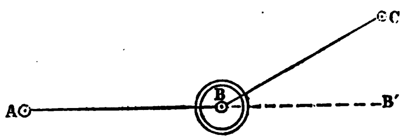
gent screw. Then unclamp above and turn the vernier plate till the telescope again points to the second object, the first reading remaining unchanged. The angle will now have been measured a second time, but on a part of the circle adjoining that on which it was first measured, the second arc beginning where the first ended. The difference between the first and last readings will therefore be twice the angle.

This operation may be repeated a third, a fourth, or any number of times, always turning the telescope back to the first object by the lower movement, (so as to start with the reading at which the preceding observation left off) and turning it to the second object by the upper movement. Take the difference of the first and last readings and divide by the number of observations.

The advantage of this method is that the errors of *observation* (i. e. sighting sometimes to the right and sometimes to the left of the true point) balance each other in a number of repetitions; while the constant error of *graduation* is reduced in proportion to this number. This beautiful principle has some imperfections in practice, probably arising from the slipping and straining of the clamps.

(371) Angles of deflection. The angle of deflection of one line from another, is the angle which one line makes with the other line produced. Thus, in the figure, the angle of deflection of BC from AB, is B′BC. It is evidently the supplement of the angle ABC.

Fig. 253.



To measure it with the *Transit*, set the instrument at B, direct the telescope to A, and then turn it over. It will now point in the direction of AB produced, or to B′, if the 2d adjustment of Chapter II, has been performed. Note the reading. Then direct the telescope to C. Note the new reading, and their difference will be the required angle of deflection, B′BC.

If the vernier be set at zero, before taking the first observation, the readings for objects on the right of the first line will be less than

180°, and more than 180° for objects on the left; conversely to Art. (368).

(372) Line surveying. The survey of a line, such as a road, &c., can be made by the Theodolite or Transit, with great precision; measuring the angle which each line makes with the preceding line, and noting their lengths, and the necessary offsets on each side.

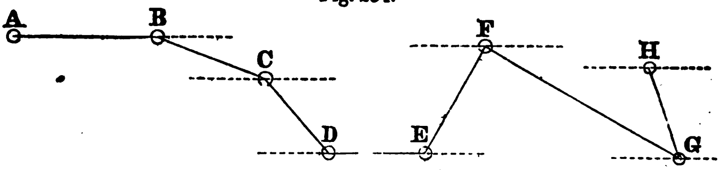
Short lines of sight should be avoided, since a slight inaccuracy in setting the centre of the instrument exactly over or under the point previously sighted to, would then much affect the angle, as noticed at close of Art. (366). Very great accuracy can be obtained by using three tripods. One would be set at the first station and sighted back to from the instrument placed at the second station, and a forward sight be then taken to the third tripod placed at the third station. The instrument would then be set on this third tripod, a back sight taken to the tripod remaining on the second station, and a foresight taken to the tripod brought from the first station to the fourth station; to which the instrument is next taken: and so on. This is especially valuable in surveys of mines.

The field-notes may be taken as directed in Chapter III of Compass Surveying, pages 149, &c., the angles taking the place of the Bearings. The "Checks by intersecting Bearings," explained in Art. (246), should also be employed. The angles made on each side of the stations may both be measured, and the equality of their sum to 360°, would at once prove the accuracy of the work.

If the magnetic Bearing of any one of the lines be given, and that of any of the other lines of the series be required, it can be deduced by constructing a diagram, or by modifications of the rules given for the reverse object, in Art. (243).

(373) Traversing: Or Surveying by the back-angle. This is a method of observing and recording the different directions of successive portions of a line, (such as a road, the boundaries of a farm, &c.,) so as to read off on the instrument, at each station, the angle which each line makes—not with the preceding line, but—with the first line observed. This line is, therefore, called the *meridian* of that survey.

Fig. 254.



Set up the instrument at the first angle, or second station, (B, in the figure), of the line to be surveyed. Sight to A and then to C. Clamp the vernier, and take the instrument to C. Loosen the lower clamp, and direct the telescope to B, the reading remaining as it was at B. Clamp below, loosen above, and sight to D. The reading of the instrument will be the angle which the line CD makes with the first line, or Meridian, AB.

Take the instrument to D. Sight back to C, and then forward to E, as before directed, and the reading of the instrument will be the angle which DE makes with AB.

So proceed for any number of lines.

When the Transit is used, the angles of deflection of each line from the first, obtained by reversing the telescope, may be used in "Traversing," and with much advantage when the successive lines do not differ greatly in their directions.

A	0°
B	200°
C	250°
D	180°
E	300°
F	210°
G	250°

The survey represented in the figure, is recorded in the first of the accompanying Tables, as observed with the Theodolite; and in the second Table, as observed with the Transit.

A	0°
B	20°
C	50°
D	0°
E	300°
F	30°
G	250°

The chief advantage of this method is its greater rapidity in the field and in platting, the angles being all laid down from one meridian, as in Compass-surveying. This also increases the accuracy of the plat, since any error in the direction of one line does not affect the directions of the following lines.*

(374) Use of the Compass. The chief use of the Compass attached to a Transit or Theodolite, is as a check on the observations; for the difference between the magnetic Bearings of any

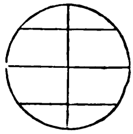
* If there are two verniers; take care always to read the degrees from the same vernier. Mark it A.

two lines should be the same, approximately, as the angle between them, measured by the more accurate instruments. The Bearing also prevents any ambiguity, as to whether an angle was taken to the right or to the left.

The instrument may also be used like a simple compass, the telescope taking the place of the sights, and requiring similar tests of accuracy. A more precise way of taking a Bearing is to turn the plate to which the compass box is attached, till the needle points to zero, and note the reading of the vernier; then sight to the object, and again read the vernier. The Bearing will thus be obtained more minutely than the divisions on the compass box could give it.

(375) Measuring distances with a telescope and rod. On the cross-hair ring, described in Art. (**330**), stretch two more horizontal spider-threads at equal distances above and below the original one; or all may be replaced by a plate of thin glass, placed precisely in the focus, with the necessary lines, as in the figure, etched by fluoric acid. Let a rod, 10 or 15 feet long, be held up at 1000 feet off, and let there be marked on it precisely the length which the distance between two of these lines covers. Let this be subdivided as minutely as the spaces, painted alternately white and red and numbered, can be seen. If ten subdivisions are made, each will represent a distance of 100 feet off, and so on. Continue these divisions over the whole length of the rod. It is now ready for use. The French call it a *stadia*. When it is held up at any unknown distance, the number of divisions on it intercepted between the two lines, will indicate the distance with considerable precision. It should be tested at various distances.

Fig. 255.



A "Levelling-rod," divided into feet, tenths and hundredths, may be used as a *stadia*, with less convenience but more precision. Experiments must previously determine at what distances the space between the lines in the telescope covers one foot, &c. Then, at any unknown distance, let the sliding "target" of the rod be moved till one line bisects it, and its place on the rod be read off; let the target be then moved so that the other line bisects it and let its place be again noted. Then the required distance will be equal to the difference of the readings on the rod, in feet, multiplied by the distance at which a foot was intercepted between the lines.

One of the horizontal hairs may be made movable, and its distance from the other, when the space between them exactly covers an object of known height, can be very precisely measured by counting the number of turns and fractions of a turn, of a screw by which this movable hair is raised or lowered. A simple proportion will then give the distance.

On sloping ground a double correction is necessary to reduce the slope to the horizon and to correct the oblique view of the rod. The horizontal distance is, in consequence, approximately equal to the observed distance multiplied by the *square* of the cosine of the slope of the ground.

The latter of the above two corrections will be dispensed with by holding the rod perpendicular to the line of sight, with the aid of a right angled triangle, one side of which coincides with the rod at the height of the telescope, and the other side of which adjoining the right angle, is caused, by leaning the rod, to point to the telescope.

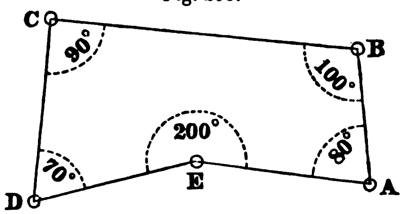
Other contrivances have been used for the same object, such as a Binocular Telescope with two eye-pieces inclined at a certain angle; a Telescope with an object-glass cut into two movable parts; &c.

(376) Ranging out lines. This is the converse of Surveying lines. The instrument is fixed over the first station with great precision, its telescope being very carefully adjusted to move in a vertical plane. A series of stakes, with nails driven in their tops, or otherwise well defined, are then set in the desired line as far as the power of the instrument extends. It is then taken forward to a stake three or four from the last one set, and is fixed over it, first by the plumb and then by sighting backward and forward to the first and last stake. The line is then continued as before. A good object for a long sight is a board painted like a target, with black and white concentric rings, and made to slide in grooves cut in the tops of two stakes set in the ground about in the line. It is moved till the vertical hair bisects the circles (which the eye can determine with great precision) and a plumb-line dropped from their centre, gives the place of the stake. "Mason & Dixon's Line" was thus ranged.

If a Transit be used for ranging, its "Second Adjustment" is most important to ensure the accuracy of the reversal of its Telescope. If a Theodolite be used, the line is continued by turning the vernier 180°, or by reversing the telescope in its Ys, as noticed in Arts. (**325**) and (**362**).

(377) Farm Surveying, &c. A large farm can be most easily and accurately surveyed, by measuring the angles of its main boundaries (and a few main diagonals, if it be very large,) with a Theodolite or Transit, as in Arts. (**366**) or (**371**), and filling up the interior details, as fences, &c., with the Compass and Chain.

Fig. 256.



If the *Theodolite* be used, keep the field on the left hand, as in following the order of the letters in this figure, and turn the telescope around "with the sun," and the angles measured as in Art. (**366**), will be the interior angles of the field, as noted in the figure.

The accuracy of the work will be proved, as alluded to in Art. (**257**), if the sum of all the interior angles be equal to the product of 180° by the number of sides of the figure less two. Thus in the figure, the sum of all the interior angles $= 540° = 180° \times (5 - 2)$. The sum of the exterior angles would of course equal $180° \times (5 + 2) = 1260°$.

If the *Transit* be used, the farm should be kept on the right hand, and then the angles measured will be the supplements of the interior angles. If the angles to the right be called *positive*, and those to the left *negative*, their algebraic sum should equal 360°.

If the boundary lines be surveyed by "Traversing," as in Art. (**373**), the reading, on getting back to the last station and looking back to the first line, should be 360°, or 0°.

The content of any surface surveyed by "Traversing" with the Transit can be calculated by the Traverse Table, as in Chapter VI, of Part III, by the following modification. When the angle of deflection of any side from the first side, or Meridian, is less than 90°, call this angle the Bearing, find its Latitude and Departure, and call them both *plus*. When the angle is between 90° and 180°, call the difference between the angle and 180° the Bearing, and call its Latitude *minus* and its Departure *plus*. When the angle is between 180° and 270°, call its difference from 180° the Bearing, and call its Latitude *minus* and its Departure *minus*. When the angle is more than 270°, call its difference from 360° the Bearing, and call its Latitude *plus* and its Departure *minus*. Then use these as in getting the content of a Compass-survey. The signs of the Latitudes and Departures follow those of the cosines and sines in the successive quadrants.

Town-Surveying would be performed as directed in Art. (**261**), substituting "angles" for "Bearings." "Traversing" is the best method in all these cases.

Inaccessible areas would be surveyed nearly as in Art. (**134**), except that the angles of the lines enclosing the space would be measured with the instrument, instead of with the chain.

(378) Platting. Any of these surveys can be platted by any of the methods explained and characterized in Chapter IV, of the preceding Part. A circular Protractor, Art. (**264**), may be regarded as a Theodolite placed on the paper. "Platting Bearings," Art. (**265**), can be employed when the survey has been made by "Traversing." But the method of "Latitudes and Departures," Art. (**285**), is by far the most accurate.

PART V.

TRIANGULAR SURVEYING;

OR

By the Fourth Method.

(379) TRIANGULAR SURVEYING is founded on the *Fourth Method* of determining the position of a point, by the intersection of two known lines, as given in Art. (8). By an extension of the principle, a field, a farm, or a country, can be surveyed by measuring only one line, and calculating all the other desired distances, which are made sides of a connected series of imaginary *Triangles*, whose angles are carefully measured. The district surveyed is covered with a sort of net-work of such triangles, whence the name given to this kind of Surveying. It is more commonly called "Trigonometrical Surveying;" and sometimes "Geodesic Surveying," but improperly, since it does not necessarily take into account the curvature of the earth, though always adopted in the great surveys in which that is considered.

(380) Outline of operations. A *base line*, as long as possible, (5 or 10 miles in surveys of countries), is measured with extreme accuracy.

From its extremities, angles are taken to the most distant objects visible, such as steeples, signals on mountain tops, &c.

The distances to these and between these are then calculated by the rules of Trigonometry.

The instrument is then placed at each of these new stations, and angles are taken from them to still more distant stations, the calculated lines being used as new base lines.

This process is repeated and extended till the whole district is embraced by these "primary triangles" of as large sides as possible.

One side of the last triangle is so located that its length can be obtained by measurement as well as by calculation, and the agreement of the two proves the accuracy of the whole work.

Within these primary triangles, *secondary* or smaller triangles are formed, to fix the position of the minor local details, and to serve as starting points for common surveys with chain and compass, &c. Tertiary triangles may also be required.

The larger triangles are first formed, and the smaller ones based on them, in accordance with the important principle in all surveying operations, always to work from the whole to the parts, and from greater to less.

Each of these steps will now be considered in turn, in the following order:

1. The Base; articles (**381**), (**382**).
2. The Triangulation; articles (**383**) to (**390**).
3. Modifications of the method; articles (**391**) to (**395**).

(381) Measuring a Base. Extreme accuracy in this is necessary, because any error in it will be *multiplied* in the subsequent work. The ground on which it is located must be smooth and nearly level, and its extremities must be in sight of the chief points in the neighborhood. Its point of beginning must be marked by a stone set in the ground with a bolt let into it. Over this a Theodolite or Transit is to be set, and the line "ranged out" as directed in Art. (**376**). The measurement may be made with chains, (which should be formed like that of a watch,) &c. but best with rods. We will notice in turn their *Materials*, *Supports*, *Alinement*, *Levelling*, and *Contact*.

As to *Materials*, iron, brass and other metals have been used, but are greatly lengthened and shortened by changes of temperature. Wood is affected by moisture. Glass rods and tubes are preferable on both these accounts. But wood is the most convenient. Wooden rods should be straight-grained white pine, &c.; well seasoned, baked, soaked in boiling oil, painted and varnished. They may be trussed, or framed like a mason's plumb-line level, to prevent their bending. Ten or fifteen feet is a convenient length. Three are required, which may be of different colors, to prevent mistakes in recording. They must be very carefully compared with a standard measure.

Supports must be provided for the rods, in accurate work. Posts set in line at distances equal to the length of the rods, may be driven or sawed to a uniform line, and the rods laid on them, either directly, or on beams a little shorter. Tripods, or trestles, with screws in their tops to raise or lower the ends of the rods resting on them, or blocks with three long screws passing through them and serving as legs, may also be used. Staves, or legs, for the rods have been used, up and down which sockets carrying the rods could slide.

The *Alinement* of the rods can be effected, if they are laid on the ground, by strings, two or three hundred feet long, stretched between the stakes set in the line, a notched peg being driven when the measurement has reached the end of one string, which is then taken on to the next pair of stakes; or, if the rods rest on supports, by projecting points on the rods being alined by the instrument.

The *Levelling* of the rods can be performed with a common mason's level; or their angle measured, if not horizontal, by a "slope-level."

The *Contacts* of the rods may be effected by bringing them end to end. The third rod must be applied to the second before the first has been removed, to detect any movement. The ends must be protected by metal, and should be rounded (with radius equal to length of rod) so as to touch in only one point. Round-headed nails will answer tolerably. Better are small steel cylinders, horizontal on one end and vertical on the other. Sliding ends, with verniers, have been used. If one rod be higher than the next one, one must be brought to touch a plumb-line which touches the other, and its thickness be added. To prevent a shock from contact, the rods may be brought not quite in contact, and a wedge be let down between them till it touches both at known points on its graduated edges. The rods may be laid side by side, and lines drawn across the end of each be made to coincide or form one line. This is more accurate. Still better is a "visual contact," a double microscope with cross-hairs being used, so placed that one tube bisects a dot at the end of one rod, and the other tube bisects a dot at the end of the next rod. The rods thus never touch. The distance between the two sets of cross-hairs is of course to be added. The combination of this principle with an arrangement by which bars of two metals preserve a length invariable at all temperatures, as used on the U. S. Coast Survey, leaves nothing to desire in precision.

A Base could be measured over very uneven ground, or even water, by suspending a series of rods from a stretched rope by rings in which they can move, and levelling them and bringing them into contact as above.

(382) Corrections of Base. If the rods were not level, their length must be reduced to its horizontal projection. This would be the square root of the difference of the squares of the length of the rod (or of the base) and of the height of one end above the other; or the product of the same length by the cosine of the angle which it makes with the horizon.*

If the rods were metallic, they would need to be corrected for temperature. Thus, if an iron bar expands $\frac{7}{1000000}$ of its length for 1° Fahrenheit, and had been tested at 32°, and a Base had been measured at 72° with such a bar 10 feet long, and found to contain 3000 of them, its apparent length would be 30,000 feet, but its real length would be 8.4 feet more.

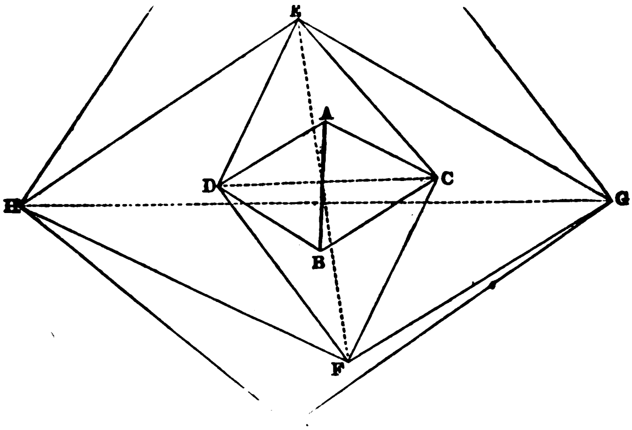
(383) Choice of Stations. The stations, or "Trigonometrical points," which are to form the vertices of the triangles, and to be observed to and from, must be so selected that the resulting triangles may be "well-conditioned," i. e. may have such sides and angles that a small error in any of the measured quantities will cause the least possible errors in the quantities calculated from them. The higher Calculus shows that the triangles should be as nearly equilateral as possible. This is seldom attainable, but no angle should be admitted less than 30°, or more than 120°.†

* More precisely, A being this angle, and not more than 2° or 3°, the difference between the inclined and horizontal lengths, equals the inclined or real length multiplied by the square of the minutes in A, and that by the decimal 0.00000004231; as shewn in Appendix B. In a Geodesic survey, the base would also be required to be reduced to the level of the sea.

† When two angles only are observed, as is often the case in the secondary triangulation, the unobserved angle ought to be nearly a right angle.

To extend the triangulation, by continually increasing the sides of the triangles, without introducing "ill-conditioned" triangles, may be effected as in the figure. AB is the measured base.

Fig. 257.



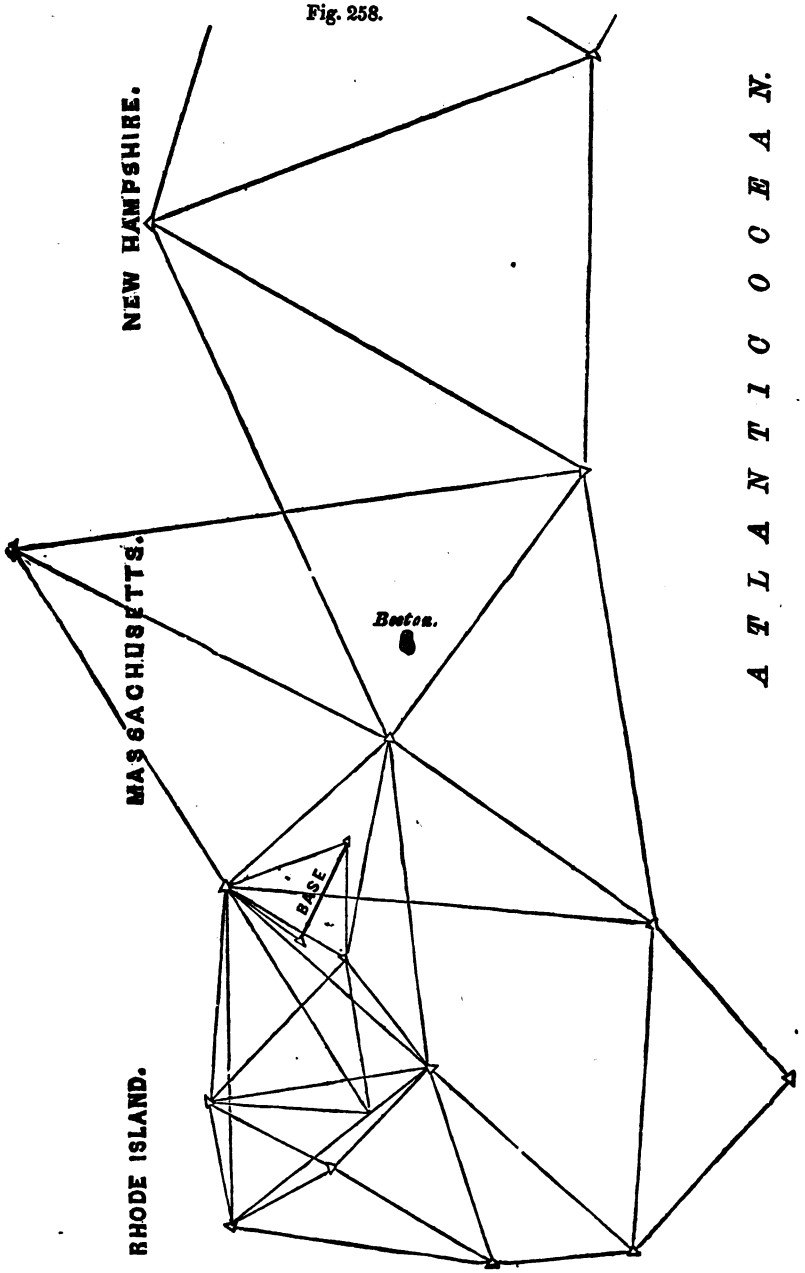
C and D are the nearest stations. In the triangles ABC and ABD, all the angles being observed and the side AB known, the other sides can be readily calculated. Then in each of the triangles DAC and DBC, two sides and the contained angles are given to find DC, one calculation checking the other. DC then becomes a base to calculate EF; which is then used to find GH; and so on.

The fewer primary stations used, the better; both to prevent confusion and because the smaller number of triangles makes the correctness of the results more "probable."

The United States Coast Survey, under the superintendence of Prof. A. D. Bache, displays some fine illustrations of these principles, and of the modifications they may undergo to suit various localities. The figure on the opposite page represents part of the scheme of the primary triangulation resting on the Massachusetts base and including some remarkably well-conditioned triangles, as well as the system of quadrilaterals which is a valuable feature of the scheme when the sides of the triangles are extended to considerable lengths, and quadrilaterals, with both diagonals determined, take the place of simple triangles.

The engraving is on a scale of 1:1200,000.

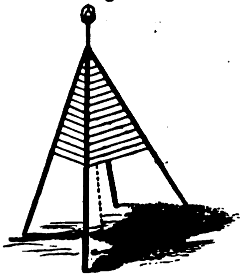
Fig. 258.


NEW HAMPSHIRE.
MASSACHUSETTS.
RHODE ISLAND.
ATLANTIC OCEAN.
Boston.
BASE

(384) Signals. They must be high, conspicuous, and so made that the instrument can be placed precisely under them.

Three or four timbers framed into a pyramid, as in the figure, with a long mast projecting above, fulfil the first and last conditions. The mast may be made vertical by directing two theodolites to it and adjusting it so that their telescopes follow it up and down, their lines of sight being at right angles to each other. Guy ropes may be used to keep it vertical.

Fig. 259.



A very excellent signal, used on the Massachusetts State Survey, by Mr. Borden, is represented in the three following figures. It

Fig. 260.

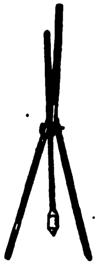


Fig. 261.

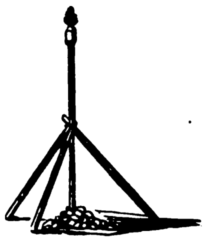
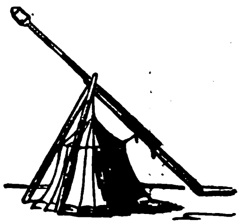
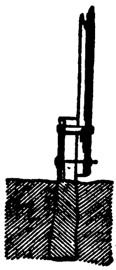


Fig. 262.



consists merely of three stout sticks, which form a tripod, framed with the signal staff, by a bolt passing through their ends and its middle. Fig. 260 represents the signal as framed on the ground; Fig. 261 shews it erected and ready for observation, its base being steadied with stones; and Fig. 262 shews it with the staff turned aside, to make room for the Theodolite and its protecting tent. The heights of these signals varied between 15 and 80 feet.

Fig. 263



Another good signal consists of a stout post let into the ground, with a mast fastened to it by a bolt below and a collar above. By opening the collar, the mast can be turned down and the Theodolite set exactly under the former summit of the signal, i. e. in its vertical axis.

Signals should have a height equal to at least $\frac{1}{7000}$ of their distance, so as to subtend an angle of half a minute, which experience has shown to be the least allowable

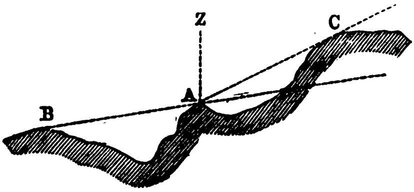
To make the tops of the signal-masts conspicuous, flags may be attached to them; white and red, if to be seen against the ground, and red and green if to be seen against the sky.* The motion of flags renders them visible, when mnch larger motionless objects are not. But they are useless in calm weather. A disc of sheet-iron, with a hole in it, is very conspicuous. It should be arranged so as to be turned to face each station. A barrel, formed of muslin sewn together four or five feet long, with two hoops in it two feet apart, and its loose ends sewn to the signal-staff, which passes through it, is a cheap and good arrangement. A tuft of pine boughs fastened to the top of the staff, will be well seen against the sky.

In sunshine, a number of pieces of tin nailed to the staff at different angles, will be very conspicuous. A truncated cone of burnished tin will reflect the sun's rays to the eye in almost every situation. But a "heliotrope," which is a piece of looking-glass, so adjusted as to reflect the sun directly to any desired point, is the most perfect arrangement.

For night signals, an Argand lamp is used; or, best of all, Drummond's light, produced by a stream of oxygen gas directed through a flame of alcohol upon a ball of lime. Its distinctness is exceedingly increased by a parabolic reflector behind it, or a lens in front of it. Such a light was brilliantly visible at 66 miles distance.

(385) Observations of the Angles. These should be repeated as often as possible. In extended surveys, three sets, of ten each, are recommended. They should be taken on different parts of the circle. In ordinary surveys, it is well to employ the method of "Traversing," Art. (373). In long sights, the state of the atmos-

Fig. 264.



* To determine at a station A, whether its signal can be seen from B, projected against the sky or not, measure the vertical angles BAZ and ZAC. If their sum equals or exceeds 180°, A will be thus seen from B. If not, the signal at A must be raised till this sum equals 180°.

phere has a very remarkable effect on both the visibility of the signals, and on the correctness of the observations.

When many angles are taken from one station, it is important to record them by some uniform system. The form given below is convenient. It will be noticed that only the minutes and seconds of the second vernier are employed, the degrees being all taken from the first.

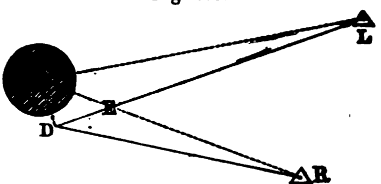
Observations at ——————.

STATION OBSERVED TO	READINGS. VERNIER A.	VERNIER B.	MEAN READING.	RIGHT OR LEFT OF PRECED'G OBJ'T.	REMARKS.
A	70° 19′ 0″	18′ 40″	70° 18′ 50″		
B	103° 32′ 20″	32′ 40″	103° 32′ 30″	R.	
C	115° 14′ 20″	14′ 50″	115° 14′ 35″	R.	

When the angles are "repeated," Art. (370), the multiple arcs will be registered under each other, and the mean of the seconds shewn by all the verniers at the first and last readings be adopted.

(384) Reduction to the centre. It is often impossible to set the instrument precisely at or under the signal which has been observed. In such cases proceed thus. Let C be the centre of the signal, and RCL the desired angle, R being the right hand object and L the left hand one. Set the instrument at D, as near as possible to C, and measure the angle RDL. It may be less than RCL, or greater than it, or equal to it, according as D lies without the circle passing through C, L and R, or within it, or in its circumference. The instrument should be set as nearly as possible in this last position. To find the proper correction for the observed angle, observe also the angle LDC, (called the angle of direction), counting it from 0° to 360°, going from the left-hand object toward the left; and measure the distance DC. Calculate the distances CR and CL with the angle RDL instead of RCL, since they are sufficiently nearly equal. Then

Fig. 265.



$$RCL = RDL + \frac{CD \,.\, \sin.\,(RDL + LDC)}{CR \,.\, \sin.\,1''} - \frac{CD \,.\, \sin.\,LDC}{CL \,.\, \sin.1''}.^{*}$$

The last two terms will be the number of seconds to be added or subtracted. The Trigonometrical signs of the sines must be attended to. The log. sin. $1'' = 4.6855749$. Instead of dividing by sin. $1''$, the correction without it, which will be a very small fraction, may be reduced to seconds by multiplying it by 206265.

Example. Let $RDL = 32° \, 20' \, 18''.06$; $LDC = 101° \, 15' \, 32''.4$; $CD = 0.9$; $CR = 35845.12$; $CL = 29783.1$.

The first term of the correction will be $+ 3''.750$, and the second term $- 6''.113$. Therefore, the observed angle RDL must be diminished by $2''.363$, to reduce it to the desired angle RCL.

Much calculation may be saved by taking the station D so that all the signals to be observed can be seen from it. Then only a single distance and angle of direction need be measured.

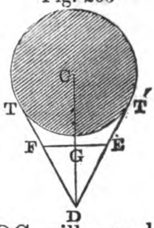


Fig. 266

It may also happen that the centre, C, of the signal cannot be seen from D. Thus, if the signal be a solid circular tower, set the Theodolite at D, and turn its telescope so that its line of sight becomes tangent to the tower at T, T'; measure on these tangents equal distances DE, DF, and direct the telescope to the middle, G, of the line EF. It will then point to the centre, C; and the distance DC will equal the distance from D to the tower plus the radius obtained by measuring the circumference.

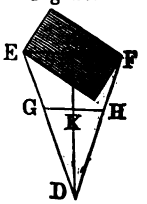


Fig. 267.

If the signal be rectangular, measure DE, DF. Take any point G on DE, and on DF set off $DH = DG \frac{DF}{DE}$. Then is GH parallel to EF, (since $DG : DH :: DE : DF$) and the telescope directed to its middle, K, will point to the middle of the diagonal EF. We shall also have $DC = DK \frac{DE}{DG}$.

Any such case may be solved by similar methods.

* For the investigation, see Appendix B.

The "*Phase*" of objects is the effect produced by the sun shining on only one side of them, so that the telescope will be directed from a distant station to the middle of that bright side instead of to the true centre. It is a source of error to be guarded against. Its effect may however be calculated.

(387) Correction of the angles. When all the angles of any triangle can be observed, their sum should equal 180.* If not they must be corrected. If all the observations are considered equally accurate, one-third of the difference of their sum from 180°, is to be added to, or subtracted from, each of them. But if the angles are the means of unequal numbers of observations, their errors may be considered to be inversely as those numbers, and they may be corrected by this proportion; *As* the sum of the reciprocals of each of the three numbers of observations *Is to* the whole error, *So is* the reciprocal of the number of observations of one of the angles *To* its correction. Thus if one angle was the mean of three observations, another of four, and the third of ten, and the sum of all the angles was 180° 3′, the first named angle must be diminished by the fourth term of this proportion; $\frac{1}{3} + \frac{1}{4} + \frac{1}{10} : 3' :: \frac{1}{3} : 1' \, 27''.8$. The second angle must in like manner be diminished by 1′ 5″.9; and the third by 26″.3. Their corrected sum will then be 180°.

It is still more accurate but laborious, to apportion the total error, or difference from 180°, among the angles inversely as the "*Weights*," explained in Art. (369). On the U. S. Coast Survey, in six triangles measured in 1844 by Prof. Bache, the *greatest* error was six-tenths of a second.

(388) Calculation and platting. The lengths of the sides of the triangles should be calculated with extreme accuracy, in two ways if possible, and by at least two persons. Plane Trigonometry may be used for even large surveys; for, though these sides are really arcs and not straight lines, the difference will be only one-

* If the triangles were very large, they would have to be regarded as spherical, and the sum of their angles would be more than 180°; but this "spherical excess" would be only 1″ for a triangle containing 76 square miles, 1′ for 4500 square miles, &c.; and may therefore be neglected in all ordinary surveying operations.

twentieth of a foot in a distance of $11\frac{1}{2}$ miles; half a foot in 23 miles; a foot in $34\frac{1}{2}$ miles, &c.

The platting is most correctly done by constructing the triangles, as in Art. (90), by means of the calculated lengths of their sides. If the measured angles are platted, the best method is that of chords, Art. (275). If many triangles are successively based on one another, they will be platted most accurately, by referring all their sides to some one meridian line by means of "Rectangular Co-ordinates," the Method of Art. (6), and platting as in Art. (277.) In the survey of a country, this Meridian would be the true North and South line passing through some well determined point.

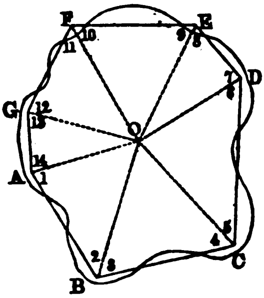
(389) Base of Verification. As mentioned in Art. (380), a side of the last triangle is so located that it can be measured, as was the first base. If the measured and calculated lengths agree, this proves the accuracy of all the previous work of measurement and calculation, since the whole is a chain of which this is the last link, and any error in any previous part would affect the very last line, except by some improbable compensation. How near the agreement should be, will depend on the nicety desired and attained in the previous operations. Two bases 60 miles distant differed on one great English survey 28 inches; on another one inch; and on a French triangulation extending over 500 miles, the difference was less than two feet. Results of equal or greater accuracy are obtained on the U. S. Coast Survey.

(390) Interior filling up. The stations whose positions have been determined by the triangulation are so many fixed points, from which more minute surveys may start and interpolate any other points. The Trigonometrical points are like the observed Latitudes and Longitudes which the mariner obtains at every opportunity, so as to take a new departure from them and determine his course in the intervals by the less precise methods of his compass and log. The chief interior points may be obtained by "Secondary Triangulation," and the minor details be then filled in by any of the methods of surveying, with Chain, Compass, or Transit, already explained, or by the Plane Table, described in Part VIII.

With the Transit, or Theodolite, "Traversing" is the best mode of surveying, the instrument being set at zero, and being then directed from one of the Trigonometrical points to another, which line therefore becomes the "Meridian" of that survey. On reaching this second point, in the course of the survey, and sighting back to the first, the reading should of course be 0°, as explained in Art. (377).

(391) Radiating Triangulation. This name may be given to a method shown in the figure. Choose a conspicuous point, O, nearly in the centre of the field or farm to be surveyed. Find other points, A, B, C, D, &c. such that the signal at O can be seen from all of them, and that the triangles ABO, BCO, &c, shall be as nearly equilateral as possible. Measure one side, AB for example. At A measure the angles OAB, and OAG; at B measure the angles OBA and OBC; and so on, around the polygon. The correctness of these measurements may be tested by the sum of the angles, as in Art. (377). It may also be tested by the Trigonometrical principle that the product of the sines of every alternate angle, or the odd numbers in the figure, should equal the product of the sines of the remaining angles, the even numbers in the figure.*

Fig. 268.



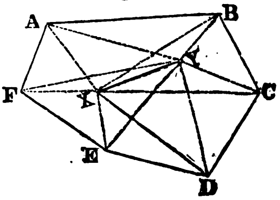
The calculations of the unknown sides are readily made. In the triangle ABO, one side and all the angles are given to find AO and BO. In the triangle BCO, BO and all the angles are given to find BC and CO; and so with the rest. Another proof of the accuracy of the work will be given by the calculation of the length of the side AO in the last triangle, agreeing with its length as obtained in the first triangle.

(392) Farm Triangulation. A Farm or Field may be surveyed by the previous methods, but the following plan will often be more

* For the demonstration, see Appendix B.

convenient. Choose a base, as XY, within the field, and from its ends measure the angles between it and the direction of each corner of the field, if the Theodolite or Transit be used, or take the bearing of each, if the Compass be used. Consider first the triangles which have XY for a base, and the corners of the field, A, B, C, &c., for vertices. In each of them one side and the angles will be known to find the other sides, XA, XB, &c. Then consider the field as made up of triangles which have their vertices at X. In each of them two sides and the included angle will be given to find its content, as in Art. (65). If Y be then taken for the common vertex, a test of the former work will be obtained.

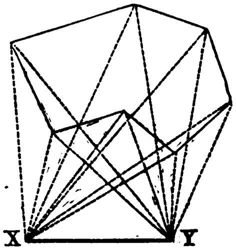
Fig. 269.



The operation will be somewhat simplified by taking for the base line a diagonal of the field, or one of its sides.

(393) Inaccessible Areas. A field or farm may be surveyed, by this "Fourth Method," without entering it. Choose a base line XY, from which all the corners of the field can be seen. Take their Bearings, or the angles between the Base line and their directions. The distances from X and Y to each of them can be calculated as in the last article. The figure will then shew in what manner the content of the field is the difference between the contents of the triangles, having X (or Y) for a vortex, which lie outside of it, and those which lie partly within the field and partly outside of it. Their contents can be calculated as in the last article, and their difference will be the desired content. If the figure be regarded as generated by the revolution of a line one end of which is at X, while its other end passes along the boundaries of the field, shortening and lengthening accordingly, and if those triangles generated by its movement in one direction be called *plus* and those generated by the contrary movement be called *minus*, their algebraic sum will be the content.

Fig, 270.



(394) Inversion of the Fourth Method. In all the operations which have been explained, the position of a point has been determined, as in Art. (8), by taking the angles, or bearings, of two lines passing from the two ends of a Base line to the unknown point. But the same determination may be effected inversely, by taking from the point the bearings, by compass, of the two ends of the Base line, or of any two known points. The unknown point will then be fixed by platting from the two known points the *opposite* bearings, for it will be at the intersection of the lines thus determined.

(395) Defects of the Method of Intersection. The determination of a point by the Fourth Method (enunciated in Art. (8), and developed in this Part) founded on the intersection of lines, has the serious defect that the point sighted to will be very indefinitely determined if the lines which fix it meet at a very acute or a very obtuse angle, which the relative positions of the points observed from and to, often render unavoidable. Intersections at right angles should therefore be sought for, so far as other considerations will permit.

PART VI.

TRILINEAR SURVEYING;

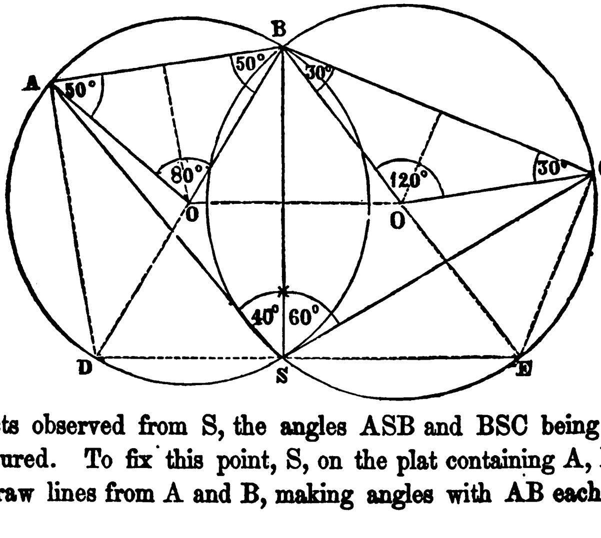
By the Fifth Method.

(396) Trilinear Surveying is founded on the Fifth Method of determining the position of a point, by measuring the angles betwen three lines conceived to pass from the required point to three known points, as illustrated in Art. (**10**).

To fix the place of the point from these data is much more difficult than in the preceding methods, and is known as the "Problem of the three points." It will be here solved Geometrically, Instrumentally and Analytically.

(397) Geometrical Solution. Let A, B and C be the known

Fig. 271.



objects observed from S, the angles ASB and BSC being there measured. To fix this point, S, on the plat containing A, B and C, draw lines from A and B, making angles with AB each equal to 90° — ASB. The intersection of these lines at O will be the centre of a circle passing through A and B, in the circumference of which the point S will be situated.* Describe this circle. Also, draw lines from B and C, making angles with BC, each equal to 90° — BSC. Their intersection, O′, will be the centre of a circle passing through B and C. The point S will lie somewhere in its circumference, and therefore in its intersection with the former circumference. The point is thus determined.

In the figure the observed angles, ASB and BSC, are supposed to have been respectively 40° and 60°. The angles set off are therefore 50° and 30°. The central angles are consequently 80° and 120°, twice the observed angles.

The dotted lines refer to the checks explained in the latter part of this article.

When one of the angles is obtuse, set off its difference from 90° on the opposite side of the line joining the two objects to that on which the point of observation lies.

When the angle ABC is equal to the supplement of the sum of the observed angles, the position of the point will be indeterminate; for the two centres obtained will coincide, and the circle described from this common centre will pass through the three points, and any point of the circumference will fulfil the conditions of the problem.

A third angle, between one of the three points and a fourth point, should always be observed if possible, and used like the others, to serve as a check.

Many tests of the correctness of the position of the point determined may be employed. The simplest one is that the centres of the circles, O and O′, should lie in the perpendiculars drawn through the middle points of the lines AB and BC.

Another is that the line BS should be bisected perpendicularly by the line OO′.

A third check is obtained by drawing at A and C perpendiculars to AB and CB, and producing them to meet BO and BO′ produced,

* For, the arc AB measures the angle AOB at the centre, which angle = 180° — 2 (90° — ASB) = 2 ASB. Therefore, any angle inscribed in the circumference and measured by the same arc is equal to ASB.

in D and E. The line DE should pass through S; for, the angles BSD and BSE being right angles, the lines DS and SE form one straight line.

The figure shews these three checks by its dotted lines.

(398) Instrumental Solution. The preceding process is tedious where many stations are to be determined. They can be more readily found by an instrument called a *Station-pointer*, or *Chorograph*. It consists of three arms, or straight-edges, turning about a common centre, and capable of being set so as to make with each other any angles desired. This is effected by means of graduated arcs carried on their ends, or by taking off with their points (as with a pair of dividers) the proper distance from a scale of chords (see Art. (274)) constructed to a radius of their length. Being thus set so as to make the two observed angles, the instrument is laid on a map containing the three given points, and is turned about till the three edges pass through these points. Then their centre is at the place of the station, for the three points there subtend on the paper the angles observed in the field.

A simple and useful substitute is a piece of transparent paper, or ground glass, on which three lines may be drawn at the proper angles and moved about on the paper as before.

(399) Analytical Solution. The distances of the required point from each of the known points may be obtained analytically. Let $AB = c$; $BC = a$; $ABC = B$; $ASB = S$; $BSC = S'$. Also, make $T = 360° - S - S' - B$. Let $BAS = U$; $BCS = V$. Then we shall have (as will be shewn in Appendix B)

$$\text{Cot. } U = \text{cot. } T \left(\frac{c \,.\, \text{sin. } S'}{a \,.\, \text{sin. } S \,.\, \text{cos. } T} + 1 \right)$$

$$V = T - U$$

$$SB = \frac{c \,.\, \text{sin. } U}{\text{sin. } S}; \text{ or, } = \frac{a \,.\, \text{sin. } V}{\text{sin. } S'}$$

$$SA = \frac{c \,.\, \text{sin. } ABS}{\text{sin. } S}. \qquad SC = \frac{a \,.\, \text{sin. } CBS}{\text{sin. } S'}$$

Attention must be given to the algebraic signs of the trigonometrical functions.

Example. ASB = 33° 45′; BSC = 22° 30′; AB = 600 feet; BC = 400 feet; AC = 800 feet. Required the distances and directions of the point S from each of the stations.

In the triangle ABC, the three sides being known, the angle ABC is found to be 104° 28′ 39″. The formula then gives the angle BAS = U = 105° 8′ 10″; whence BCS is found to be 94° 8′ 11″; and SB = 1042.51; SA = 710.193; and SC = 934.291.

(400) Maritime Surveying. The chief application of the Trilinear Method is to *Maritime* or *Hydrographical* Surveying, the object of which is to fix the positions of the deep and shallow points in harbors, rivers, &c., and thus to discover and record the shoals, rocks, channels and other important features of the locality. To effect this, a series of signals are established on the neighboring shore, any three of which may be represented by our points A, B, C. They are observed to from a boat, by means of a sextant, and the position of the boat is thus fixed as just shewn. The boat is then rowed in any desired direction, and soundings are taken at regular intervals, till it is found convenient to fix the new position of the boat as before. The precise point where each sounding was taken can now be platted on the map or chart. A repetition of this process will determine the depths and the places of each point of the bottom.

PART VII.

OBSTACLES IN ANGULAR SURVEYING.

(401) The obstacles, such as trees, houses, hills, vallies, rivers, &c., which prevent the direct alinement or measurement of any desired course, can be overcome much more easily and precisely with any angular instrument, than with the chain, methods for using which were explained in Part II, Chapter V. They will however be taken up in the same order.* As before, the given and measured lines are drawn with fine full lines; the visual lines with broken lines; and the lines of the result with heavy full lines.

CHAPTER I.

PERPENDICULARS AND PARALLELS.

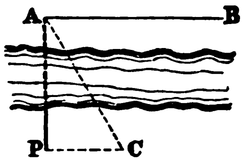
(402) Erecting Perpendiculars. *To erect a perpendicular to a line at a given point*, set the instrument at the given point, and, if it be a *Compass*, direct its sights on the line, and then turn them till the new Bearing differs 90° from the original one, as explained in Art. (**243**). A convenient approximation is to file notches in the Compass-plate, at the 90° points, and stretch over them a thread, sighting across which will give a perpendicular to the direction of the sights.

The *Transit* or *Theodolite* being set as above, note the reading of the vernier and then turn it till the new reading is 90° more or less than the former one.

* The Demonstrations of the Problems which require them, and from which they can conveniently be separated, will be found in Appendix B.

(403) *To erect a perpendicular to an inaccessible line, at a given point of it.* Let AB be the line and A the point. Calculate the distance from A to any point C, and the angle CAB, by the method of Art. (400). Set the instrument at C, sight to A, turn an angle = CAB, and measure in the direction thus obtained a distance CP = CA . cos. CAB. PA will be the required perpendicular.

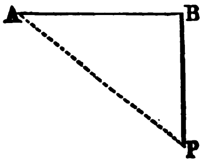
Fig. 272.



(404) Letting fall perpendiculars. *To let fall a perpendicular to a line from a given point.* With the *Compass*, take the Bearing of the given line and then from the given point run a line, with a Bearing differing 90° from the original Bearing, till it reaches the given line.

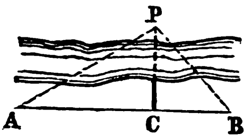
With the *Transit* or *Theodolite*, set it at any point of the given line, as A, and observe the angle between this line and a line thence to the given point, P. Then set at P, sight to the former position of the instrument, and turn a number of degrees equal to what the observed angle at A wanted of 90°. The instrument will then point in the direction of the required perpendicular PB.

Fig. 273.



(405) *To let fall a perpendicular to a line from an inaccessible point.* Let AB be the line and P the point. Measure the angles PAB, and PBA. Measure AB. The angles APC and BPC are known, being the complements of the angles measured. Then is

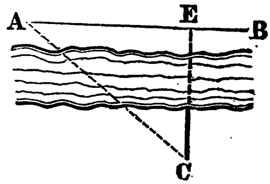
Fig. 274.



$$AC = AB \cdot \frac{\tan.\ APC}{\tan.\ APC + \tan.\ BPC}.$$

(406) *To let fall a perpendicular to an inaccessible line from a given point.* Let C be the point and AB the line. Calculate the angle CAB by the method of Art. (**430**). Set the instrument at C, sight to A, and turn an angle = 90 — CAB. It will then point in the direction of the required perpendicular CE.

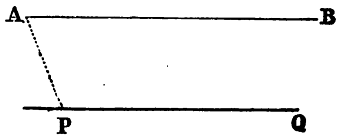
Fig. 275.



(407) Running Parallels. *To trace a line through a given point parallel to a given line.* With the *Compass*, take the Bearing of the given line, and then, from the given point, run a line with the same Bearing.

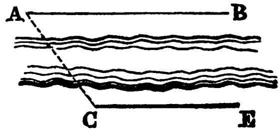
With the *Transit* or *Theodolite*, set it at any convenient point of the given line, as A, direct it on this line, and note the reading. Then turn the vernier till the cross-hairs bisect the given point, P. Take the instrument to this point and sight back to the former station, by the lower motion, without changing the reading. Then move the vernier till the reading is the supplement of what it was when the telescope was directed on the given line. It will then be directed on PQ, a parallel to AB, since equal angles have been measured at A and P. The manner of reading them is similar to the method of "Traversing," Art. (**373**).

Fig. 276.



(408) *To trace a line through a given point parallel to an inaccessible line.* Let C be the given point, and AB the inaccessible line. Find the angle CAB, as in Art. (**430**). Set the instrument at C, direct it to A, and then turn it so as to make an angle with CA equal to the supplement of the angle CAB. It will then point in a direction, CE, parallel to AB.

Fig. 277.



CHAPTER II.

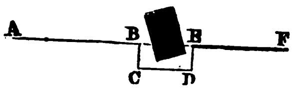
OBSTACLES TO ALINEMENT.

A. To prolong a line.

(409) The instrument being set at the farther end of a line, and directed back to its beginning, the sights of the *Compass*, if that be used, will at once give the forward direction of the line. They serve the purpose of the rods described in Art. (169). A distant point being thus obtained, the Compass is taken to it and the process repeated. The use of the *Transit* or *Theodolite*, for this purpose, was fully explained in Art. (376).

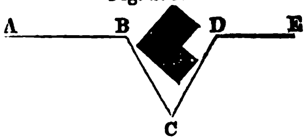
(410) By perpendiculars. When a tree, or house, obstructing the line, is met with, place the instrument at a point B of the line, and set off there a perpendicular, to C; set off another at C to D, a third at D to E, and a fourth at E, which last will be in the direction of AB prolonged. If perpendiculars cannot be conveniently used, let BC and DE make any equal angles with the line AB, so as to make CD parallel to it.

Fig. 278.



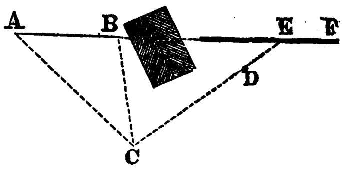
(411) By an equilateral triangle. At B, turn aside from the line at an angle of 60°, and measure some convenient distance BC. At C, turn 60° in the contrary direction, and measure a distance CD = BC. Then will D be a point in the line AB prolonged. At D, turn 60° from CD prolonged, and the new direction will be in the line of AB prolonged. This method requires the measurement of one angle less than the preceding.

Fig. 279.



(412) By triangulation. Let AB be the line to be prolonged. Choose some station C, whence can be seen A, B, and a point beyond the obstacle. Measure AB and the angles A and B, of the triangle ABC, and thence calculate the side AC. Set the instrument at C, and measure the angle ACD, CD being any line which will clear the obstacle. Let E be the desired point in the lines AB and CD prolonged. Then in the triangle ACE, will be known the side AC and its including angles, whence CE can be calculated. Measure the resulting distance on the ground, and its extremity will be the desired point E. Set the instrument at E, sight to C, and turn an angle equal to the supplement of the angle AEC, and you will have the direction, EF, of AB prolonged.

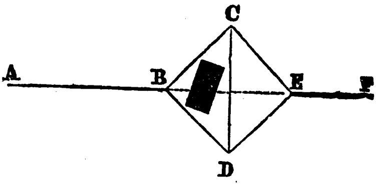
Fig. 280.



(413) When the line to be prolonged is inaccessible. In this case, before the preceding method can be applied, it will be necessary to determine the lengths of the lines AB and AC, and the angle A, by the method given in Art. (430).

(414) To prolong a line with only an angular instrument. This may be done when no means of measuring any distance can be obtained. Let AB be the line to be prolonged: Set the instrument at B and deflect angles of 45° in the directions C and D. Set at some point, C, on one of these lines and deflect from CB 45°, and mark the point D where this direction intersects the direction BD. Also, at C, deflect 90° from B. Then, at D, deflect 90° from DB. The intersections of these last directions will fix a point E. At E deflect 135° from EC or ED, and a line EF, in the direction of AB will be obtained and may be continued.

Fig. 281.



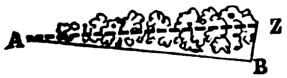
* This ingenious contrivance is due to a former student, Mr. R. Hood, in whose practice, while running an air line for a railroad, the necessity occurred.

B. To interpolate points in a line.

(415) The instrument being set at one end of a line and directed to the other, intermediate points can be found as in Art. (177), &c. If a valley intervenes, the sights of the Compass, (if the Compass-plate be very carefully kept level cross-ways), or the telescope of the Transit or Theodolite, answer as substitutes for the plumb-line of Art. (179).

(416) By a random line. When a wood, hill, or other obstacle, prevents one end of the line, Z, from being seen from the other, A, run a random line AB with the Compass or Transit, &c., as nearly in the desired direction as can be guessed, till you arrive opposite the point Z. Measure the error, BZ, at right angles to AB, as an offset. Multiply this error by $57\frac{3}{10}$, and divide the product by the distance AB. The quotient will be the degrees and decimal parts of a degree, contained in the angle BAZ. Add or subtract this angle to or from the Bearing or reading with which AB was run, according to the side on which the error was, and start from A, with this corrected Bearing or reading, to run another line, which will come out at Z, if no error has been committed.*

Fig. 282.



Example. A random line was run, by compass, with a Bearing of S. 80° E. At 20 chains' distance a point was reached opposite to the desired point, and 10 links distant from it on its right. Required the correct Bearing.

Ans. By the rule, $\frac{10 \times 57^\circ.3}{2000} = 0^\circ.2865 = 17'$. The correct Bearing is therefore S. 80° 17′ E. If the Transit had been used, its reading would have been changed for the new line by the same 17′. A simple diagram of the case will at once shew whether the correction is to be added to the original Bearing or angle, or subtracted from it.

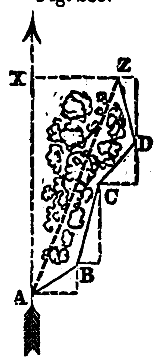
* This rule is substantially identical with that of Art. (319), where its reason is given.

If Trigonometrical Tables are at hand, the correction will be more precisely obtained from this equation; Tan. $BAZ = \frac{BZ}{AB}$. In this example, $\frac{BZ}{AB} = \frac{10}{2000} = .005 = \tan. 17'$.

The 57°.3 rule, as it is sometimes called, may be variously modified. Thus, multiply the error by 86°, and divide by one and a half times the distance; or, to get the correction in minutes, multiply by 3438 and divide by the distance; or, if the error is given in feet and the distance in four-rod chains, multiply the former by 52 and divide by the distance, to get the correction in minutes.

The correct line may be run with the Bearing of the random line, by turning the vernier for the correction, as in Art. (**312**).

Fig. 283.



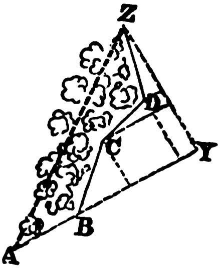
(417) By Latitudes and Departures. When a single line, such as AB, cannot be run so as to come opposite to the given point Z, proceed thus, with the *Compass*. Run any number of zig-zag courses, AB, BC, CD, DZ, in any convenient direction, so as at last to arrive at the desired point. Calculate the Latitude and Departure of each of these courses and take their *algebraic* sums. The sum of the Latitudes will be equal to AX, and that of the Departures to XZ. Then is Tan. $ZAX = \frac{XZ}{XA}$; i. e. the algebraic sum of the Departures divided by the algebraic sum of the Latitudes is equal to the tangent of the Bearing.*

(418) When the *Transit* or *Theodolite* is used, any line may be taken as a Meridian, i. e. as the line to which the following lines are referred; as in "Traversing," Art. (**373**), page 254, all the successive lines were referred to the first line. In the figure, on the next page, the same lines as in the preceding figure are repre-

* The length of the line AZ can also be at once obtained, since it is equal to the square root of the sum of the squares of AX and XZ; or to the Latitude divided by the cosine of the Bearing.

sented, but they are referred to the first course, AB, instead of to the Magnetic Meridian as before, and their Latitudes are measured along its produced line, and its Departures perpendicular to it. As before, a right-angled triangle will be formed, and the angle ZAY will be the angle at A between the first line AB and the desired line AZ.

Fig. 284.

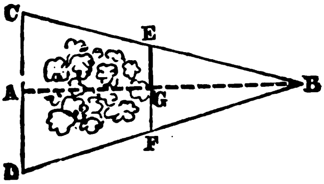


This method of operation has many useful applications, such as in obtaining data for running Railroad Curves, &c., and the student should master it thoroughly.

The desired angle (and at the same time the distance) can be obtained, approximately, in this and the preceding case, by finding in a Traverse Table, the final Latitude and Departure of the desired line (or a Latitude and Departure having the same ratio) and the Bearing and Distance corresponding to these will be the angle and distance desired.

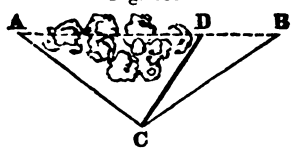
(419) By similar triangles. Through A measure any line CD. Take a point E, on the line CB, beyond the obstacle, and from it set off a parallel to CD, to some point, F, in the line DB. Measure EF, CD, and CA. Then this proportion, CD : CA :: EF : EG, will give the distance EG, from E to a point in the line AB. So for other points.

Fig. 285.



(420) By triangulation. When obstacles prevent the preceding methods being used, if a point, C, can be found, from which A and B are accessible, measure the distances CA, CB, and the angle ACB, and thence calculate the angle CAB. Then observe any angle ACD, beyond the obstacle. In the triangle ACD, a side and its including angles are known, to find CD. Measure it, and a point, D, in the desired line, will be obtained.

Fig. 286.



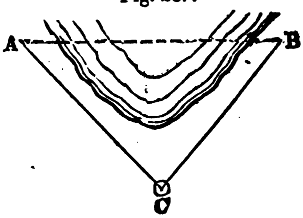
CHAPTER III.

OBSTACLES TO MEASUREMENT.

A. When both ends of the line are accessible.

(421) The methods given in the preceding Chapter for prolonging a line and for interpolating points in it, will generally give the length of the line by the same operation. Thus, in Fig. 278, the inaccessible distance BE is equal to CD; in Fig. 279, BD = BC = CD; in Fig. 280, the distance BE can be calculated from the same data as CE; in Fig. 282, $AZ = \sqrt{(AB^2 + BZ^2)}$; in Fig. 283, $AZ = \sqrt{(AX^2 + XZ^2)}$; in Fig. 284, $AZ = \sqrt{(AY^2 + YZ^2)}$; in Fig. 285, $AG = \frac{GB\ (CA - EG)}{EG}$; in Fig. 286, the triangle ACD will give the distance AD. The method of Latitudes and Departures, Arts. (**417**) and (**418**), is very generally applicable. So is the following.

Fig. 287.



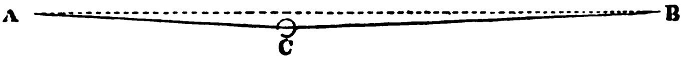
(422) By triangulation. Let AB be the inaccessible distance. From any point, C, from which both A and B are accessible, measure CA, CB, and the angle ACB. Then in the triangle ABC two sides and the included angle are known to find the side AB. If all the angles can be measured, they may be corrected, as in Art. (**387**).*

(423) A broken Base. When the angle C is very obtuse, the preceding problem may be modified as follows. Naming the lines as is usual in Trigonometry, by small letters corresponding to the

* In this figure, and the following ones, the angular point enclosed in a circle indicates the place at which the instrument is set.

capital letters at the angles to which they are opposite, and letting K = the number of minutes in the supplement of the angle C, we

Fig. 288.



shall have

$$AB = c = a + b - 0.000000042308 \times \frac{abK^2}{a+b}.$$

This formula is chiefly used in the case of what is called in Triangular Surveying "A broken Base;" such as above; AC and CB being measured and forming very nearly a straight line, and the length of AB being required.

Log. 0.000000042308 = 2.6264222 — 10.

(424) By angles to known points. The length of a line, both ends of which are accessible, may also be determined by angles measured at its extremities between it and the directions of two or more known points. But as the methods of calculation involve subsequent problems, they will be postponed to Articles (**435**), (**436**) and (**437**).

B. When one end of the line is inaccessible.

(425) By Perpendiculars. Many of the methods given for the chain, in Part II, Chapter V, may be still more advantageously employed with angular instruments, which can so much more easily and precisely set off the Perpendiculars required in Articles (**191**), (**192**), (**193**), &c.

(426) By equal angles. Let AB be the inaccessible line. At A set off AC, perpendicular to AB, and as nearly equal to it, by estimation, as the ground will permit. At C, measure the angle ACB, and turn the sights, or vernier, till ACD = ACB. Find the point, D, at the intersections of the lines CD and BA produced. Then is AD = AB.

Fig. 289.

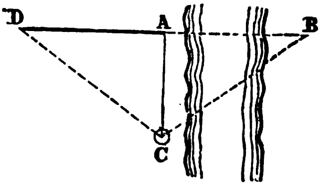
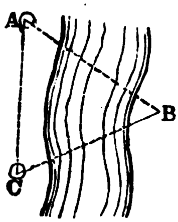


Fig. 290.



(427) By triangulation. Measure a distance AC, about equal to AB. Measure the angles at A and C. Then in the triangle ABC, two angles and the included side are known, to find another side, $AB = \frac{AC \sin. ACB}{\sin. ABC}$.

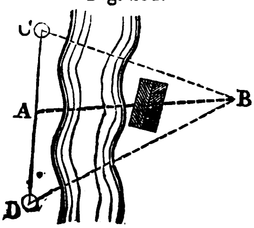
When the compass is used, the angles between the lines will be deduced from their respective Bearings, by the principles of Art. (**243**).

If the angle at A is 90°, $AB = AC \,.\, \text{tang. } ACB$.

If the angle $ACB = 45°$, then $AC = AB$; but this position could not easily be obtained, except by the use of the Sextant, a reflecting instrument, not described in this volume.

(428) When one point cannot be seen from the other.—Choose two points, C and D, in the line of A, and such that from C, A and B can be seen, and from D, A and B. Measure AC, AD, and the angles C and D. Then, in the triangle BCD, are known two angles and the included side, to find CB. Then, in the triangle ABC, are known two sides and the included angle, to find the third side, AB.

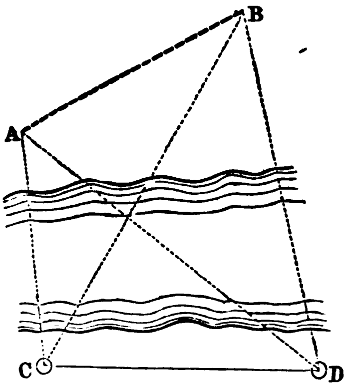
Fig. 291.



(429) To find the distance from a given point to an inaccessible line. In Fig. 275, Art. (**406**), the required distance is CE. The operations therein directed give the line CA and the angle CAB, or CAE. The required distance $CE = CA \,.\, \sin. CAE$.

C. When both ends of the line are inaccessible.

Fig. 292.



(430) General Method. Let AB be the inaccessible line. Measure any convenient distance CD, and the angles ACD, BCD, ADC, BDC.

Then, in the triangle CDA, two angles and the included side are given, to find CA. In the triangle CDB, two angles and the included side are given, to find CB. Then, in the triangle ABC, two sides and the included angle are given, to find AB.

The work may be verified by taking another set of triangles, and finding AB from the triangle ABD instead of ABC.

The following formulas will however give the desired distance with less labor.

Find an angle K, such that $\text{tang. } K = \dfrac{\text{sin. } ADC \, . \, \text{sin. } CBD}{\text{sin. } CAD \, . \, \text{sin. } BDC}$.

Then find the difference of the unknown angles in the triangle CAB from the formula

$\text{Tang. } \tfrac{1}{2}\,(CAB - ABC) = \text{tang. } (45° - K) \, . \, \text{cot. } \tfrac{1}{2}\, ACB.$

Then is $CAB = \tfrac{1}{2}\,(CAB - ABC) + \tfrac{1}{2}\,(CAB + ABC)$.

Finally, $AB = CD \, \dfrac{\text{sin. } BDC \, . \, \text{sin. } ACB}{\text{sin. } CBD \, . \, \text{sin. } CAB}$.

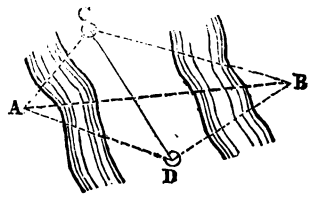
Example. Let $CD = 7106.25$ feet; $ACD = 95° \; 17' \; 20''$; $BCD = 61° \; 41' \; 50''$; $ADC = 39° \; 38' \; 40''$; $BDC = 78° \; 35' \; 10''$; required AB.

The figure is constructed with these data on a scale of 5000 feet to 1 inch = 1 : 60000.

By the above formulas, K is found to be $30° \; 26' \; 5''$; $CAB = 113° \; 55' \; 37''$; and lastly $AB = 6598.32$.

Both the methods may be used as mutual checks in any important case.

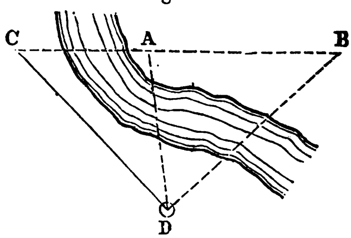
Fig. 293.



If the lines AB and CD crossed each other, as in Fig. 293, instead of being situated as in the preceding figure, the same method of calculation would apply.

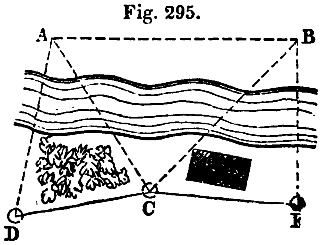
(431) Problem. *To measure an inaccessible distance*, AB, *when a point*, C, *in its line can be obtained.* Set the instrument at a point, D, from which A, B and C can be seen, and measure the angles CDA and ADB. Measure also the line DC and the angle C. Then in the triangle ACD two angles and the included side are given to find AD. In the triangle DAB, the angle DAB is known, (being equal to ACD + CDA), and AD having been found, we again have two angles and the included side to find AB.

Fig. 294.



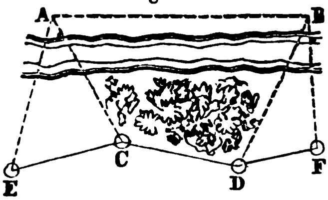
(432) Problem. *To measure an inaccessible distance*, AB, *when only one point*, C, *can be found from which both ends of the line can be seen.* Consider CA and CB as distances to be determined, having one end accessible. Determine them, as in Art. (427), by choosing a point D, from which C and A are visble, and a point E from which C and B are visible. At C observe the angles DCA, ACB and BCE. Measure the distances CD and CE. Observe the angles ADC and BEC. Then in the triangle ADC, two angles and the included side are given, to find CA; and the same in the triangle CBE, to find CB. Lastly, in the triangle ACB two sides and the included angle are known, to find AB.

Fig. 295.



(433) **Problem.** *To measure an inaccessible distance*, AB, *when no point can be found from which the two ends can be seen.*

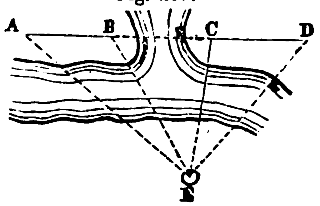
Fig. 296.



Let C be a point from which A is visible, and D a point from which B is visible, and also C. Measure CD. Find the distances CA and DB, as in the preceding problem; i. e. choose a point E, from which A and C are visible, and another point, F, from which D and B are visible. Measure EC and DF. Observe the angles AEC, ECA, BDF and DFB; and at the same time the angles ACD and CDB, for the subsequent work. Then CA and DB will be found, as were CA and CB in the last problem. Then in the triangle CDB, two sides and the included angle are known to find CB and the angle DCB; and, lastly, in the triangle ACB, two sides and the included angle (the difference of ACD and DCB) to find AB.

(434) **Problem. To interpolate a Base.** *Four inaccessible objects*, A, B, C, D, *being in a right line, and visible from only one point*, E, *it is required to determine the distance between the middle points*, B *and* C, *the exterior distances*, AB *and* CD, *being known.*

Fig. 297.



Let $AB = a$, $CD = b$, $BC = x$; $AEB = P$, $AEC = Q$, $AED = R$.

Calculate an auxiliary angle, K, such that

$$\text{tang.}^2\, K = \frac{4ab}{(a-b)^2} \cdot \frac{\sin.\, Q \,.\, \sin.\, (R-P)}{\sin.\, P \,.\, \sin.\, (R-Q)}.$$

$$\text{Then is } x = -\frac{a+b}{2} \pm \frac{a-b}{2\,.\,\cos.\, K}.$$

Of the two values of x, the positive one is alone to be taken.

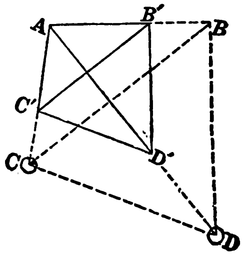
This problem is used in Triangular Surveying when a portion of Base line passes over water, &c.

(435) Problem. *Given the angles observed, at the ends of a line which cannot be measured, between it and the ends of a line of known length but inaccessible, required the length of the former line.* This Problem is the converse of that given in Art. (**430**). Its figure, 292, may represent the case, if the distance AB be regarded as known and CD as that to be found. Use the first and second formulas as before, and invert the last formula, obtaining

$$CD = AB \frac{\sin.\ CBD\ .\ \sin.\ CAB}{\sin.\ BDC\ .\ \sin.\ ACB}.$$

This problem may also be solved, indirectly, by assuming any length for CD, and thence calculating as in the first part of Art. (**430**), the length of AB on this hypothesis. The imaginary figure thus calculated is *similar* to the true one; and the true length of CD will be given by this proportion; calculated length of AB : true length of AB :: assumed length of CD : true length of CD.

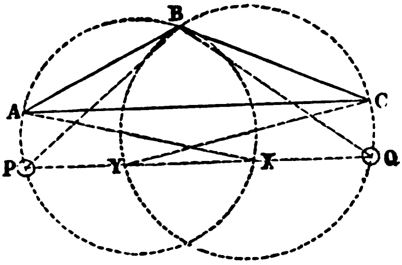
Fig. 298.



The length of AB can also be obtained graphically. Take a line of any length, as C′D′, and from C′ and D′ lay off angles equal to those observed at C and D, and thus fix points A, B′. Produce AB′ till it equals the given distance AB, on any desired scale. From B draw a parallel to B′D′, meeting AD′ produced in D; and from D draw a parallel to D′C′ meeting AC′ produced in C. Then CD will be the required distance to the same scale as AB.

(436) Problem. *Three points,* A, B, C, *being given by their distances from each other, and two other points,* P *and* Q, *being so situated that from each of them two of the three points can be seen and the angles* APQ, BPQ, CQP, BQP, *be measured, it is required to determine the positions of* P *and* Q.

Fig. 299.



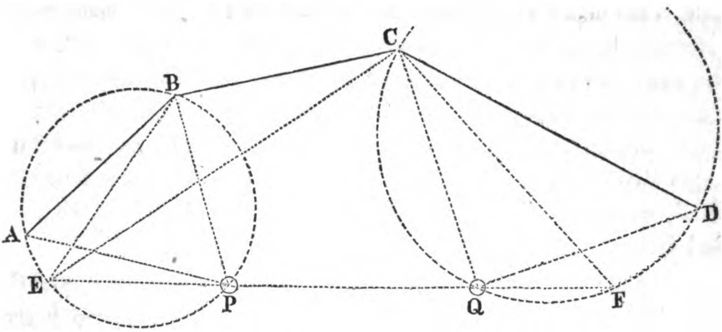
CONSTRUCTION. Begin, as in Art. (397), by describing a circle passing through A and B, and having the central angle subtended by AB, equal to twice the given angle APB, and thus containing that angle. The point P will lie somewhere in its circumference. Describe another circle passing through B and C, and having a central angle subtended by BC equal to twice the given angle BQC. The point Q will lie somewhere in its circumference. From A draw a line making with AB an angle = BPQ, and meeting at X the circle first drawn. From C draw a line making with CB an angle = BQP, and meeting the second circle in Y. Join XY and produce it till it cuts the circles in points P and Q, which will be those required; since BPX = BAX = BPQ; and BQY = BCY = BQP.

CALCULATION. In the triangle ABC, the sides being given, the angle ABC is known. In the triangle ABX, a side and all the angles are known, to find BX. In the triangle CBY, BY is similarly found. By subtracting the angle ABC from the sum of the angles ABX and CBY, the angle XBY can be obtained. Then in the triangle XBY, the sides BX, BY, and the included angle are given to find the other angles. Then in the triangle BPX are known all the angles and the side BX to find BP. In the triangle BQY, BQ is found in like manner. Finally, in the triangle BPQ, PQ can then be found.

If desired, we can also obtain AP in the triangle APB; and CQ in the triangle CBQ.

(437) Problem. *Four points,* A, B, C, D, *being given in position, by their mutual distances and directions, and two other points,* P *and* Q, *being so situated that from each of them two of the four points can be seen and the angles* APB, APQ, PQC *and* PQD *measured, it is required to determine the position of* P *and* Q.

Fig. 300.



CONSTRUCTION. Begin as in the last article, by describing on AB the segment of a circle to contain an angle equal to APB. From B draw a chord BE, making an angle with BA equal to the supplement of the angle APQ. On CD describe another segment to contain an angle equal to CQD. From C draw a chord CF, making an angle with CD equal to the supplement of the angle DQP. Draw the line EF, and it will cut the two circles in the required points P and Q.*

CALCULATION. To obtain PQ = EF — EP — QF, we proceed to find those three lines thus. In the triangle ABE, we know the side AB, the angle ABE, and the angle AEB = APB; whence to find EB. In the same way, the triangle CFD gives FC. In the triangle EBC are known EB and BC, and the angle EBC = ABC — ABE; whence EC and the angle ECB are found. In the triangle ECF are known EC, FC, and the angle ECF = BCD — ECB — FCD; whence we find EF, and the angles CEF and CFE.

In the triangle BEP, we have EB, the angle BEP = BEC + CEP, and the angle BPE = BPA + APE; to find EP and PB. In the triangle QCF, we have CF, and the angles CQF and CFQ, to find QC and QF. Then we know PQ = EF — EP — QF.

* For, the angle APQ in the figure equals the measured angle APQ, because the supplement of the former, EPA, equals the supplement of the latter, since it is measured by the same arc as the angle ABE, equal to that supplement by construction. So too with the angle DQP.

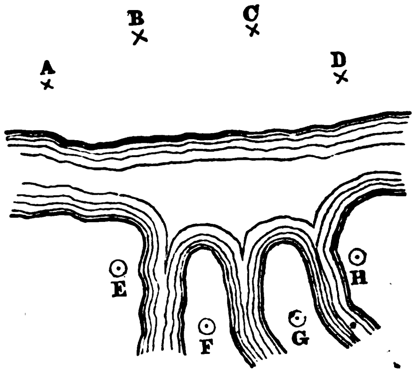
The other distances, if desired, can be easily found from the above data, some of the calculations, not needed for PQ, being made with reference to them. In the triangle ABP, we know AB, BP, and the angle BAP, to find the angle ABP and AP. In the triangle QDC we know QC, CD, and the angle CQD, to find the angle QCD and QD. In the triangle PBC, we know PB, BC, and the angle PBC = ABC — ABP, to find PC. Lastly, in the triangle QCB, we know QC, CB, and the angle QCB = DCB — DCQ, to find QB.

The solution of this problem includes the two preceding; for, let the line BC be reduced to a point so that its two ends come together and the three lines become two, and we have the problem of Art. (436); and let the line AB be reduced to a point, B, and CD to a point, C, and we have but one line, and the problem becomes that of Art. (435).

In these three problems, if the two stations lie in a right line with one of the given points, the problem is indeterminate.

(438) Problem of the eight points. *Four points*, A, B, C, D, *are inaccessible, but visible from four other points*, E, F, G, H; *it is required to find the respective distances of these eight points; the only data being the observation, from each of the points of the second system, of the angles under which are seen the points of the first system.*

Fig. 301.



This problem can be solved, but the great length and complication of the investigation and resulting formulas render it more a matter of curiosity than of utility. It may be found in Puissant's "*Topographie*," page 55; Lefevre's "*Trigonometrie*," p. 90, and Lefevre's "*Arpentage*," No. 387.

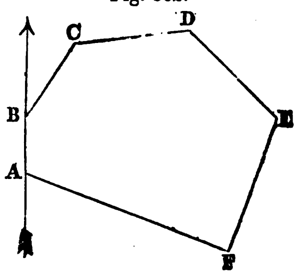
CHAPTER IV.

TO SUPPLY OMISSIONS.

(439) Any two omissions in a closed survey, whether of the direction or of the length, or of both, of one or more of the sides bounding the area surveyed, can always be supplied by a suitable application of the principle of Latitudes and Departures, as was stated in Art. (283); although this means should be resorted to only in cases of absolute necessity, since any omission renders it impossible to "Test the survey," as directed in Art. (282). In the following articles the survey will be considered to have been made with the Compass. All the rules will however apply to a Transit or Theodolite survey, the angles being referred to any line as a meridian, as in "Traversing."

To save unnecessary labor, the examples in the various cases now to be examined, will all be taken from the same survey, a plat of which is given in the margin on the scale of 40 chains to 1 inch (1:31,680), and the Field-notes of which, with the Latitudes and Departures carried out to five decimal places, are given on the following page.*

Fig. 302.



* The teacher can make any number of examples for his own use by taking a tolerably accurate survey, striking out the bearing and distance of any one course, and calculating it precisely as in Case 1, given below. He can then omit any two quantities at will, to be supplied by the student by means of the rules now to be given.

STA.	BEARING.	DIST. IN LINKS.	LATITUDES.		DEPARTURES.	
			N.	S.	E.	W.
A	North.	1284	1284.00000		0	0
B	N. 32° E.	1782	1511.22171		944.31619	
C	N. 80° E.	2400	416.75568		2363.53872	
D	S. 48° E.	2700		1806.65262	2006.49096	
E	S. 18° W.	2860		2720.02159		883.78862
F	N. 73° 28′ 21″ W.	4621½	1314.69682			4430.55725
			4526.67421	4526.67421	5314.34587	5314.34587

CASE 1. *When the length and the Bearing of any one side are wanting.*

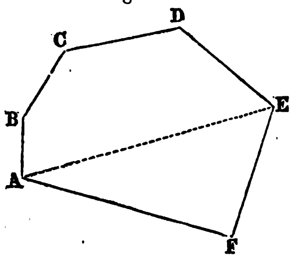
(440) Find the Latitudes and the Departures of the remaining sides. The difference of the North and South Latitudes of these lines, is the Latitude of the omitted line, and the difference of their Departures is its Departure. This Latitude and Departure are two sides of a right angled triangle of which the omitted line is the hypothenuse. Its length is therefore equal to the square root of the sum of their squares, and the quotient of the Departure divided by the Latitude is the tangent of its Bearing; as in Art. (**417**).

In the above survey, suppose the course from F to A to have been omitted or lost. The difference of the Latitudes of the remaining courses will be found to be 1314.69682, and the difference of the Departures to be 4430.55725. The square root of the sum of their squares is 4621.5; and the quotient of the Departure divided by the Latitude is the tangent of 73° 28′ 21″. The deficiencies were in North Latitude and West Departure; and the omitted course is therefore N. 73° 28′ 21″ W., 4621.5

CASE 2. *When the length of one side and the Bearing of another are wanting.*

(441) When the deficient sides adjoin each other. Find, as in Case 1, the length and Bearing of the line joining the ends of the remaining courses. This line and the deficient lines will form a triangle, in which two sides will be known, and the angle between the calculated side and the side whose Bearing is given can be found by Art. (**243**). The parts wanting can then be obtained by the common rules of Trigonometry.

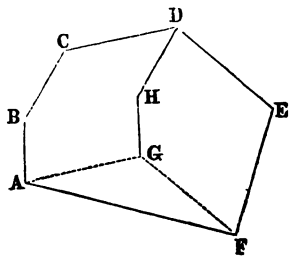
Fig. 303.



In the figure, let the length of EF, and the Bearing of FA be the omitted parts. The difference of the sums of the N. and S. Latitudes, and the E. and W. Departures of the complete courses from A to E, are respectively 1405.32477 North Latitude, and 5314.34587 East Departure. The course, EA, corresponding to this deficiency we find, by proceeding as in case 1, to be S. 75° 11′ 15″ W., 5497.026. The angle AEF is therefore $= 75° \ 11' \ 15'' - 18° = 57° \ 11' \ 15''$. Then in the triangle AEF are given the sides AE, AF, and the angle AEF to find the remaining parts; viz. the angle AFE $= 91° \ 28' \ 21''$, whence the Bearing of FA $= 91° \ 28' \ 21'' - 18° =$ N. 73° 28′ 21″ W.; and the side EF $= 28.60$.

(442) When the deficient sides are separated from each other. A modification of the preceding method will still apply. In this figure let the omissions be the Bearing of FA and the length of CD. Imagine the courses to change places without changing Bearings or lengths, so as to bring the deficient lines next to each other, by transferring CD to AG, AB to GH, and BC to HD. This will not affect their Latitudes or Departures. Join GF. Then in the figure DEFGH, the Latitudes and Departures of all the sides but FG are known, whence its length and Bearing can be found as in Case 1. Then the triangle AGF may be treated like the triangle AEF in the last article, to obtain the length of AG = CD, and the Bearing of FA.

Fig. 304.



(443) *Otherwise, by changing the Meridian.* Imagine the field to turn around, till the side of which the distance is unknown, becomes the Meridian, i. e. comes to be due North and South;

all the other sides retaining their *relative* positions, and continuing to make the same angles with each other. Change their Bearings, accordingly, as directed in Art. (244). Find the Latitudes and Departures of the sides in their new positions. Since the side whose length was unknown has been made the Meridian, it has no Departure, whatever may be its unknown length; and the difference of the columns of Departure will therefore be the Departure of the side whose Bearing is unknown. The length of this side is given. It is the hypothenuse of a right angled triangle, of which the Departure is one side. Hence the other side, which is the Latitude, can be at once found; and also the unknown Bearing.

Put this Latitude in the Table in the blank where it belongs. Then add up the columns of Latitude, and the difference of their sums will be the unknown length of the side which had been made a Meridian.*

Let the omitted quantities be, as in the last article, the length of CD and the Bearing of FA. Make CD the Meridian. The changed Bearings will then be found by Art. (244) to be as in the margin. To aid the imagination, turn the book around till CD points up and down, as North lines are usually placed on a map. Then obtain the Latitudes of the courses with their new Bearings and old distances, and proceed as has been directed.

STA.	OLD BEARING.	NEW BEARING.
A	North.	N. 80° W.
B	N. 32° E.	N. 48° W.
C	N. 80° E.	*North.*
D	S. 48° E.	N. 52° E.
E	S. 18° W.	S. 62° E.
F		

CASE 3. *When the lengths of two sides are wanting.*

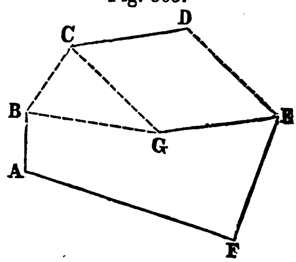
(444) When the deficient sides adjoin each other. Find the Latitudes and Departures of the other courses, and then, by Case 1, find the length and Bearing of the line joining the extremities of the deficient courses. Then, in the triangle thus formed, are known one side and all the angles (deduced from the Bearings) to find the lengths of the other two sides.

* This conception of thus changing the Bearings is stated to be due to Prof. Robert Patterson, of Philadelphia, by whom it was communicated to Mr. John Gummere, and published by him, in 1814, in his "Treatise on Surveying."

Thus, in Fig. 303, page 299, let EF and FA be the sides whose lengths are unknown. EA is then to be calculated, and its length will be found, as in Art. (441), to be 5497.026, and its bearing S. 75° 11′ 15″ W., whence the angle AEF $= 75° 11' 15'' - 18° = 57° 11' 15''$; AFE $= 18° + 73° 28' 21'' = 91° 28' 21''$; and EAF $= 31° 20' 24''$; whence can be obtained EF $= 28.60$ and FA $= 46.215$.

(445) When the deficient sides are separated from each other. Let the lengths of BC and DE be those omitted. Again imagine the courses to change places, so as to bring the deficient lines together, DE being transferred to CG, and CD to GE. Join BG. Then in the figure ABGEFA, are known the Latitudes and Departures of all the courses except BG, whence its length and Bearing can be found as in Case 1. Then in the triangle BCG, the angle CBG can be found from the Bearings of CB and BG, and the angle CGB from the Bearings of BG and GC. Then all the angles of the triangle are known and one side, BG, whence to find the required sides, BC $= 1782$, and CG $=$ DE $= 2700$.

Fig. 305.



(446) *Otherwise, by changing the Meridian.* As in Art. (443), imagine the field to turn around, till one of the sides whose length is wanting, becomes a Meridian or due North and South. Change all the Bearings correspondingly. Find the Latitudes and Departures of the changed courses. The difference of the columns of Departure will be the Departure of the second course of unknown length, since the course made Meridian has now no Departure. The new Bearing of this second course being given, in the right angled triangle formed by this course, as an hypothenuse, and its Departure and Latitude, we know one side, the Departure, and the acute angles, which are the Bearing and its complement. The length of the course is then readily calculated; and also its Latitude. This Latitude being inserted in its proper place, the difference of the columns of Latitude will be the length of that wanting side which had been made a Meridian.

Thus, let the lengths of BC and DE be wanting, as in the preceding example. Make BC a Meridian. The other Bearings are then changed as in the margin. Calculate new Latitudes and Departures. The difference of the Departures will be the Departure of DE, since BC, being a Meridian, has no Departure. Hence the length and Latitude of DE are readily obtained. This Latitude being put in the table, and the columns of Latitude then added up, their difference will be the length of BC.

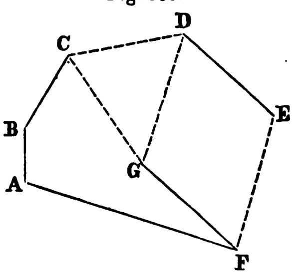
STA.	OLD BEARING.	NEW BEARING.
A	North.	N. 32° W.
B	N. 32° E.	*North.*
C	N. 80° E.	N. 48° E.
D	S. 48° E.	S. 80° E.
E	S. 18° W.	S. 14° E.
F	N. 73° 28′ 21″ W.	S. 74° 31′ 39″ W.

CASE 4. *When the Bearings of two sides are wanting.*

(447) When the deficient sides adjoin each other. Find the Latitudes and Departures of the other sides, and then, as in Case 1, find the length and bearing of the line joining the extremities of the deficient sides. Then in the triangle thus formed we have the three sides to find the angles and thence the Bearings.

(448) When the deficient sides are separated from each other. Change the places of the sides so as to bring the deficient ones next to each other. Thus, in the figure, supposing the Bearings of CD, and EF to be wanting, transfer EF to DG, and DE to GF. Then calculate, as in Case 1, the length and Bearing of the line joining the extremities of the deficient sides, CG in the figure. This line and the deficient sides form a triangle in which the three sides are given to determine the angles and thence the required Bearings.*

Fig. 306.



* The fullest investigation of this subject, developing many curious points, will be found in Mascheroni's "*Problemes de Geometrie pour les Arpenteurs,*" and Lhuillier's "*Polygonometrie.*"

PART VIII.

PLANE TABLE SURVEYING.

(449) THE Plane Table is in substance merely a drawing board fixed on a tripod, so that lines may be drawn on it by a ruler placed so as to point to any object in sight. All its parts are mere additions to render this operation more convenient and precise.

Such an arrangement may be applied to any kind of "Angular Surveying"; such as the Third Method, "Polar Surveying," in its two modifications of *Radiation* and *Progression*, (characterized in Art. (**220**)), and the Fourth Method, by *Intersections*. Each of these will be successively explained. The instrument is very convenient for filling in the details of a survey, when the principal points have been determined by the more precise method of "Triangular Surveying" and can then be platted on the paper in advance. It has the great advantage of dispensing with all notes and records of the measurements, since they are platted as they are made. It thus saves time and lessens mistakes, but is wanting in precision.

(450) The Table. It is usually a rectangular board of well seasoned pine, about 20 inches wide and 30 long. The paper to be drawn upon may be attached to it by drawing-pins, or by clamping plates fixed on its sides for that purpose, or by springs pressed upon it, or it may be held between rollers at opposite sides of the table. Tinted paper is less dazzling in the sun. Cugnot's joint, described on page 134, is the best for connecting it with its tripod, though a pair of parallel plates, like those of the Theodolite, are often used. A detached level is placed on the board to test its horizontality; though a smooth ball, as a marble, will answer the same purpose approximately.

A pair of sights, like those of the compass, are sometimes placed under the board, serving, like a "Watch Telescope," (Art. (**339**), to detect any movement of the instrument. To find what point on the lower side of the board is exactly under a point on the upper side, so that by suspending a plumb-line from the former the latter may be exactly over any desired point of ground, a large pair of "callipers," or dividers with curved legs, may be used, one of their points being placed on the upper point of the board, and their other point then determining the corresponding under point; or a frame forming three sides of a rectangle, like a slate frame, may be placed so that one end of one side of it touches the upper point, and the end of the corresponding side is under the table precisely below the given point, so that from this end a plumb-line can be dropped. A compass is sometimes attached to the table, or a detached compass, consisting of a needle in a narrow box, (called a Declinator), is placed upon it, as desired. The edges of the table are sometimes divided into degrees, like the "Drawing board Protractor," Art. (**278**). It then becomes a sort of Goniometer, like that of Art. (**213**).

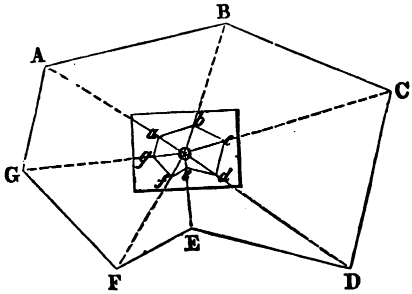
(451) The Alidade. The ruler has a fiducial or feather edge, which may be divided into inches, tenths, &c. At each end it carries a sight like those of the compass. Two needles would be tolerable substitutes. The sights project beyond its edge so that their centre lines shall be precisely in the same vertical plane as this edge, in order that the lines drawn by it may correspond to the lines sighted on by them. To test this, fix a needle in the board, place the alidade against it, sight to some near point, draw a line by the ruler, turn it end for end, again place it against the needle, again sight to the same point, and draw a new line. If it coincides with the former line, the above condition is satisfied. The ruler and sights together take the name of *Alidade*. If a point should be too high or too low to be seen with the alidade, a plumb-line, held between the eye and the object, will remove the difficulty.

A telescope is sometimes substituted for the sights, being supported above the ruler by a standard, and capable of pointing upward or downward. It admits of adjustments similar in principle to the 2d and 3d adjustments of the Transit, Part IV, Chapter 3, pages 242 and 246.

But even without these adjustments, whether of the sights or of the telescope, a survey could be made which would be perfectly correct as to the relative position of its parts, however far the line of sight might be from lying in the same vertical plane as the edge of the ruler, or even from being parallel to it; just as in the Transit or Theodolite the index or vernier need not to be exactly under the vertical hair of the telescope, since the angular deviation affects all the observed directions equally.

(452) Method of Radiation. This is the simplest, though not the best, method of surveying with the Plane-table. It is especially applicable to surveying a field, as in the figure. In it and the following figures, the size of the Table is much exaggerated. Set the instrument at any convenient point, as O; level it, and fix a needle (having a head of sealing-wax) in the board to represent the station.

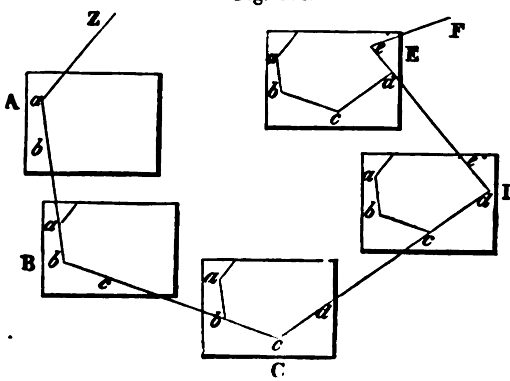
Fig. 307.



Direct the alidade to any corner of the field, as A, the fiducial edge of the ruler touching the needle, and draw an indefinite line by it. Measure OA, and set off the distance, to any desired scale, from the needle point, along the line just drawn, to *a*. The line OA is thus platted on the paper of the table as soon as determined in the field. Determine and plat in the same way, OB, OC, &c., to *b*, *c*, &c. Join *ab*, *bc*, &c., and a complete plat of the field is obtained. Trees, houses, hills, bends of rivers, &c., may be determnied in the same manner. The corresponding method with the Compass or Transit, was described in Articles **(258)** and **(391)**. The table may be set at one of the angles of the field, if more convenient. If the alidade has a telescope, the method of measuring distances with a stadia, described in Art. **(375)**, may be here applied with great advantage.

(453) Method of Progression. Let ABCD, &c., be the line to be surveyed. Fix a needle at a convenient point of the Plane-table, near a corner so as to leave room for the plat, and set up the table at B, the second angle of the line, so that the needle, whose point repre-

Fig. 308.



sents B, and which should be named *b*, shall be exactly over that station. Sight to A, pressing the fiducial edge of the ruler against the needle, and draw a line by it. Measure BA, and set off its length, to the desired scale, on the line just drawn, from *b* to a point *a*, representing A. Then sight to C, draw an indefinite line by the ruler, and on it set off the length of BC from *b* to *c*. Fix the needle at *c*. Set up at C, the point *c* being over this station, and make the line *cb* of the plat coincide in direction with CB on the ground, by placing the edge of the ruler on *cb*, and turning the table till the sights point to B. The compass, if the table have one, will facilitate this. Then sight forward from C to D, and fix CD, *cd* on the plat, as *bc* was fixed. Set up at D, make *dc* coincide with DC, and proceed as before. The figure shews the lines drawn at each successive station. The Table drawn at A shews how the survey might be commenced there.

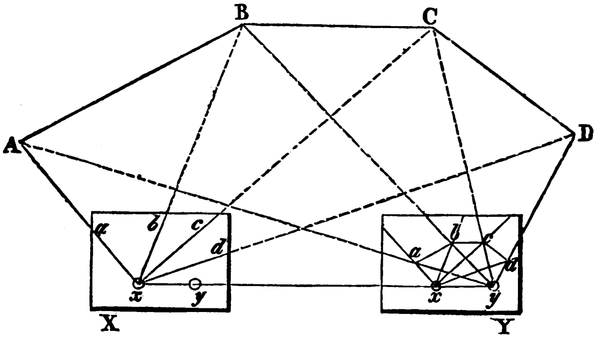
In going around a field, the work would be proved by the last line "closing" at the starting point, and, during the progress of the survey, by any direction, as from C to A on the ground, coinciding with the corresponding line, *ca*, on the plat.

This method is substantially the same as the method of surveying a line with the Transit, explained in Art. (372). It requires all the points to be accessible. It is especially suited to the survey of a road, a brook, a winding path through woods, &c. The offsets required may often be sketched in by eye with sufficient precision.

When the paper is filled, put on a new sheet, and begin by fixing on it two points, such as C and D, which were on the former sheet, and from them proceed as before. The sheets can then be afterwards united, so that all the points on both shall be in their true relative positions.

(454) Method of Intersection. This is the most usual and the most rapid method of using the Plane-table. The principle was referred to in Articles (**259**) and (**392**). Set up the instrument at any convenient point, as X in the figure, and sight to all

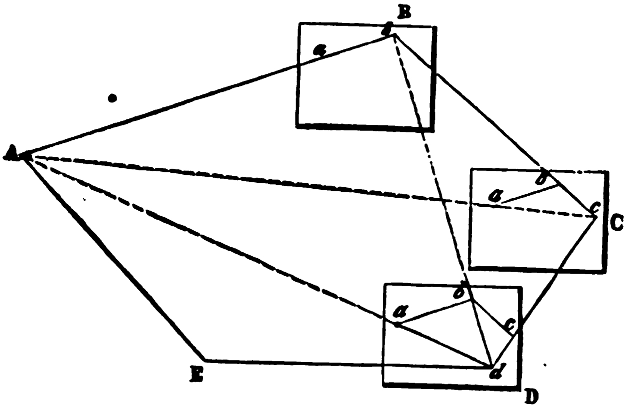
Fig. 309.



the desired points A, B, C, &c., which are visible, and draw indefinite lines in their directions. Measure any line XY, Y being one of the points sighted to, and set off this line on the paper to any scale. Set up at Y, and turn the table till the line XY on the paper lies in the direction of XY, on the ground, as at C in the last method. Sight to all the former points and draw lines in their directions, and the intersections of the two lines of sight to each point will determine them, by the Fourth Method, Art. (8). Points on the other side of the line XY could be determined at the same time. In surveying a field, one side of it may be taken for the base XY. Very acute or obtuse intersections should be avoided. 30° and 150° should be the *extreme* limits. The impossibility of always doing this, renders this method often deficient in precision. When the paper is filled, put on a new sheet, by fixing on it two known points, as in the preceding method.

(455) Method of Resection. This method (called by the French *Recoupement*) is a modification of the preceding method of Inter-

Fig. 310.



section. It requires the measurement of only one distance, but all the points must be accessible. Let AB be the measured distance. Lay it off on the paper as *ab*. Set the table up at B, and turn it till the line *ba* on the paper coincides with BA on the ground, as in the Method of Progression. Then sight to C, and draw an indefinite line by the ruler. Set up at C, and turn the line last drawn so as to point to B. Fix a needle at *a* on the table, place the alidade against the needle and turn it till it sights to A. Then the point in which the edge of the ruler cuts the line drawn from B will be the point *c* on the table. Next sight to D, and draw an indefinite line. Set up at D, and make the line last drawn point to C. Then fix the needle at *a* or *b*, and by the alidade, as at the last station, get a new line back from either of them, to cut the last drawn line at a point which will be *d*. So proceed as far as desired.

(456) To orient the table.* The operation of orientation consists in placing the table at any point so that its lines shall have the same directions as when it was at previous stations in the same survey.

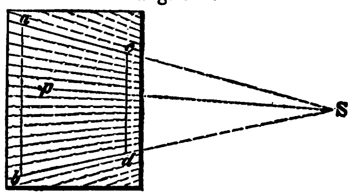
* The French phrase, To *orient* one's self, meaning to determine one's position, usually with respect to the four quarters of the heavens, of which the Orient is the leading one, well deserves naturalization in our language.

With a compass, this is very easily effected by turning the table till the needle of the attached compass, or that of the Declinator, placed in a fixed position, points to the same degree as when at the previous station.

Without a compass the table is oriented, when set at one end of a line previously determined, by sighting back on this line, as at C in the Method of Progression, Art. (**453**).

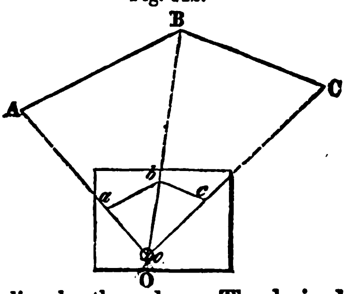
To orient the table, when at a station unconnected with others, is more difficult. It may be effected thus. Let *ab* on the table represent a line AB on the ground. Set up at A, make *ab* coincide with AB, and draw a line from *a* directed towards a steeple, or other conspicuous object, as S. Do the same at B. Draw a line *cd*, parallel to *ab*, and intercepted between *a*S, and *b*S. Divide *ab* and *cd* into the same number of equal parts. The table is then prepared. Now let there be a station, P, *p* on the table, at which the table is to be oriented. Set the table, so that *p* is over P, apply the edge of the ruler to *p*, and turn it till this edge cuts *cd* in the division corresponding to that in which it cuts *ab*. Then turn the table till the sights point to S, and the table will be oriented.

Fig. 311.



(457) To find one's place on the ground. This problem may be otherwise expressed as Interpolating a point in a plat. It is most easily performed by reversing the Method of Intersection. Set up the table over the station, O in the figure, whose place on the plat already on the table is desired, and *orient* it, by one of the means described in the last article. Make the edge of the ruler pass through some point, *a* on the table, and turn it till the sights point to the corresponding station, A on the ground. Draw a line by the ruler. The desired

Fig. 312.



point is somewhere in this line. Make the ruler pass through another point, *b* on the table, and make the sights point to B on the ground. Draw a second line, and its intersection with the first will be the point desired. Using C in the same way would give a third line to prove the work. This operation may be used as a new method of surveying with the plane-table, since any number of points can have their places fixed in the same manner.

This problem may also be executed on the principle of Trilinear Surveying. Three points being given on the table, lay on it a piece of transparent paper, fix a needle any where on this, and with the alidade sight and draw lines towards each of these three points on the ground. Then use this paper to find the desired point, precisely as directed in the last sentence of Art. (**398**), page 277.

(458) Inaccessible distances. Many of the problems in Part VII. can be at once solved on the ground by the plane-table, since it is at the same time a Goniometer and a Protractor. Thus, the Problem of Art. (**435**) may be solved as follows, on the principle of the construction in the last paragraph of that article. Set the table at C. Mark on it a point, *c'*, to represent C, placing *c'* vertically over C. Sight to A, B and D, and draw corresponding lines from *c'*. Set up at D, mark any point on the line drawn from *c'* towards D, and call it *d'*. Let *d'* be exactly over D, and direct *d'c'* toward C. Then sight to A and B, and draw corresponding lines, and their intersections with the lines before drawn towards A and B will fix points *a'* and *b'*. Then on the line joining *a* and *b*, given on the paper to represent A and B, *ab* being equal to AB on any scale, construct a figure, *abcd*, *similar* to *a'b'c'd'*, and the line *cd* thus determined will represent CD on the same scale as AB.

PART IX.

SURVEYING WITHOUT INSTRUMENTS.

(459) THE Principles which were established in Part I, and subsequently applied to surveying with various instruments, may also be employed, with tolerable correctness, for determining and representing the relative positions of larger or smaller portions of the earth's surface without any Instruments but such as can be extemporized.

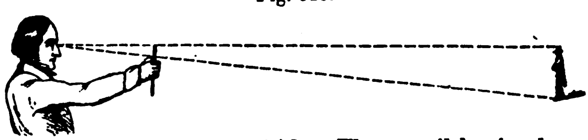
The prominent objects on the ground, such as houses, trees, the summits of hills, the bends of rivers, the crossings of roads, &c., are regarded as "points" to be "determined." Distances and angles are consequently required. Approximate methods of obtaining these will therefore be first given.

(460) Distances by pacing. Quite an accurate measurement of a line of ground may be made by walking over it at a uniform pace, and counting the steps taken. But the art of walking in a straight line must first be acquired. To do this, fix the eye on two objects in the desired line, such as two trees, or bushes, or stones, or tufts of grass. Walk forward, keeping the nearest of these objects steadily covering the other. Before getting up to the nearest object, choose a new one in line farther ahead, and then proceed as before, and so on. It is better not to attempt to make each of the paces three feet, but to take steps of the natural length, and to ascertain the value of each by walking over a known distance, and dividing it by the number of paces required to traverse it. Every person should thus determine the usual length of his own steps, repeating the experiment sufficiently often. The French "Geographical Engineers" accustom themselves to take regular steps of eight-tenths of a *metre*, equal to two feet seven and a half inches. The English military pace is two feet and six inches. This is regarded as a usual average. 108 such paces per minute give 3.07 English miles per hour. Quick pacing of 120 such paces per minute gives 3.41 miles per hour. Slow paces, of three feet each and 60 per minute, give 2.04 miles per hour.*

An instrument, called a Pedometer, has been contrived, which counts the steps taken by one wearing it, without any attention on his part. It is attached to the body, and a cord, passing from it to the foot, at each step moves a toothed wheel one division, and some intermediate wheelwork records the whole number upon a dial.

(461) Distances by visual angles. Prepare *a scale*, by marking off on a pencil what length of it, when it is held off at arm's length, a man's height appears to cover at different distances (previously measured with accuracy) of 100, 500, 1000 feet, &c. To apply this, when a man is seen at any unknown distance, hold up the pencil at arm's length, making the top of it come in the line from the eye to his head, and placing the thumb nail in the line from

Fig. 313.



the eye to his feet, as in Fig. 313. The pencil having been previously graduated by the method above explained, the portion of it now intercepted between these two lines will indicate the corresponding distance.

If no previous scale have been prepared, and the distance of a man be required, take a foot-rule, or any measure minutely divided, hold it off at arm's length as before, and see how much a man's height covers. Then knowing the distance from the eye to the rule, a statement by the Rule of Three (on the principle of similar triangles) will give the distance required. Suppose a man's height, of 70 inches, covers 1 inch of the rule. He is then 70 times as far

* A horse, on a walk, averages 330 feet per minute, on a trot 650, and on a common gallop 1040. For longer times, the difference in horses is more apparent.

from the eye as the rule; and if its distance be 2 feet, that of the man is 140 feet. Instead of a man's height, that of an ordinary house, of an apple-tree, the length of a fence-rail, &c., may be be taken as the standard of comparison.

To keep the arm immovable, tie a string of known length to the pencil, and hold between the teeth a knot tied at the other end of the string.

(462) Distances by visibility. The degree of visibility of various well-known objects will indicate approximately how far distant they are. Thus, by ordinary eyes, the windows of a large house can be counted at a distance of about 13000 feet, or $2\frac{1}{2}$ miles; men and horses will be perceived as points at about half that distance, or $1\frac{1}{4}$ miles; a horse can be clearly distinguished at about 4000 feet; the movements of men at 2600 feet, or half a mile; and the head of a man, occasionally, at 2300 feet, and very plainly at 1300 feet, or a quarter of a mile. The Arabs of Algeria define a mile as "the distance at which you can no longer distinguish a man from a woman." These distances of visibility will of course vary somewhat with the state of the atmosphere, and still more with individual acuteness of sight, but each person should make a corresponding scale for himself.

(463) Distances by sound. Sound passes through the air with a moderate and known velocity; light passes almost instantaneously. If, then, two distant points be visible from each other, and a gun be fired at night from one of them, an observer at the other, noting by a stop-watch the time at which the flash is seen, and then that at which the report is heard, can tell by the intervening number of seconds how far apart the points are, knowing how far sound travels in a second. Sound moves about 1090 feet per second in dry air, with the temperature at the freezing point, 32° Fahrenheit. For higher or lower temperatures add or subtract $1\frac{1}{7}$ foot for each degree of Fahrenheit. If a wind blows with or against the movement of the sound, its velocity must be added or subtracted. If it blows obliquely, the correction will evidently equal its velocity multiplied by the cosine of the angle which the direction of the wind makes with the direction of the sound.* If the gun be fired at each end of the base in turn, and the means of the times taken, the effect of the wind will be eliminated.

If a watch is not at hand, suspend a pebble to a string (such as a thread drawn from a handkerchief) and count its vibrations. If it be $39\frac{1}{8}$ inches long, it will vibrate in one second; if $9\frac{3}{4}$ inches long, in half a second, &c. If its length is unknown at the time, still count its vibrations; measure it subsequently; and then will the time of its vibration, in seconds, $= \sqrt{\left(\frac{\text{length of string}}{39\frac{1}{8}}\right)}$.

(464) Angles. Right angles are those most frequently required in this kind of survey, and they can be estimated by the eye with much accuracy. If other angles are desired, they will be determined by measuring equal distances along the lines which make the angle, and then the line, or chord, joining the ends of these distances, thus forming chain angles, explained in Art. (**100**).

(465) Methods of operation. The "First Method" of determining the position of a point, Art. (**5**), is the one most generally applicable. Some line, as AB in Fig. 1, is paced, or otherwise measured, and then the lines AS and BS; the point S is thus determined.

The "Second Method," Art. (**6**), is also much employed, the right angles being obtained by eye, or by the easy methods given in Part II, Chapter V, Arts. (**140**), &c. It is used for offsets, as in Part II, Chapter III, Arts. (**114**), &c.

The "Third Method," Art. (**7**), may also be used, the angles being determined as in Art. (**464**).

The "Fourth Method," Art. (**8**), may also be employed, the angles being similarly determined.

The "Fifth Method," Art. (**10**), would seldom be used, unless by making an extempore plane-table, and proceeding as directed in the last paragraph of Art. (**457**).

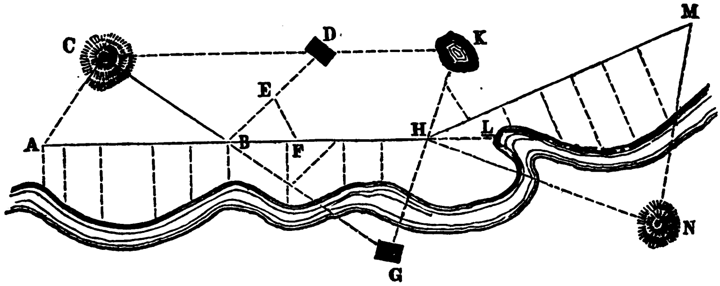
* A gentle, pleasant wind has a velocity of 10 feet per second; a brisk gale 20 feet per second; a very brisk gale 30 feet; a high wind 50 feet; a very high wind 70 feet; a storm or tempest 80 feet; a great storm 100 feet; a hurricane 120 feet; and a violent hurricane, that tears up trees, &c., 150 feet per second

The method referred to in Art. (**11**) may also be employed.

When a sketch has made some progress, new points may be fixed on it by their being in line with others already determined.

All these methods of operation are shown in the following figure. AB is a line paced, or otherwise measured approximately.

Fig. 314.



The hill C is determined by the first method. The river on the other side of AB is determined by offsets according to the Second Method. The house D is determined by the Third Method, EBF being a chain angle. The house G is determined by the Fourth Method, chain angles being measured at B and H, a point in AB prolonged. The pond K is determined, as in Art. (**11**), by the intersection of the alinements CD and GH prolonged. The bend of the river at L is determined by its distance from H in the line of AH prolonged. A new base line, HM, is fixed by a chain angle at H, and employed like the former one so as to fix the hill at N, &c. All these methods may thus be used collectively and successively. The necessary lines may always be ranged with rods, as directed in Art. (**169**), and very many of the instrumental methods already explained, may be practiced with extempore contrivances. The use of the Plane-table is an admirable preparation for this style of surveying or sketching, which is most frequently employed by Military Engineers, though they generally use a prismatic Compass, or pocket Sextant, and a sketching case, which may serve as a Plane-table.

PART X.

MAPPING.

CHAPTER I.

COPYING PLATS.

(466) THE Plat of a survey necessarily has many lines of construction drawn upon it, which are not needed in the finished map. These lines, and the marks of instruments, so disfigure the paper that a fair copy of the plat is usually made before the map is finished. The various methods of copying plats, &c., whether on the same scale, or reduced or enlarged, will therefore now be described.

(467) Stretching the paper. If the map is to be colored, the paper must first be wetted and stretched, or the application of the wet colors will cause its surface to swell or blister and become uneven. Therefore, with a soft sponge and clean water wet the back of the paper, working from the centre outward in all directions. The "water-mark" reads correctly only when looked at from the front side, which it thus distinguishes. When the paper is thoroughly wet and thus greatly expanded, glue its edges to the drawing board, for half an inch in width, turning them up against a ruler, passing the glue along them, and then turning them down and pressing them with the ruler. Some prefer gluing down opposite edges in succession, and others adjoining edges. The paper must be moderately stretched smooth during the process. Hot glue is best. Paste or gum may be used, if the paper be kept wet by a damp cloth, so that the edges may dry first. "Mouth-glue" may be used by rubbing it (moistened in the mouth or in boiling water) along the turned up edges, and then rubbing them dry by an ivory folder, a piece of dry paper being interposed. As this is a slower process, the middle of each side should first be fastened down, then the four angles, and lastly the intermediate portions. When the paper becomes dry, the creases and puckerings will have disappeared, and it will be as smooth and tight as a drum-head.

(468) Copying by tracing. Fix a large pane of clear glass in a frame, so that it can be supported at any angle before a window, or, at night, in front of a lamp. Place the plat to be copied on this glass, and the clean paper upon it. Connect them by pins, &c. Trace all the desired lines of the original with a sharp pencil, as lightly as they can be easily seen. Take care that the paper does not slip. If the plat is larger than the glass, copy its parts successively, being very careful to fix each part in its true relative position. Ink the lines with India ink, making them very fine and pale, if the map is to be afterwards colored.

(469) Copying on tracing paper. A thin transparent paper is prepared expressly for the purpose of making copies of maps and drawings, but it is too delicate for much handling. It may be prepared by soaking tissue paper in a mixture of turpentine and Canada balsam or balsam of fir (two parts of the former to one of the latter), and drying very slowly. Cold drawn linseed oil will answer tolerably, the sheets being hung up for some weeks to dry. Linen is also similarly prepared, and sold under the name of "Vellum tracing paper." It is less transparent than the tracing paper, but is very strong and durable. Both of these are used rather for preserving duplicates than for finished maps.

(470) Copying by transfer paper. This is thin paper, one side of which is rubbed with blacklead, &c., smoothly spread by cotton. It is laid on the clean paper, the blackened side downward, and the plat is placed upon it. All the lines of the plat are then gone over with moderate pressure by a blunt point, such as the eye-end of a small needle. A faint tracing of these lines will then be found on the clean paper, and can be inked at leisure. If the original cannot be thus treated, it may first be copied on tracing paper, and this copy be thus transferred. If the transfer paper be prepared by rubbing it with lampblack ground up with hard soap, its lines will be ineffaceable. It is then called "Camp-paper."

(471) Copying by punctures. Fix the clean paper on a drawing board and the plat over it. Prepare a fine needle with a sealing-wax head. Hold it very truly perpendicular to the board, and prick through every angle of the plat, and every corner and intersection of its other lines, such as houses, fences, &c., or at least the two ends of every line. For circles, the centre and one point of the circumference are sufficient. For irregular curves, such as rivers, &c., enough points must be pricked to indicate all their sinuosities. Work with system, finishing up one strip at a time, so as not to omit any necessary points nor to prick through any twice, though the latter is safer. When completed, remove the plat. The copy will present a wilderness of fine points. Select those which determine the leading lines, and then the rest will be easily recognized. A beginner should first pencil the lines lightly, and then ink them. An experienced draftsman will omit the pencilling. Two or three copies may be thus pricked through at once. The holes in the original plat may be made nearly invisible by rubbing them on the back of the sheet with a paper-folder, or the thumb nail.

(472) Copying by intersections. Draw a line on the clean paper equal in length to some important line of the original. Two starting points are thus obtained. Take in the dividers the distance from one end of the line on the original to a third point. From the corresponding end on the copy, describe an arc with this distance for radius and about where the point will come. Take the distance on the original from the other end of the line to the point, and describe a corresponding arc on the copy to intersect the former arc in a point which will be that desired. The principle of the operation is that of our "First Method," Art. (5). Two pairs of dividers may be used as explained in Art. (90). "Triangular compasses," having three legs, are used by fixing two of their legs on the two given points of the original, and the third leg on the point to be copied, and then transferring them to the copy. All the points of the original can thus be accurately reproduced. The operation is however very slow. Only the chief points of a plat may be thus transferred, and the details filled in by the following method.

(473) Copying by squares. On the original plat draw a series of parallel and equidistant lines. The T square does this most readily. Draw a similar series at right angles to these. The plat will then be covered with squares, as in Fig. 38, page 48. On the clean paper draw a similar series of squares. The important points may now be fixed as in the last article, and the rest copied by eye, all the points in each square of the original being properly placed in the corresponding square of the copy, noticing whether they are near the top or bottom of each square, on its right or left side, &c. This method is rapid, and in skilful hands quite accurate.

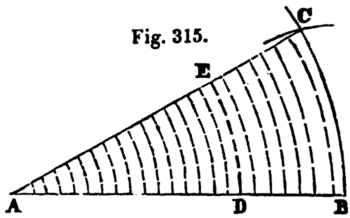
Instead of drawing lines on the original, a sheet of transparent paper containing them may be placed over it; or an open frame with threads stretched across it at equal distances and at right angles.

This method supplies a transition to the *Reduction* and *Enlargement* of plats in any desired ratio; under which head copying by the Pantagraph and Camera Lucida will be noticed.

(474) Reducing by squares. Begin, as in the preceding article, by drawing squares on the original, or placing them over it. Then on the clean paper draw a similar set of squares, but with their sides one-half, one third, &c., (according to the desired reduction), of those of the original plat. Then proceed as before to copy into each small square all the points and lines found in the large square of the plat in their true positions relative to the sides and corners of the square, observing to reduce each distance by eye, or as directed in the following article, in the given ratio.

(475) Reducing by proportional scales. Many graphical methods of finding the proportionate length on the copy, of any line of the original, may be used. The "Angle of reduction" is constructed thus. Draw any line AB. With it for radius and A for centre, describe an indefinite arc. With B for centre and a radius equal to one-half, one-third, &c., of AB according to the desired reduction describe another arc intersecting the former arc in C. Join AC. From A as centre describe a series of arcs. Now to reduce any distance, take it in the dividers, and set it off from A on AB, as to D. Then the distance from D to E, the other end of the arc passing through D, will be the proportionate length to be set off on the copy, in the manner directed in Art. (472).

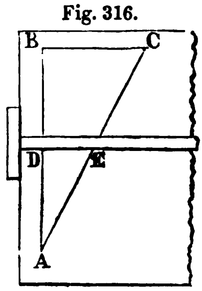
Fig. 315.



The Sector, or "Compass of proportion," described in Art. (52), presents such an "Angle of reduction," always ready to be used in this manner.

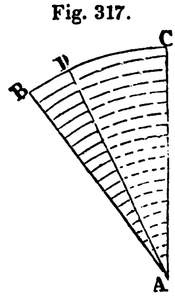
The "Angle of reduction" may be simplified thus. Draw a line, AB, parallel to one side of the drawing board, and another, BC, at right angles to it, and one-half, &c., of it, as desired. Join AC. Then let AD be the distance required to be reduced. Apply a T square so as to pass through D. It will meet AC in some point E, and DE will be the reduced length required.

Fig. 316.



Another arrangement for the same object is shown in Fig. 317. Draw two lines, AB, AC, at any angle, and describe a series of arcs from their intersection, A, as in the figure. Suppose the reduced scale is to be half the original scale. Divide the outermost arc into three equal parts, and draw a line from A to one of the points of division, as D. Then *each* arc will be divided into parts, one of which is twice the other. Take any distance on the original scale, and find by trial which of the arcs on

Fig. 317.



the right hand side of the figure it corresponds to. The other part of that arc will be half of it, as desired.

"Proportional compasses," being properly set, reduce lines in any desired ratio. A simple form of them, known as "Wholes and halves," is often useful. It consists of two slender bars, pointed at each end, and united by a pivot which is twice as far from one pair of the points as from the other pair. The long ends being set to any distance, the short ends will give precisely half that distance.

(476) Reducing by a pantagraph. This instrument consists of two long and two short rulers, connected so as to form a parallelogram, and capable of being so adjusted that when a tracing point attached to it is moved over the lines of a map, &c., a pencil attached to another part of it will mark on paper a precise copy, reduced on any scale desired. It is made in various forms. It is troublesome to use, though rapid in its work.

(477) Reducing by a camera lucida. This is used in the Coast Survey Office. It cannot reduce smaller than one-fourth, without losing distinctness, and is very trying to the eyes. Squares drawn on the original are brought to apparently coincide with squares on the reduction, and the details are then filled in with the pencil, as seen through the prism of the instrument.

(478) Enlarging plats. Plats may be enlarged by the principal methods which have been given for reducing them, but this should be done as seldom as possible, since every inaccuracy in the original becomes magnified in the copy. It is better to make a new plat from the original data.

CHAPTER II.

CONVENTIONAL SIGNS.

(479) Various conventional signs or marks have been adopted, more or less generally, to represent on maps the inequalities of the surface of the ground, its different kinds of culture or natural products, and the objects upon it, so as not to encumber and disfigure it with much writing or many descriptive legends. This is the purpose of what is called *Topographical Mapping*.

(480) The relief of ground. The inequalities of the surface of the earth, its elevations and depressions, its hills and hollows, constitute its "Relief." The representation of this is sometimes called "Hill drawing." Its difficulty arises from our being accustomed to see hills sideways, or "in elevation," while they must be represented as they would be seen from above, or "in plan." Various modes of thus drawing them are used; their positions being laid down in pencil as previously sketched by eye or measured.

If light be supposed to fall *vertically*, the slopes of the ground will receive less light in proportion to their steepness. The relief of ground will be indicated on this principle by making the steep slopes very dark, the gentler inclinations less so, and leaving the level surfaces white. The shades may be produced by tints of India ink applied with a brush, their edges, at the top and bottom of a hill or ridge, being softened off with a clean brush.

If light be supposed to fall *obliquely*, the slopes facing it will be light, and those turned from it dark. This mode is effective, but not precise. In it the light is usually supposed to come from the upper left hand corner of the map.

Horizontal contour lines are however the best convention for this purpose. Imagine a hill to be sliced off by a number of equidistant horizontal planes, and their intersections with it to be drawn as they would be seen from above, or horizontally projected on the map. These are "Contour lines." They are the same lines as would be formed by water surrounding the hill, and rising one foot at a time (or any other height) till it reached the top of the hill. The edge of the water, or its shore, at each successive rise, would be one of these horizontal contour lines. It is plain that their nearness or distance on the map would indicate the steepness or gentleness of the slopes. A right cone would thus be represented by a series of concentric circles, as in Fig. 318; an oblique cone by circles not concentric, but nearer to each other on the steep side than on the other, as in Fig. 319; and a half-egg, somewhat as in Fig. 320.

Fig. 318.

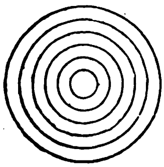


Fig. 319.

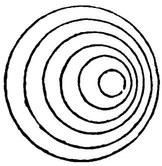
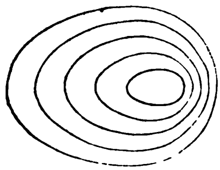
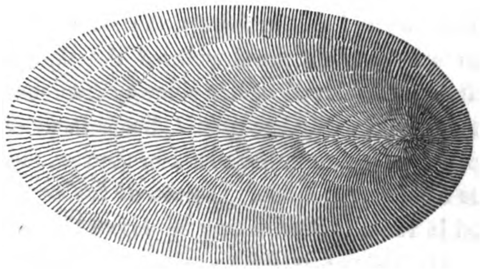


Fig. 320.



Vertical sections, perpendicular to these contour lines, are usually combined with them. They are the "Lines of greatest slope," and may be supposed to represent water running down the sides of the hill. They are also made thicker and nearer together on the steeper slopes, to produce the effect required by the convention of vertical light already referred to. The marginal figure shews an elongated half-egg, or oval hill, thus represented.

Fig. 321.



The spaces between the rows of vertical "Hatchings" indicate the contour lines, which are not actually drawn. The beauty of the graphical execution of this work depends on the uniformity of the strokes representing uniform slopes, on their perfectly regular gradation in thickness and nearness, for varying slopes, and on their being made precisely at right angles to the contour lines between which they are situated.

The methods of determining the contour lines are applications of Levelling, and will therefore be postponed, together with the farther details of "Hill-drawing," to the volume treating of that subject, which is announced in the Preface.

(481) Signs for natural surface. *Sand* is represented by fine dots made with the point of the pen; *gravel* by coarser dots. *Rocks* are drawn in their proper places in irregular angular forms, imitating their true appearance as seen from above. The nature of the rocks, or the *Geology* of the country, may be shown by applying the proper colors, as agreed on by geologists, to the back of the map, so that they may be seen by holding it up against the light, while they will thus not confuse the usual details.

(482) Signs for vegetation. *Woods* are represented by scolloped circles, irregularly disposed, imitating trees seen "in plan," and closer or farther apart according to the thickness of the forest. It is usual to shade their lower and right hand sides and to represent their shadows, as in the figure, though, in strictness, this is inconsistent with the hypothesis of vertical light, adopted for "hill-drawing." For pine and similar forests, the signs may have a star-like form, as on the right hand side of the figure. Trees are sometimes drawn "in elevation," or sideways, as usually seen. This makes them more easily recognized, but is in utter violation of the principles of mapping in horizontal projection, though it may be defended as a pure convention. *Orchards* are represented by trees arranged in rows. *Bushes* may be drawn like trees, but smaller.

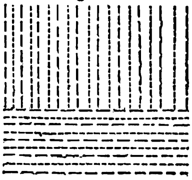
Fig. 322.

Grass-land is drawn with irregularly scattered groups of short lines, as in the figure, the lines being arranged in odd numbers, and so that the top of each group is convex and its bottom horizontal or parallel to the base of the drawing. *Meadows* are sometimes represented by pairs of diverging lines, (as on the right

Fig. 323.

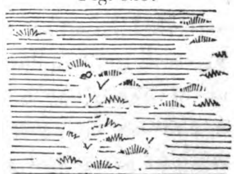
of the figure), which may be regarded as tall blades of grass. *Uncultivated* land is indicated by appropriately intermingling the signs for grass land, bushes, sand and rocks. *Cultivated* land is shown by parallel rows of broken and dotted lines, as in the figure, representing furrows. *Crops* are so temporary that signs for them are unnecessary, though often used. They are usually imitative, as for cotton, sugar, tobacco, rice, vines, hops, &c. *Gardens* are drawn with circular and other beds and walks.

Fig. 324.



(483) Signs for water. The *Sea-coast* is represented by drawing a line parallel to the shore, following all its windings and indentations, and as close to it as possible, then another parallel line a little more distant, then a third still more distant, and so on. Examples are seen in figures 287, &c. If these lines are drawn from the low tide mark, a similar set may be drawn between that and the high tide mark, and dots, for sand, be made over the included space. *Rivers* have each shore treated like the sea shore, as in the figures of Part VII.* *Brooks* would be shown by only two lines, or one, according to their magnitude. *Ponds* may be drawn like sea shores, or represented by parallel horizontal lines ruled across them. *Marshes* and *Swamps* are represented by an irregular intermingling of the preceding sign with that for grass and bushes, as in the figure.

Fig. 325.



(484) Colored Topography. The conventional signs which have been described, as made with the pen require much time and labor. Colors are generally used by the French as substitutes for them, and combine the advantages of great rapidity and effectiveness. Only three colors (besides India ink) are required; viz. *Gamboge* (yellow), *Indigo* (blue), and *Lake* (pink). Sepia, Burnt Sienna, Yellow ochre, Red lead, and Vermillion, are also sometimes used. The last three are difficult to work with. To

* Those in Part II, Chapter V, have the lines too close together in the middle.

use these paints, moisten the end of a cake and rub it up with a drop of water, afterwards diluting this to the proper tint, which should always be light and delicate. To cover any surface with a uniform flat tint, use a large camel's hair or sable brush, keep it always moderately full, incline the board towards you, previously moisten the paper with clean water if the outline is very irregular, begin at the top of the surface, apply a tint across the upper part, and continue it downwards, *never letting the edge dry*. This last is the secret of a smooth tint. It requires rapidity in returning to the beginning of a tint to continue it, and dexterity in following the outline. *Marbling*, or variegation, is produced by having a brush at each end of a stick, one for each color, and applying first one, and then the other beside it before it dries, so that they may blend but not mix, and produce an irregularly clouded appearance. Scratched parts of the paper may be painted over by first applying strong alum water to the place.

The conventions for colored Topography, adopted by the French Military Engineers, are as follows. WOODS, *yellow;* using gamboge and a very little indigo. GRASS-LAND, *green;* made of gamboge and indigo. CULTIVATED LAND, *brown;* lake, gamboge, and a little India ink. "Burnt Sienna" will answer. Adjoining fields should be slightly varied in tint. Sometimes furrows are indicated by strips of various colors. GARDENS are represented by small rectangular patches of brighter *green* and *brown*. UNCULTIVATED LAND, marbled *green* and light *brown*. BRUSH, BRAMBLES, &c., marbled *green* and *yellow*. HEATH, FURZE, &c., marbled *green* and *pink*. VINEYARDS, *purple;* lake and indigo. SANDS, a light *brown;* gamboge and lake. "Yellow ochre" will do. LAKES and RIVERS, light *blue*, with a darker tint on their upper and left hand sides. SEAS, dark *blue*, with a little yellow added. MARSHES, the *blue* of water, with spots of grass *green*, the touches all lying horizontally. ROADS, *brown;* between the tints for sand and cultivated ground, with more India ink. HILLS, *greenish brown;* gamboge, indigo, lake and India ink, instead of the pure India ink, directed in Art. (480). WOODS may be finished up by drawing the trees as in Art. (482) and coloring them green, with touches of gamboge towards the light (the upper and left hand side) and of indigo on the opposite side.

(485) Signs for detached objects. Too great a number of these will cause confusion. A few leading ones will be given, the meanings of which are apparent.

		Figs.			Figs.
Court house,		326.	*Wind mill,*		334.
Post office,		327.	*Steam mill,*		335.
Tavern,		328.	*Furnace,*		336.
Blacksmith's shop,		329.	*Woollen factory,*		337.
Guide board,		330.	*Cotton factory,*		338.
Quarry,		331.	*Glass works,*		339
Grist mill,		332.	*Church,*		340.
Saw mill,		333.	*Grave yard,*		341.

An ordinary house is drawn in its true position and size, and the ridge of its roof shown if the scale of the map is large enough. On a very small scale, a small shaded rectangle represents it. If colors are used, buildings of masonry are tinted a deep crimson, (with lake), and those of wood with India ink. Their lower and right hand sides are drawn with heavier lines. Fences of stone or wood, and hedges, may be drawn in imitation of the realities; and, if desired, colored appropriately.

Mines may be represented by the signs of the planets which were anciently associated with the various metals. The signs here given represent respectively,

Gold,	Silver,	Iron,	Copper,	Tin,	Lead,	Quicksilver.
☉	☽	♂	♀	♃	♄	☿

A large black circle, ●, may be used for Coal.

Boundary lines, of private properties, of townships, of counties, and of states, may be indicated by lines formed of various combinations of short lines, dots and crosses, as below.*

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* Very minute directions for the execution of the details described in this chapter, are given in Lieut. R. S. Smith's "Topographical Drawing." Wiley, N. Y.

CHAPTER III.

FINISHING THE MAP.

(486) Orientation. The map is usually so drawn that the top of the paper may represent the North. A Meridian line should also be drawn, both True and Magnetic, as in Fig. 199, page 189. The number of degrees and minutes in the Variation, if known, should also be placed between the two North points. Sometimes a compass-star is drawn and made very ornamental.

(487) Lettering. The style in which this is done very much affects the general appearance of the map. The young surveyor should give it much attention and careful practice. It must all be in imitation of the best printed models. No writing, however beautiful, is admissible. The usual letters are the ordinary ROMAN CAPITALS, Small Roman, *ITALIC CAPITALS*, *Small Italic*, and GOTHIC OR EGYPTIAN. This last, when well done, is very effective. For the Titles of maps, various fancy letters may be used. For very large letters, those formed only of the shades of the letters regarded as blocks (the body being rubbed out after being pencilled as a guide to the placing of the shades) are most easily made to look well. The simplest lettering is generally the best. The sizes of the names of places, &c., should be proportional to their importance. Elaborate tables for various scales have been published. It is better to make the letters too small than too large. They should not be crowded. Pencil lines should always be ruled as guides. The lettering should be in lines parallel to the bottom of the map, except the names of rivers, roads, &c., whose general course should be followed.

(488) Borders. The *Border* may be a single heavy line, enclosing the map in a rectangle, or such a line may be relieved by a finer line drawn parallel and near to it. Time should not be wasted in ornamenting the border. The simplest is the best.

(489) Joining paper. If the map is larger than the sheets of paper at hand, they should be joined with a feather-edge, by proceeding thus. Cut, with a knife guided by a ruler, about one-third through the thickness of the paper, and tear off on the under side, a strip of the remaining thickness, so as to leave a thin sharp edge. Treat the other sheet in the same way on the other side of it. When these two feather edges are then put together, (with paste, glue or varnish), they will make a neat and strong joint. The sheet which rests upon the other must be on the right hand side, if the sheets are joined lengthways, or below if they are joined in that direction, so that the thickness of the edge may not cast a shadow, when properly placed as to the light. The sheets must be joined before lines are drawn across them, or the lines will become distorted. Drawing paper is now made in rolls of great length, so as to render this operation unnecessary.

(490) Mounting maps. A map is sometimes required to be mounted, i. e. backed with canvas or muslin. To do this, wet the muslin and stretch it strongly on a board by tacks driven very near together. Cover it with strong paste, beating this in with a brush to fill up the pores of the muslin. Then spread paste over the back of the paper, and when it has soaked into it, apply it to the muslin, inclining the board, and pasting first a strip, about two inches wide, along the upper side of the paper, pressing it down with clean linen in order to drive out all air bubbles. Press down another strip in like manner, and so proceed till all is pasted. Let it dry very gradually and thoroughly before cutting the muslin from the board.

Maps may be varnished with picture varnish; or by applying four or five coats of isinglass size, letting each dry well before applying the next, and giving a full flowing coat of Canada balsam diluted with the best oil of turpentine.

PART XI.

LAYING OUT, PARTING OFF, AND DIVIDING UP LAND.*

CHAPTER I.

LAYING OUT LAND.

(491) Its nature. This operation is precisely the reverse of those of Surveying properly so called. The latter measures certain lines as they are; the former marks them out in the ground where they are required to be, in order to satisfy certain conditions. The same instruments, however, are used as in Surveying.

Perpendiculars and parallels are the lines most often employed. The *Perpendiculars* may be set out either with the chain alone, Arts. (**140**) to (**159**); still more easily with the Cross-staff, Art. (**104**), or the Optical-square, Art. (**107**); and most precisely with a Transit or Theodolite, Arts. (**402**) to (**406**). *Parallels* may also be set out with the chain alone, Arts. (**160**) to (**166**); or with Transit, &c., Arts. (**407**) and (**408**). The ranging out of lines by rods is described in Arts. (**169**) and (**178**), and with an Angular instrument, in Arts. (**376**), (**409**) and (**415**).

(492) To lay out squares. Reduce the desired content to square chains, and extract its square root. This will be the length of the required side, which is to be set out by one of the methods indicated in the preceding article.

An *Acre*, laid out in the form of a square, is frequently desired by farmers. Its side must be made $316\frac{1}{4}$ links of a Gunter's

* The Demonstrations of the Problems in this part, when required, will be found in Appendix B.

chain; or $208\frac{71}{100}$ feet; or $69\frac{57}{100}$ yards. It is often taken at 70 paces.

The number of plants, hills of corn, loads of manure, &c., which an acre will contain at any uniform distance apart, can be at once found by dividing 209 by this distance in feet, and multiplying the quotient by itself; or by dividing 43560 by the square of the distance in feet. Thus, at 3 feet apart, an acre would contain 4840 plants, &c.; at 10 feet apart, 436; at a rod apart, 160; and so on. If the distances apart be unequal, divide 43560 by the product of these distances in feet; thus, if the plants were in rows 6 feet apart, and the plants in the rows were 3 feet apart, 2420 of them would grow on one acre.

(493) To lay out rectangles. *The content and length being given*, both as measured by the same unit, divide the former by the latter, and the quotient will be the required breadth. Thus, 1 acre or 10 square chains, if 5 chains long, must be 2 chains wide.

The content being given and the length to be a certain number of times the breadth. Divide the content in square chains, &c., by the ratio of the length to the breadth, and the square root of the quotient will be the shorter side desired, whence the longer side is also known. Thus, let it be required to lay out 30 acres in the form of a rectangle 3 times as long as broad. 30 acres = 300 square chains. The desired rectangle will contain 3 squares, each of 100 sq. chs., having sides of 10 chs. The rectangle will therefore be 10 chs. wide and 30 long.

An *Acre* laid out in a rectangle twice as long as broad, will be 224 links by 448 links, nearly; or $147\frac{1}{2}$ feet by 295 feet; or $49\frac{1}{5}$ yards by $98\frac{2}{5}$ yards. 50 paces by 100 is often used as an approximation, easy to be remembered.

The content being given, and the difference between the length and breadth. Let c represent this content, and d this difference. Then the longer side $= \frac{1}{2}d + \frac{1}{2}\sqrt{(d^2 + 4c)}$.

Example. Let the content be 6.4 acres, and the difference 12 chains. Then the sides of the rectangle will be respectively 16 chains and 4 chains.

The content being given, and the sum of the length and breadth. Let c represent this content, and s this sum. Then the longer side $= \frac{1}{2}s + \frac{1}{2}\sqrt{(s^2 - 4c)}$.

Example. Let the content be 6.4 acres, and the sum 20 chains. The above formula gives the sides of the rectangle 16 chains and 4 chains as before.

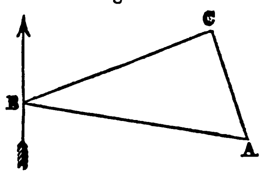
(494) To lay out triangles. The content and the base being given, divide the former by half the latter to get the height. At any point of the base erect a perpendicular of the length thus obtained, and it will be the vertex of the required triangle.

The content being given and the base having to be m times the height, the height will equal the square root of the quotient obtained by dividing twice the given area by m.

The content being given and the triangle to be equilateral, take the square root of the content and multiply it by 1.520. The product will be the length of the side required. This rule makes the sides of an equilateral triangle containing *one acre* to be $480\frac{1}{2}$ links. A quarter of an acre laid out in the same form would have each side 240 links long. An equilateral triangle is very easily set out on the ground, as directed in Art. (**90**), under "Platting," using a rope or chain for compasses.

(495) The content and base being given, and one side having to make a given angle, as B, with the base AB, the length of the side $BC = \frac{2 \times ABC}{AB \,.\, \sin.\, B}$.

Fig. 342.



Example. Eighty acres are to be laid out in the form of a triangle, on a base, AB, of sixty chains, bearing N. 80° W. the bearing of the side BC being N. 70° E. Here the angle B is found from the Bearings (by Art. (**243**), reversing one of them) to be 30°. Hence $BC = 53.33$. The figure is on a scale of 50 chains to 1 inch $= 1:39600$.

Any right-line figure may be laid out by analogous methods.

(496) To lay out circles. Multiply the given content by 7, divide the product by 22, and take the square root of the quotient.

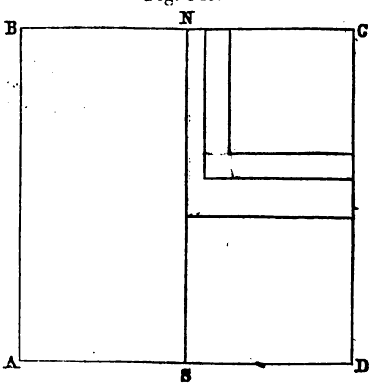
This will give the radius, with which the circle can be described on the ground with a rope or chain. A circle containing one acre has a radius of $178\frac{1}{4}$ links. A circle containing a quarter of an acre will have a radius of 89 links.

(497) Town lots. House lots in cities are usually laid off as rectangles of 25 feet front and 100 feet depth, variously combined in blocks. Part of New-York is laid out in blocks 200 feet by 800, each containing 64 lots, and separated by streets, 60 feet wide, running along their long sides, and avenues, 100 feet wide, on their short sides. The eight lots on each short side of the block, front on the avenues, and the remaining forty-eight lots front on the streets. Such a block covers almost precisely $3\frac{2}{3}$ acres, and $17\frac{1}{2}$ such lots about make an acre. But, allowing for the streets, land laid out into lots, 25 by 100, arranged as above, would contain only 11.9, or not quite 12 lots per acre.

Lots in small towns and villages are laid out of greater size and less uniformity. 50 feet by 100 is a frequent size for new villages, the blocks being 200 feet by 500, each therefore containing 20 lots.

(498) Land sold for taxes. A case occurring in the State of New-York will serve as an application of the modes of laying out squares and rectangles. Land on which taxes are unpaid is sold at auction to the *lowest* bidder; i. e. to him who will accept the smallest portion of it in return for paying the taxes on the whole. The lot in question was originally the east half of the square lot ABCD, containing 500 acres. At a sale for taxes in 1830, 70 acres were bid off, and this area was set off to the purchaser in a square lot, from the north-east corner. Required the side of the square in links. Again, in 1834, 29 acres more were thus sold, to be set off in a strip of equal width

Fig. 343.



around the square previously sold. Required the width of this strip. Once more, in 1839, 42 acres more were sold, to be set off around the preceding piece. Required the dimensions of this third portion. The answer can be proved by calculating if the dimensions of the remaining rectangle will give the content which it should have, viz. $250 - (70 + 29 + 42) = 109$ Acres.

The figure is on a scale of 40 chains to 1 inch $= 1:31680$.

(499) New countries. The operations of laying out land for the purposes of settlers, are required on a large scale in new countries, in combination with their survey. There is great difficulty in uniting the necessary precision, rapidity and cheapness. "Triangular Surveying" will ensure the first of these qualities, but is deficient in the last two, and leaves the laying out of lots to be subsequently executed. "Compass Surveying" possesses the last two qualities, but not the first. The United States system for surveying and laying out the Public Lands admirably combines an accurate determination of standard lines (Meridians and Parallels) with a cheap and rapid subdivision by compass. The subject is so important and extensive that it will be explained by itself in Part XII.

CHAPTER II.

PARTING OFF LAND.

(500) It is often required to part off from a field, or from an indefinite space, a certain number of acres by a fence or other boundary line, which is also required to run in a particular direction, to start from a certain point, or to fulfil some other condition. The various cases most likely to occur will be here arranged according to these conditions. Both graphical and numerical methods will generally be given.*

* The given lines will be represented by fine full lines; the lines of construction by broken lines, and the lines of the result by heavy full lines.

The given content is always supposed to be reduced to square chains and decimal parts, and the lines to be in chains and decimals.

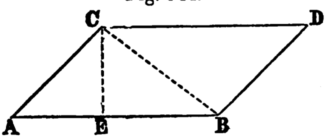
A. By a line parallel to a side.

(501) To part off a rectangle. If the sides of the field adjacent to the given side make right angles with it, the figure parted off by a parallel to the given side will be a rectangle, and its breadth will equal the required content divided by that side, as in Art. (493).

If the field be bounded by a curved or zigzag line outside of the given side, find the content between these irregular lines and the given straight side, by the method of offsets, subtract it from the content required to be parted off, and proceed with the *remainder* as above. The same directions apply to the subsequent problems.

(502) To part off a parallelogram. If the sides adjacent to the given side be parallel, the figure parted off will be a parallelogram, and its perpendicular width, CE, will be obtained as above. The length of one of the parallel sides, as $AC = \frac{CE}{\sin. A} = \frac{ABDC}{AB \,.\, \sin. A}$.

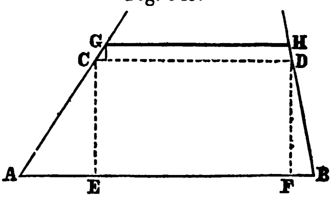
Fig. 344.



(503) To part off a trapezoid. When the sides of the field adjacent to the given side are not parallel, the figure parted off will be a trapezoid.

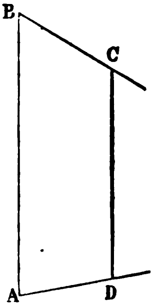
When the field or figure is given on the ground, or on a plat, begin as if the sides were parallel, dividing the given content by the base AB. The quotient will be an *approximate* breadth, CE, or DF; too small if the sides converge, as in the figure, and *vice versa*. Measure CD. Calculate the content of ABDC. Divide the difference of it and the required

Fig. 345.



content by CD. Set off the quotient perpendicular to CD, (in this figure, outside of it,) and it will give a new line, GH, a still nearer approximation to that desired. The operation may be repeated, if found necessary.

Fig. 346.



(504) When the field is given by Bearings, deduce from them, as in Art. (**243**), the angles at A and B. The required sides will then be given by these formulas:

$$CD = \sqrt{\left(AB^2 - \frac{2 \times ABCD \,.\, \sin.\,(A+B)}{\sin.\,A \,.\, \sin.\,B}\right)}.$$

$$AD = (AB - CD)\,\frac{\sin.\,B}{\sin.\,(A+B)}.$$

$$BC = (AB - CD)\,\frac{\sin.\,A}{\sin.\,(A+B)}.$$

When the sides AD and BC diverge, instead of converging, as in the figure, the negative term, in the expression for CD, becomes positive; and in the expressions for both AD and BC, the first factor becomes (CD — AB).

The perpendicular breadth of the trapezoid $= AD \,.\, \sin.\,A$; or, $= BC \,.\, \sin.\,B$.

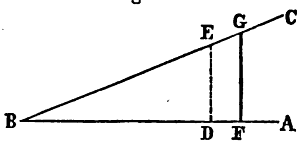
Example. Let AB run North, six chains; AD, N. 80° E.; BC, S. 60° E. Let it be required to part off one acre by a fence parallel to AB. Here $AB = 6.00$, $ABCD = 10$ square chains, $A = 80°$, $B = 60°$. *Ans.* $CD = 4.57$, $AD = 1.92$, $BC = 2.18$, and the breadth $= 1.89$.

The figure is on a scale of 4 chains to 1 inch $= 1:3168$.

B. BY A LINE PERPENDICULAR TO A SIDE.

(505) To part off a triangle. Let FG be the required line. When the field is given on the ground, or on a plat, at any point, as D, of the given side AB, set out a "guess line," DE, perpendicular to AB, and calculate the content of DEB. Then the required distance BF, from the angular point to the foot of the desired perpendicular, $= BD\sqrt{\left(\frac{BFG}{BDE}\right)}$.

Fig. 347.

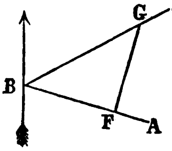


Example. Let BD = 30 chains; ED = 12 chains; and the desired area = 24.8 acres. Then BF = 35.22 chains.

The scale of the figure is 30 chains to 1 inch = 1 : 23760.

(506) When the field is given by Bearings, find the angle B from the Bearings; then is $BF = \sqrt{\left(\frac{2 \times BFG}{\text{tang. } B}\right)}$.

Fig. 348.

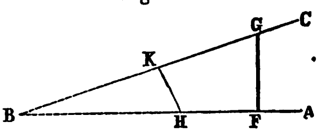


Example. Let BA bear S. 75° E., and BC N. 60° E., and let five acres be required to be parted off from the field by a perpendicular to BA. Here the angle B = 45°, and BF = 10.00 chains.

The scale of the figure is 20 chains to 1 inch = 1 : 15840.

(507) To part off a quadrilateral. Produce the converging sides to meet at B. Calculate the content of the triangle HKB, whether on the ground or plat, or from Bearings. Add it to the content of the quadrilateral required to be parted off, and it will give that of the triangle FGB, and the method of the preceding case can then be applied.

Fig. 369.



(508) To part off any figure. If the field be very irregularly shaped, find by trial any line which will part off a little less than the required area. This trial line will represent HK in the preceding figure, and the problem is reduced to parting off, according to the required condition, a *quadrilateral*, comprised between the trial line, two sides of the field, and the required line, and containing the difference between the required content and that parted off by the trial-line.

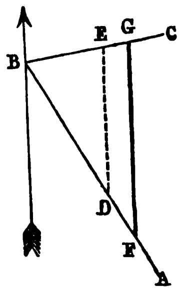
C. By a line running in any given direction.

(509) To part off a triangle. By construction, on the ground or the plat, proceed nearly as in Art. (505), setting out a line in the required direction, calculating the triangle thus formed, and obtaining BF by the same formula as in that Article.

(510) If the field be given by Bearings, find from them the angles CBA and GFB; then is

$$BF = \sqrt{\left(\frac{2 \times BFG \sin.(B + F)}{\sin. B . \sin. F}\right)}.$$

Fig. 350.

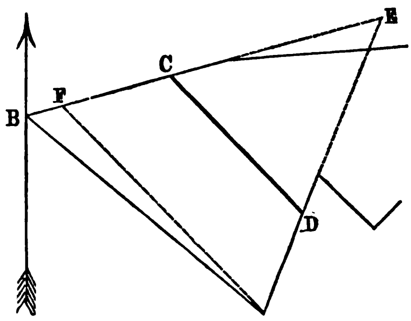


Example. Let BA bear S. 30° E.; BC, N. 80° E.; and a fence be required to run from some point in BA, a due North course, and to part off one acre. Required the distance from B to the point F, whence it must start. *Ans.* The angle $B = 70°$, and $F = 30°$. Then $BF = 6.47$.

The scale of Fig. 350 is 6 chains to 1 inch $= 1:4752$.

(511) To part off a quadrilateral. Let it be required to part off, by a line running in a given direction, a quadrilateral from a field in which are given the side AB, and the directions of the two other sides running from A and from B.

Fig. 351.



On the ground or plat produce the two converging sides to meet at some point E. Calculate the content of the triangle ABE. Measure the side AE. From ABE subtract the area to be cut off, and the remainder will be the content of the triangle CDE. From A set out a line AF parallel to the given direction. Find the content of ABF. Take it from ABE, and thus obtain AFE. Then this formula, $ED = AE\sqrt{\frac{CDE}{FAE}}$, will fix the point D, since $AD = AE - ED$.

(512) When the field and the dividing line are given by Bearings, produce the sides as in the last article. Find all the angles from the Bearings. Calculate the content of the triangle ABE, by the formula for one side and its including angles. Take the desired content from this to obtain CDE. Calculate the side $AE = AB \frac{\sin. B}{\sin. E}$. Then is $AD = AE - \sqrt{\left(\frac{2 \times CDE . \sin. DCE}{\sin. E . \sin. DCE}\right)}$.*

Example. Let DA bear S. 20¼° W.; AB, N. 51½° W., 8.19; BC, N. 73½° E.; and let it be required to part off two acres by a fence, DC, running N. 45° W. *Ans.* ABE = 32.50 sq. chains; whence CDE = 12.50 sq. chs. Also, AE = 8.37; and finally AD = 8.37 — 5.49 = 2.88 chains.

The scale of Fig. 351 is 5 chains to 1 inch = 1:3960.

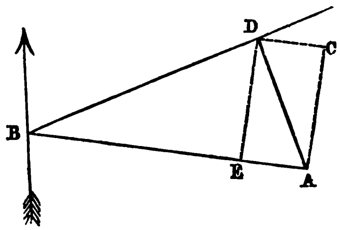
If the sum of the angles at A and B was more than two right angles, the point E would lie on the other side of AB. The necessary modifications are apparent.

(513) To part off any figure. Proceed in a similar manner to that described in Art. (**508**), by getting a suitable trial-line, producing the sides it intersects, and then applying the method just given.

D. By a line starting from a given point in a side.

(514) To part off a triangle. Let it be required to cut off from a corner of a field a triangular space of given content, by a line starting from a given point on one of the sides, A in the figure, the base, AB, of the desired triangle being thus given. If the field be given on the ground or on a plat, divide the given content by half the base, and the quotient will be the height of the triangle. Set off this distance from any point of AB, perpendicular to it, as from A to C; from C set out a parallel to AB, and its intersection with the second side, as at D, will be the vertex of the required triangle.

Fig. 352.



Otherwise, divide the required content by half of the perpendicular distance from A to BD, and the quotient will be BD.

* This original formula is very convenient for logarithmic computation.

(515) If the field be given by the Bearings of two sides and the length of one of them, deduce the angle B (Fig. 352) from the Bearings, as in Art. (243). Then is $BD = \frac{2 \times ABD}{AB \,.\, \sin. B}$.

If it is more convenient to fix the point D, by the Second Method, Art. (6), that of rectangular co-ordinates, we shall have $BE = BD \,.\, \cos. B$; and $ED = BD \,.\, \sin. B$.

The Bearing of AD is obtained from the angle BAD; which is known, since $\frac{ED}{EA} = \frac{ED}{AB - BE} = \text{tang. } BAD$.

Example. Eighty acres are to be set off from a corner of a field, the course AB being N. 80° W., sixty chains; and the Bearing of BD being N. 70° E. *Ans.* BD = 53.33; BE = 46.19; ED = 26.67; and the Bearing of AD, N. 6° 48′ W.

The scale of Fig. 352 is 40 chains to 1 inch = 1 : 31680.

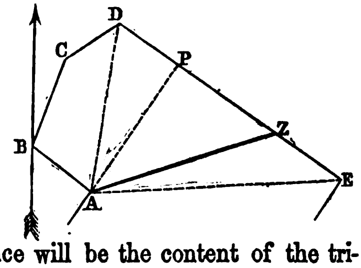
If the field were right angled at B, of course $BD = \frac{2\, ABD}{AB}$.

(516) To part off a quadrilateral. Imagine the two converging sides of the field produced to meet, as in Art. (511). Calculate the content of the triangle thus formed, and the question will then be reduced to the one explained in the last two articles.

(517) To part off any figure. Proceed as directed in Art. (513). Otherwise, proceed as follows.

The field being given on the ground or on a plat, find on which side of it the required line will end, by drawing or running "guess lines" from the given point to various angles, and roughly measuring the content thus parted off.

Fig. 353.



If, as in the figure, A being the given point, the guess line AD parts off less than the required content, and AE parts off more, then the desired division line AZ will end in the side DE. Subtract the area parted off by AD from the required content, and the difference will be the content of the triangle ADZ. Divide this by half the perpendicular let fall from the given point A to the side DE, and the quotient will be the base, or distance from D to Z.

Or, find the content of ADE and make this proportion; ADE : ADZ :: DE : DZ.

(518) The field being given by Bearings and distances, find as before, by approximate trials on the plat, or otherwise, which side the desired line of division will terminate in, as DE in the last figure. Draw AD. Find the Latitude and Departure of this line, and thence its length and Bearing, as in Art. **(440)**. Then calculate the area of the space this line parts off, ABCD in the figure, by the usual method, explained in Part III, Chapter VI. Subtract this area from that required to be cut off, and the remainder will be the area of the triangle ADZ. Then, as in Art. **(515)**,

$$DZ = \frac{2\ ADZ}{AD \cdot \sin.\ ADZ}.$$

This problem may be executed without any other Table than that of Latitudes and Departures, thus. Find the Latitude and Departure of DA, as before, the area of the space ABCD, and thence the content of ADZ. Then find the Latitude and Departure of EA, and the content of ADE. Lastly, make this proportion: ADE : ADZ :: DE : DZ.*

Example. In the field ABCDE, &c., part of which is shown in Fig. 353, (on a scale of 4 chains to 1 inch = 1 : 3168), one acre is to be parted off on the west side, by a line starting from the angle A. Required the distance from D to Z, the other end of this dividing line.†

The only courses needed are these. AB, N. 53° W., 1.55; BC, N. 20° E., 2.00; CD, N. $53\frac{1}{2}$° E., 1.32; DE, S. 57° E., 5.79. A rough measurement will at once shew that ABCD is less than an acre, and that ABCDE is more; hence the desired line will fall

* The problem may also be performed by making the side on which the division line is to fall, a Meridian, and changing the Bearings as in Art. (244). The difference of the new Departures will be the Departure of the Division line. Its position can then be easily determined, by calculations resembling those in Part VII, Chapter IV, Arts. (443), &c.

† If the whole field has been surveyed and balanced, the balanced Latitudes and Departures should be used. We will here suppose the survey to have proved perfectly correct.

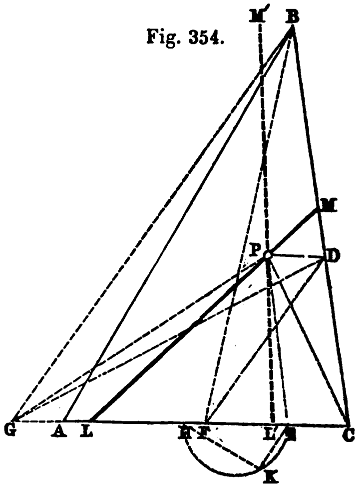
on DE. The Latitudes and Departures of AB, BC and CD are then found. From them the course AD is found to be N. 8° E., 3.63. The content of ABCD will be 3.19 square chains. Subtracting this from one acre, the remainder, 6.81 sq. chs., is the content of ADZ. AP = 3.63 × sin. 65° = 3.29. Dividing ADZ by half of this, we obtain DZ = 4.14 chains.

By the Second Method, the Latitude and Departure of DA, the area of ABCD, and of ADZ, being found as before, we next find the Latitude and Departure of EA, from those of AD and DE, and thence the area of ADE = 9.53. Lastly, we have the proportion 9.53 : 6.81 : : 5.79 : DZ = 4.14, as before.

E. BY A LINE PASSING THROUGH A GIVEN POINT WITHIN THE FIELD.

(519) To part off a triangle. Let P be a point within a field through which it is required to run a line so as to part off from the field, a given area in the form of a triangle.

Fig. 354.



When the field is given on the ground or on a plat, the division can be made by construction, thus. From P draw PE, parallel to the side BC. Divide the given area by half of the perpendicular distance from P to AC, and set off the quotient from C to G. Bisect GC in H. On HE describe a semi-circle. On it set off EK = EC. Join KH. Set off HL = HK. The line LM, drawn from L through P, will be the division line required.* If HK be set off in the contrary direction, it will fix another line L′PM′, meeting CB produced, and thus parting off another triangle of the required content.

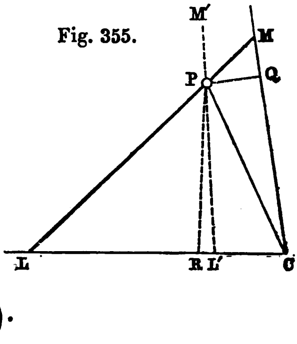
Example. Let it be required to part off 31.175 acres by a fence passing through a point P, the distance PD of P from the

* As some lines in the figure are not used in the construction, though needed for the Demonstration, the student should draw it himself to a large scale.

side BC, measured parallel to AC, being 6 chains, and DC 18 chains. The angle at C is fixed by a "tie-line" AB = 48.00, BC being 42.00, and CA being 30.00. *Ans.* CL = 27.31 chains, or CL′ = 7.69 chains.

The figure is on a scale of 20 chains to 1 inch = 1 : 15840.

Fig. 355.



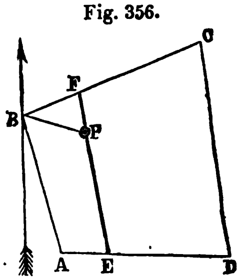
(520) If the angle of the field and the position of the point P are given by Bearings or angles, proceed thus. Find the perpendicular distances, PQ and PR, from the given point to the sides, by the formulas PQ = PC . sin. PCQ; and PR = PC . sin. PCR. Let PQ = q, PR = p, and the required content = c. Then $\text{CL} = \frac{c}{p} \pm \sqrt{\left(\frac{c^2}{p^2} - \frac{2\,qc}{\text{sin. LCM}}\right)}$.

Example. Let the angle LCM = 82°. Let it be required to part off the same area as in the preceding example. Let PC = 19.75, PCQ = 17° 30½′, PCR = 64° 29½′. Required CL. *Ans.* PQ = 5.94, PR = 17.82, and therefore, by the formula, CL = 27.31, or CL′ = 7.69; corresponding to the graphical solution. The figure is on the same scale.

If the given point were *without* the field, the division line could be determined in a similar manner.

(521) To part off a quadrilateral. Conceive the two sides of the field which the division line will intersect, DA and CB, produced till they meet at a point G, not shown in the figure. Calculate the triangle thus formed outside of the field. Its area increased by the required area, will be that of the triangle EFG. Then the problem is identical with that in the last article. The following example is that given in Gummere's Surveying. The figure represents it on a scale of 20 chains to 1 inch = 1 : 15840.

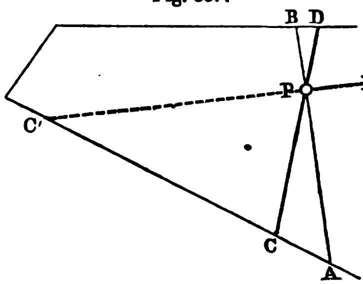
Fig. 356.



Example. A field is bounded thus: N. 14° W., 15.20; N. 70½° E., 20.43; S. 6° E., 22.79; N. 86½° W., 18.00. A spring within it bears from the second corner S. 75° E., 7.90. It is required to cut off 10 acres from the West side of the field by a straight fence through the spring. How far will it be from the first corner to the point at which the division fence meets the fourth side? *Ans.* 4.6357 chains.

(522) To part off any figure. Let it be required to part off from a field a certain area by a line passing through a given point P within the field. Run a guess-line AB through P. Calculate the area which it parts off. Call the difference between it and the required area $= d$. Let CD be the desired line of division, and let P represent the angle, APC or BPD, which it makes with the given line. Obtain the angles PAC = A, and PBD = B, either by measurement, or by deduction from Bearings. Measure PA and PB. Then the desired angle P will be given by the following formula.

Fig. 357.



$$\text{Cot. P} = -\tfrac{1}{2}\left(\text{cot. A} + \text{cot. B} - \frac{AP^2 - BP^2}{2\,d}\right) \pm$$

$$\sqrt{\left[\frac{AP^2 \,.\, \text{cot. B} - BP^2 \,.\, \text{cot. A}}{2\,d} - \text{cot. A} \,.\, \text{cot. B} + \tfrac{1}{4}\left(\text{cot. A} + \text{cot. B} - \frac{AP^2 - BP^2}{2\,d}\right)^2\right]}.$$

If the guess line be run so as to be perpendicular to one of the sides of the field, at A, for example, the preceding expression reduces to the following simpler form.

$$\text{Cot. P} = -\tfrac{1}{2}\left(\text{cot. B} - \frac{AP^2 - BP^2}{2\,d}\right) \pm$$

$$\sqrt{\left[\frac{AP^2 \,.\, \text{cot. B}}{2\,d} + \tfrac{1}{4}\left(\text{cot. B} - \frac{AP^2 - BP^2}{2\,d}\right)^2\right]}.$$

Example. It was required to cut off from a field twelve acres by a line passing through a spring, P. A guess-line, AB, was run making an angle with one side of the field, at A, of 55°, and with the opposite side, at B, of 81°. The area thus cut off was found to be 13.10 acres. From the spring to A was 9.30 chains, and to B 3.30 chains. Required the angle which the required line, CD, must make with the guess line, AB, at P. *Ans.* 20° 45′; or —86° 25′. The heavy broken line, C′D′, shows the latter.

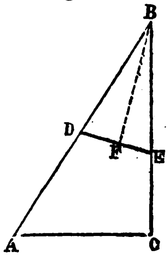
The scale of the figure is 10 chains to 1 inch = 1 : 7920.

If the given point were outside of the field, the calculations would be similar.

F. By the shortest possible line.

(523) To part off a triangle. Let it be required to part off a triangular space, BDE, of given content, from the corner of a field, ABC, by the shortest possible line, DE.

Fig. 358.



From B set off BD and BE each equal to $\sqrt{\left(\frac{2\ \text{BDE}}{\sin.\ \text{B}}\right)}$. The line DE thus obtained will be perpendicular to the line, BF, which bisects the angle B. The length of DE $= \frac{\sqrt{(2\ .\ \text{DBE}\ .\ \sin.\ \text{B})}}{\cos.\ \frac{1}{2}\ \text{B}}$.

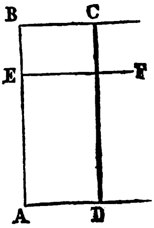
Example. Let it be required to part off 1.3 acre from the corner of a field, the angle, B, being 30°. *Ans.* BD = BE = 7.21; and DE = 3.73.

The scale of the figure is 10 chains to 1 inch = 1 : 7920.

G. Land of variable value.

(524) Let the figure represent a field in which the land is of two qualities and values, divided by the "quality line" EF. It is required to part off from it a quantity of land worth a certain sum, by a straight fence parallel to AB.

Fig. 359.



Multiply the value per acre of each part by its length (in chains) on the line AB, add the products, multiply the value to be set off by 10, divide by the above sum, and the quotient will be the desired breadth, BC or AD, in chains.

Example. Let the land on one side of EF be worth $200 per acre, and on the other side $100. Let the length of the former, BE, be 10 chains, and EA be 30 chains. It is required to part off a quantity of land worth $7500. *Ans.* The width of the desired strip will be 15 chains.

The scale of the figure is 40 chains to 1 inch = 1 : 31680.

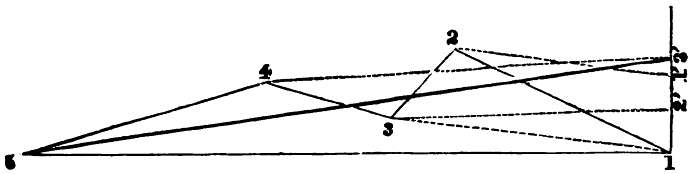
If the "quality line" be not perpendicular to AB, it may be made so by "giving and taking," as in Art. (124), or as in the article following this one.

The same method may be applied to land of any number of different qualities; and a combination of this method with the preceding problems will solve any case which may occur.

H. Straightening crooked fences.

(525) It is often required to substitute a straight fence for a crooked one, so that the former shall part off precisely the same quantity of land as did the latter. This can be done on a plat by the method given in Art. (83), by which the irregular figure

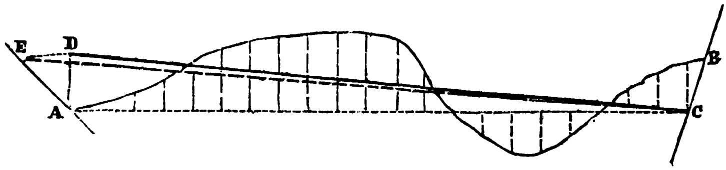
Fig. 360.



1...2...3...4...5 is reduced to the equivalent triangle 1...5...3′, and the straight line 5...3′ therefore parts off the same quantity of land on either side as did the crooked one. The distance from 1 to 3′, as found on the plat, can then be set out on the ground and the straight fence be then ranged from 3′ to 5

The work may be done on the ground more accurately by running a guess line, AC, Fig. 361, across the bends of the fence which crooks from A to B, measuring offsets to the bends on each side of the guess line, and calculating their content. If the sums of these areas on each side of AC chanced to be equal, that would be the line desired; but if, as in the figure, it passes too far on one

Fig. 361.



side, divide the difference of the areas by half of AC, and set it off at right angles to AC, from A to D. DC will then be a line parting off the same quantity of land as did the crooked fence. If the fence at A was not perpendicular to AC, but oblique, as AE, then from D run a parallel to AC, meeting the fence at E, and EC will be the required line.

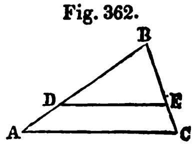
CHAPTER III.

DIVIDING UP LAND.

(526) MOST of the problems for "Dividing up" land may be brought under the cases in the preceding chapter, by regarding one of the portions into which the figure is to be divided, as an area to be "Parted off" from it. Many of them, however, can be most neatly executed by considering them as independent problems, and this will be here done. They will be arranged, firstly, according to the simplicity of the figure to be divided up, and then sub-arranged, as in the leading arrangement of Chapter II, according to the manner of the division.

DIVISION OF TRIANGLES.

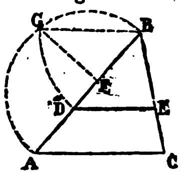
Fig. 362.



(527) By lines parallel to a side. Suppose that the triangle ABC is to be divided into two equivalent parts by a line parallel to AC. The desired point, D, from which this line is to start, will be obtained by measuring BD $=$ AB $\sqrt{\frac{1}{2}}$. So, too, E is fixed by BE $=$ BC $\sqrt{\frac{1}{2}}$.

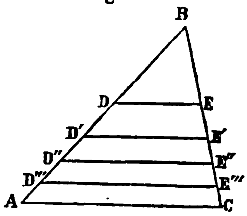
Generally, to divide the triangle into two parts, BDE and ACED, which shall have to each other a ratio $= m : n$, we have $BD = AB\sqrt{\dfrac{m}{m+n}}$.

Fig. 363.



This may be constructed thus. Describe a semicircle on AB as a diameter. From B set off $BF = \dfrac{m}{m+n} \cdot BA$. At F erect a perpendicular meeting the semicircle at G. Set off BG from B to D. D is the starting point of the division line required. In the figure, the two parts are as 2 to 3, and BF is therefore $= \frac{2}{5}$ BA.

Fig. 364.

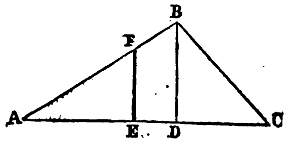


To divide the triangle ABC into five equivalent parts, we should have, similarly, $BD = AB\sqrt{\frac{1}{5}}$; $BD' = AB\sqrt{\frac{2}{5}}$; $BD'' = AB\sqrt{\frac{3}{5}}$; $BD''' = AB\sqrt{\frac{4}{5}}$.

The same method will divide the triangle into any desired number of parts having any ratios to each other.

(528) By lines perpendicular to a side. Suppose that ABC is to be divided into two parts having a ratio $= m : n$, by a line perpendicular to AC. Let EF be the dividing line whose position is required. Let BD be a perpendicular let fall from B to AC. Then is $AF = \sqrt{\left(AC \times AD \times \dfrac{m}{m+n}\right)}$. In this figure, AFE : EFBC :: $m : n$ :: 1 : 2.

Fig. 365.

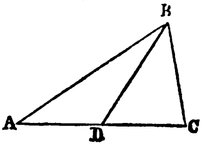


If the triangle had to be divided into two equivalent parts, the above expression would become $AF = \sqrt{(\frac{1}{2}\, AC \times AD)}$.

(529) By lines running in any given direction. Let a triangle, ABC, be given to be divided into two parts, having a ratio $= m : n$, by a line making a given angle with a side. Part off, as in Art. (**509**) or (**510**), Fig. 350, an area $BFG = \dfrac{m}{m+n} \cdot ABC$.

(530) By lines starting from an angle. Divide the side opposite to the given angle into the required number of parts, and draw lines from the angle to the points of division. In the figure the triangle is represented as being thus divided into two equivalent parts.

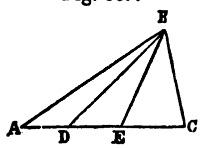
Fig. 366.



If the triangle were required to be divided into two parts, having to each other a ratio $= m : n$, we should have $AD = AC \dfrac{m}{m+n}$, and $DC = AC \dfrac{n}{m+n}$.

If the triangle had to be divided into three parts which should be to each other $:: m : n : p$, we should have $AD = AC \dfrac{m}{m+n+p}$, $DE = AC \dfrac{n}{m+n+p}$, and $EC = AC \dfrac{p}{m+n+p}$.

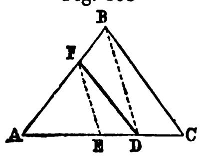
Fig. 367.



Suppose that a triangular field ABC, had to be divided among five men, two of them to have a quarter each, and three of them each a sixth. Divide AC into two equal parts, one of these again into two equal parts, and the other one into three equal parts. Run the lines from the four points thus obtained to the angle B.

(531) By lines starting from a point in a side. Suppose that the triangle ABC is to be divided into *two* equivalent parts by a line starting from a point D in the side AC. Take a point E in the middle of AC. Join BD, and from E draw a parallel to it, meeting AB in F. DF will be the dividing line required.

Fig. 368

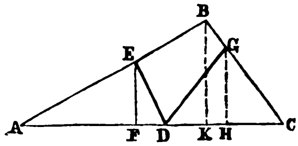


The point F will be most easily obtained on the ground by the proportion $AD : AB :: AE = \frac{1}{2} AC : AF$.

The altitude of AFD of course equals $\frac{1}{2} ABC \div \frac{1}{2} AD$.

If the triangle is to be divided into two parts having any other ratio to each other, divide AC in that ratio, and then proceed as before. Let this ratio $= m : n$, then $AF = \dfrac{AB \times AC}{AD} \cdot \dfrac{m}{m+n}$.

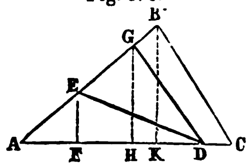
Fig. 369.



(532) Next suppose that the triangle ABC is to be divided into *three* equivalent parts, meeting at D. The altitudes, EF and GH, of the parts ADE and DCG, will be obtained by dividing $\frac{1}{3}$ ABC, by half of the respective bases AD and DC.

If one of these quotients gives an altitude greater than that of the triangle ABC, it will shew that the two lines DE and DG would both cut the same side, as in Fig. 370, in which EF is obtained as above, and GH = $\frac{2}{3}$ ABC $\div$ $\frac{1}{2}$ AD.

Fig. 370.



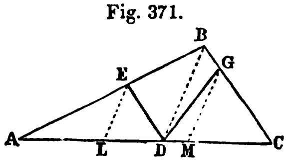
In practice it is more convenient to determine the points F and G, by these proportions;

BK : AK :: EF : AF; and BK : AK :: GH : AH.

The division of a triangle into a greater number of parts, having any ratios, may be effected in a similar manner.

(533) This problem admits of a more elegant solution, analogous to that given for the division into two parts, graphically. Divide AC into three equal parts at L and M. Join BD, and from L and M draw parallels to it, meeting AB and BC in E and G. Draw ED and GD, which will be the desired lines of division. The figure is the same triangle as Fig. 369.

Fig. 371.

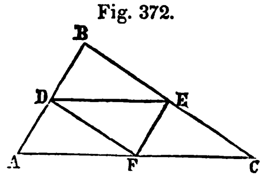


The points E and G can be obtained on the ground by measuring AD and AB, and making the proportion AD : AB :: $\frac{1}{3}$ AC : AE. The point G is similarly obtained.

The same method will divide a triangle into a greater number of parts.

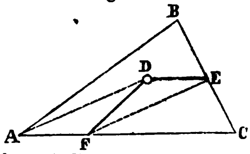
(534) To divide a triangle into *four* equivalent triangles by lines terminating in the sides, is very easy. From D, the middle point of AB, draw DE parallel to AC, and from F, the middle of AC, draw FD and FE. The problem is now solved.

Fig. 372.



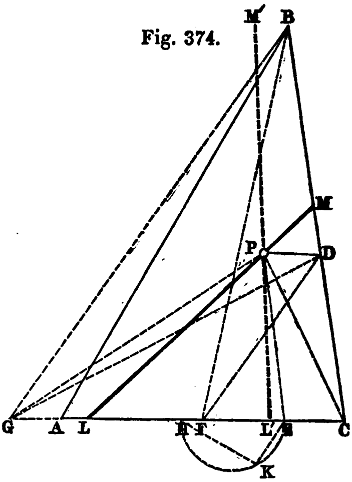
(535) By lines passing through a point within the triangle. Let D be a given point (such as a well, &c.) within a triangular field ABC, from which fences are to run so as to divide the triangle into *two* equivalent parts. Join AD. Take E in the middle of BC, and from it draw a parallel to DA, meeting AC in F. EDF is the fence required.

Fig. 373.



(536) If it be required to divide a triangle into two equivalent parts by a straight line passing through a point within it, proceed thus. Let P be the given point. From P draw PD parallel to AC, and PE parallel to BC. Bisect AC at F. Join FB. From B draw BG parallel to DF. Then bisect GC in H. On HE describe a semicircle. On it set off EK = EC. Join KH. Set off HL = HK. The line LM drawn from L, through P, will be the division line required.

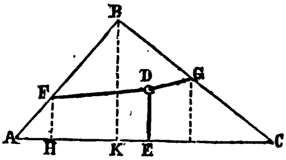
Fig. 374.



This figure is the same as that of Art. **(519)**. The triangle ABC contains 62.35 acres, and the distance CL = 27.31 chains, as in the example in that article.

(537) Next suppose that the triangle ABC is to be divided into *three* equivalent parts by lines starting from a point D, within the triangle, given by the rectangular co-ordinates AE and ~~and~~ ED. Let ED be one of the lines of division, and F and G the other points required. The point F will be determined if AH is known; AH and HF being its rectangular co-ordinates. From B let fall the perpendicular BK on AC.

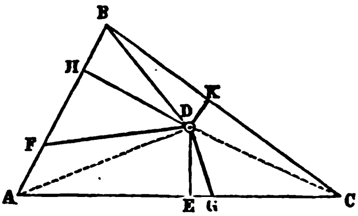
Fig. 375.



Then is $AH = \dfrac{AK\,(\frac{2}{3}\,ABC - AE \times ED)}{AE \times BK - ED \times AK}$. The position of the other point, G, is determined in a similar manner.

(538) Let DB, instead of DE, be one of the required lines of division. Divide $\frac{1}{3}$ ABC by half of the perpendicular DH, let fall from D to AB, and the quotient will be the distance BF. To find G, if, as in this figure, the triangle BDC ($= BC \times \frac{1}{2}\,DK$) is less than $\frac{1}{3}$ ABC, divide the excess of the latter (which will be CDG) by $\frac{1}{2}$ DE, and the quotient will be CG.

Fig. 376.



Example. Let AB = 30.00; BC = 45.00; CA = 50.00. Let the perpendiculars from D to the sides be these; DE = 10.00; DH = 20.00; DK = 5.17$\frac{1}{3}$. The content of the triangle ABC will be 666.6 square chains. Each of the small triangles must therefore contain 222.2 sq. chs., BD being one division line. We shall therefore have BF = 222.2 ÷ $\frac{1}{2}$ DH = 22.2 chains. BDC = 45 × $\frac{1}{2}$ × 5.17$\frac{1}{3}$ = 116.4 sq. chs., not enough for a second portion, but leaving 105.8 sq. chs. for CDG; whence CG = 21.16 chs. To prove the work, calculate the content of the remaining portion, GDFA. We shall find DGA = 144.2 sq. chs., and ADF = 78.0 sq. chs., making together 222.2 sq. chs., as required.

The scale of Fig. 376 is 30 chains to 1 inch = 1 : 23760.

(539) The preceding case may be also solved graphically, thus. Take CL = $\frac{1}{3}$ AC. Join DL, and from B draw BG parallel to DL. Join DG. It will be a second line of division. Then take a point, M, in the middle of BG, and from it draw a line, MF, parallel to DA. DF will be the third line of division. This method is neater on paper than the preceding; but less convenient on the ground.

Fig. 377.

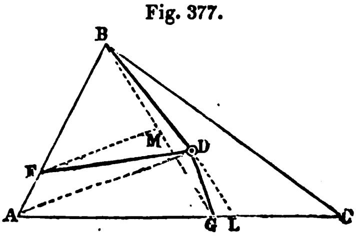
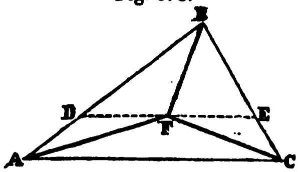


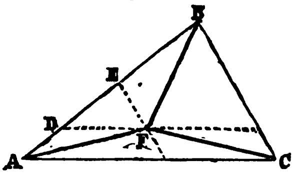
Fig. 378.



(540) Let it be required to divide the triangle ABC into three equivalent triangles, by lines drawn from the three angular points to some *unknown* point within the triangle. This point is now to be found. On any side, as AB, take AD = $\frac{1}{3}$ AB. From D draw DE parallel to AC. The middle, F, of DE, is the point required.

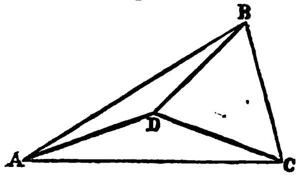
If the three small triangles are not to be equivalent, but are to have to each other the ratios :: $m : n : p$, divide a side, AB, into parts having these ratios, and through each point of division, D, E, draw a parallel to the side nearest to it. The intersection of these parallels, in F, is the point required. In the figure the parts ACF, ABF, BCF, are as 2 : 3 : 4.

Fig. 379.



(541) Let it be required to find the position of a point, D, situated within a given triangle, ABC, and equally distant from the points A, B, C; and to determine the ratios to each other of the three triangles into which the given triangle is divided.

Fig. 380.

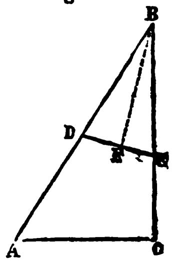


By construction, find the centre of the circle passing through A, B, C. This will be the required point.

By calculation, the distance $DA = DB = DC = \dfrac{AB \times BC \times CA}{4 \times \text{area } ABC}$.

The three small triangles will be to each other as the sines of their angles at D; i. e. ADB : ADC : BDC :: sin. ADB : sin. ADC : sin. BDC. These angles are readily found, since the sine of half of each of them equals the opposite side divided by twice one of the equal distances.

Fig. 381.



(542) By the shortest possible line. Let it be required to divide the triangle ABC by the shortest possible line, DE, into two parts, which shall be to each other :: $m : n$; or DBE : ABC :: $m : m + n$.

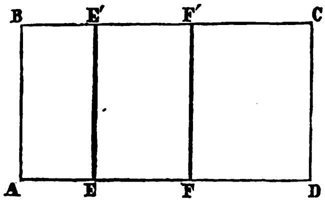
From the smallest angle, B, of the triangle, measure along the sides, BA and BC, a distance $\text{BD} = \text{BE} = \sqrt{\left(\frac{m}{m+n} \times \text{AB} \times \text{BC}\right)}$. DE is the line required. It is perpendicular to the line BF which bisects the angle ABC; and it is $= \frac{\sin. \text{B}}{\cos. \frac{1}{2}\text{B}} \sqrt{\left(\frac{m}{m+n} \times \text{AB} \times \text{BC}\right)}$.

DIVISION OF RECTANGLES.

(543) By lines parallel to a side. Divide two opposite sides into the required number of parts, either equal or in any given ratio to each other, and the lines joining the points of division will be the lines desired.

The same method is applicable to any parallelogram.

Fig. 382.



Example. A rectangular field ABCD, measuring 15.00 chains by 8.00, is bought by three men, who pay respectively \$300, \$400 and \$500. It is to be divided among them in that proportion. *Ans.* The portion of the first, AEE′B, is obtained by making the proportion 300 + 400 + 500 : 300 :: 15.00 : AE = 3.75. EF is in like manner found to be 5.00; and FD = 6.25. BE′ is made equal to AE; E′F′ to EF; and F′C to FD. Fences from E to E′, and from F to F′, will divide the land as required.

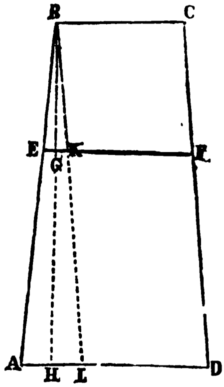
The scale of the figure is 10 chains to 1 inch = 1 : 7920.

The other modes of dividing up rectangles will be given under the head of "Quadrilaterals," Art. (548), &c.

DIVISION OF TRAPEZOIDS.

(544) By lines parallel to the bases. Given the bases and a third side of the trapezoid, ABCD, to be divided into two parts, such that BCFE : EFDA :: $m : n$.

Fig. 383.



The length of the desired dividing line,

$$EF = \sqrt{\left(\frac{m \times AD^2 + n \times BC^2}{m+n}\right)}.$$

The distance $BE = \frac{AB.(EF - BC)}{AD - BC}$.

Example. Let AD = 30 chains; BC = 20 chs.; and AB = $54\frac{1}{3}$ chs.; and the parts to be as 1 to 2; required EF and BE. *Ans.* EF = 23.80; and BE = 20.65.

The figure is on a scale of 30 chains to 1 inch = 1 : 23760.

(545) Given the bases of a trapezoid, and the perpendicular distance, BH, between them; it is required to divide it as before, and to find EF, and the altitude, BG, of one of the parts. Let BCFE : EFDA :: $m : n$. Then $BG = -\frac{BC \times BH}{AD - BC} +$

$$\sqrt{\left[\frac{m}{m+n} \times \frac{2 \times ABCD \times BH}{AD - BC} + \left(\frac{BC \times BH}{AD - BC}\right)^2\right]}.$$

$$EF = BC + BG \times \frac{AD - BC}{BH}.$$

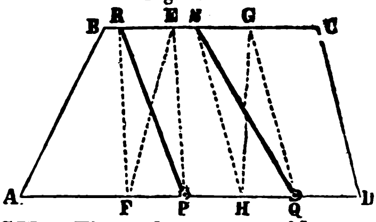
Example. Let AD = 30.00; BC = 20.00; BH = 54.00; and the two parts to be to each other :: 46 : 89.

The above data give the content of ABCD = 1350 square chains. Substituting these numbers in the above formula, we obtain BG = 20.96, and EF = 23.88.

(546) By lines starting from points in a side. To divide a trapezoid into parts equivalent, or having any ratios, divide its parallel sides in the same ratios, and join the corresponding points.

If it be also required that the division lines shall start from *given* points on a side, proceed thus. Let it be required to divide the trapezoid ABCD into three equivalent parts by fences starting from P and Q Divide the trapezoid, as above directed, into three equivalent trapezoids by the lines EF and GH. These three trapezoids must now be transformed, thus. Join EP, and from F draw FR parallel to it. Join PR, and it will be one of the division lines required.

Fig. 384.



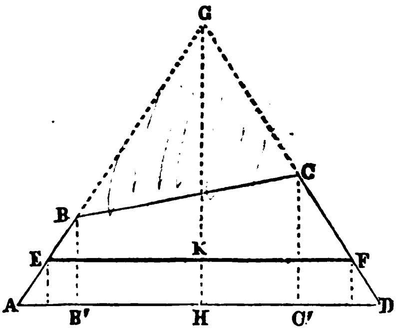
The other division line, QS, is obtained similarly.

(547) Other cases. For other cases of dividing trapezoids, apply those for quadrilaterals in general, given in the following articles.*

DIVISION OF QUADRILATERALS.

(548) By lines parallel to a side. Let ABCD be a quadrilateral which it is required to divide, by a line EF, parallel to AD, into two parts, BEFC and EFDA, which shall be to each other as $m : n$. Prolong AB and CD to intersect in G. Let a be the area of the triangle ADG, obtained by any method, graphical or trigonometrical, and a' = the area of the triangle BCG, obtained by subtracting the area of the given quadrilateral from that of the triangle ADG. Then GK = GH $\sqrt{\left(\frac{ma + na'}{(m + n)\,a}\right)}$. Having measured this length of GK from G on GH, set off at K a perpendicular to GK, and it will be the required line of division.

Fig. 385.



* If a line be drawn joining the middle points of the parallel bases of a trapezoid, *any line* drawn through the middle of the first line will divide the trapezoid into two equivalent parts.

Otherwise, take $GE = GA\sqrt{\left(\frac{ma + na'}{(m+n)\,a}\right)}$; and from E run a parallel to AD.

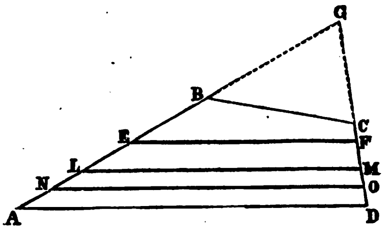
If the two parts of the quadrilateral were to be equivalent, $m = n$, and we have $GK = GH\sqrt{\left(\frac{a + a'}{2\,a}\right)}$; and consequently GE to GA in the same ratio.

Example. Let a quadrilateral, ABCD, be required to be thus divided, and let its angles, B and C, be given by rectangular co-ordinates, viz: AB′ = 6.00; B′B = 9.00; DC′ = 8.00; C′C = 13.00; B′C′ = 24.00. Here GH is readily found to be 29.64; ADG = 563.16 square chains; and BGC = 220.16 square chains. Hence, by the formula, GK = 24.72; whence KH = GH — GK = 4.92; and the abscissas for the points E and F can be obtained by a simple proportion.

The scale of the figure is 20 chains to 1 inch = 1 : 15840.

If the quadrilateral be given by Bearings, part off the desired area $= \frac{n}{m+n}$. ABCD, by the formulas of Art. (504).

Fig. 386.



Suppose now that a quadrilateral, ABCD, is to be divided into p equivalent parts, by lines parallel to AD. Measure, or calculate by Trigonometry, AG. Let Q be the quadrilateral ABCD, and, as before, $a' = BCG$. Then

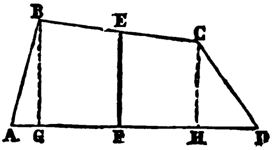
$$GE = AG\sqrt{\left\{\frac{a' + \frac{Q}{p}}{a' + Q}\right\}};\quad GL = AG\sqrt{\left\{\frac{a' + \frac{2\,Q}{p}}{a' + Q}\right\}};$$

$$GN = AG\sqrt{\left\{\frac{a' + \frac{3\,Q}{p}}{a' + Q}\right\}};\ \text{\&c.}$$

If the quadrilateral be given by Bearings, part off, by Art. (504), $\frac{1}{p}$. ABCD, then part off $\frac{2}{p}$. ABCD; &c.; so in any similar case.

(549) By lines perpendicular to a side. Let ABCD be a quadrilateral which is to be divided, by a line perpendicular to AD, into two parts having a ratio $= m : n$. By hypothesis, ABEF $= \frac{m}{m+n}$. ABCD.

Fig. 387.

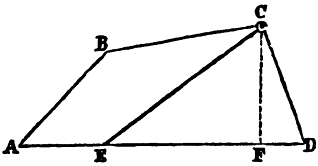


Taking away the triangle ABG, the remainder, GBEF, will be to the rest of the figure in a known ratio, and the position of EF, parallel to BG, will be found as in the last article.

(550) By lines running in any given direction. To divide a quadrilateral ABCD into two parts $:: m : n$, part off from it an area $= \frac{m}{m+n}$. ABCD, by the methods of Arts. (**509**) or (**510**), if the area parted off is to be a triangle, or Arts. (**511**) or (**512**), if the area parted off is to be a quadrilateral.

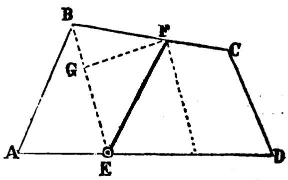
(551) By lines starting from an angle. ABCD is to be divided, by the line CE, into two parts having the ratio $m : n$. Since the area of the triangle CDE $= \frac{m}{m+n}$. ABCD, DE will be obtained by dividing this area by half of the altitude CF.

Fig. 388.



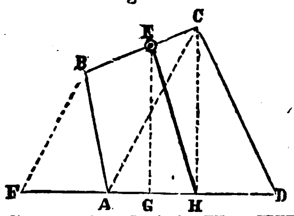
(552) By lines starting from points in a side. Let it be required to divide ABCD into two parts $:: m : n$, by a line starting from the point E. The area ABFE is known, (being $= \frac{m}{m+n}$. ABCD) as also ABE; AB, BE, and EA being given on the ground. BEF will then be known $\doteq$ ABFE — ABE. Then GF $= \frac{\text{BEF}}{\frac{1}{2}\text{BE}}$, and the point F is obtained by running a parallel to BE, at a perpendicular distance from it $=$ GF.

Fig. 389.



To divide a quadrilateral, ABCD, graphically, into *two equivalent* parts by a line from a point, E, on a side, proceed thus. Draw the diagonal CA, and from B draw a parallel to it, meeting DA prolonged in F. Mark the middle point, G, of FD. Join GE. From C draw a parallel to EG, meeting DA in H. EH is the required line. The quadrilateral could also be divided in any ratio $= m : n$, by dividing FD in that ratio.

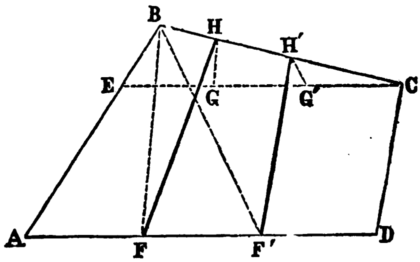
Fig. 390.



If the quadrilateral be given by Bearings, proceed to part off the desired area, as in Art. (515) or (516).

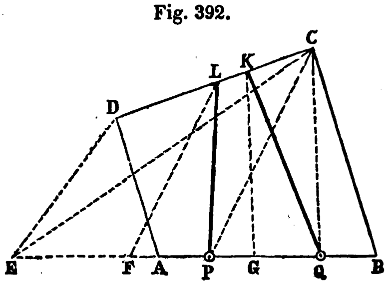
(553) Let it be required to divide a quadrilateral, ABCD, into *three equivalent* parts. From any angle, as C, draw CE, parallel to DA. Divide AD and EC, each into three equal parts, at F, F′, and G, G′. Draw BF, BF′. From G draw GH, parallel to FB, and from G′ draw G′H′, parallel to F′B. FH and F′H′ are the required lines of division.

Fig. 391.



Let it be required to make the above division by *lines starting from two given points*, P and Q. Reduce the quadrilateral to an equivalent triangle CBE, as in Art. (87). Divide EB into three equal parts at F and G. Join CQ, and, from G, draw GK parallel to it. Join CP, and from F draw FL parallel to it. Join PL and QK, and they will be the division lines required.

Fig. 392.

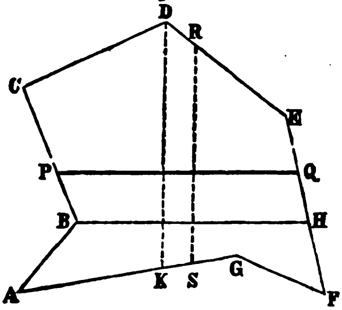


(554) By lines passing through a point within the figure. Proceed to part off the desired area as in Arts. (519), (520), or (521), according to the circumstances of the case.

DIVISION OF POLYGONS.

(555) By lines running in any direction. Let ABCDEFG be a given polygon, and BH the direction parallel to which is to be drawn a line PQ, dividing the polygon into two parts in any desired ratio $= m : n$. The area

Fig. 393.



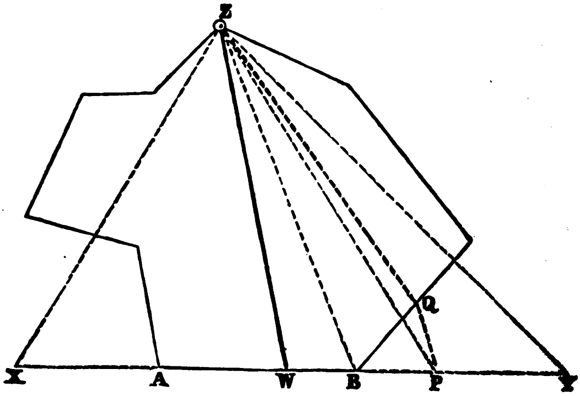
$$PCDEQ = \frac{m}{m+n} . ABCDEFG.$$

Taking it from the area BCDEH, the remainder will be the area BPQH. The quadrilateral BCEH, CE being supposed to be drawn, can then be divided by the method of Art. (548), into two parts, BPQH and PQEC, having to each other a known relation.

If DK were the given direction, at right angles to the former, the position of a dividing line RS could be similarly obtained.

(556) By lines starting from an angle. Produce one side, AB,

Fig. 394.



of the given polygon, both ways, and reduce the polygon to a single equivalent triangle, XYZ, by the method of Art. (82). Then divide the base, XY, in the required ratio, as at W, and draw ZW, which will be the division line desired. In this figure the polygon is divided into two equivalent parts.

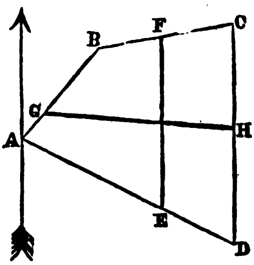
If the division line should pass outside of the polygon, as does ZP, through P draw a parallel to BZ, meeting the adjacent side of the polygon in Q, and ZQ will be the division line desired.

(557) By lines starting from a point on a side. See Articles (517) and (518) in the preceding chapter.

(558) By lines passing through a point within the figure. Part off, as in Arts. (519) or (522) in the preceding chapter, if a straight line be required; or by guess lines and the addition of triangles, as in Art. (538) of this chapter, if the lines have merely to start from the point, such as a spring or well.

(559) Other problems. The following is from Gummere's Surveying. *Question.* A tract of land is bounded thus: N. $35\frac{1}{4}°$ E., 23.00; N. $75\frac{1}{2}°$ E., 30.50; S. $3\frac{1}{4}°$ E., 46.49; N. $66\frac{1}{4}°$ W., 49.64. It is to be divided into four equivalent parts by two straight lines, one of which is to run parallel to the third side; required the distance of the parallel division line from the first corner, measured on the fourth side; also the Bearing of the other division line, and its distance from the same corner measured on the first side. *Ans.* Distance of the parallel division line from the first corner, 32.50; the Bearing of the other, S. 88° 22′ E.; and its distance from the same corner 5.99.

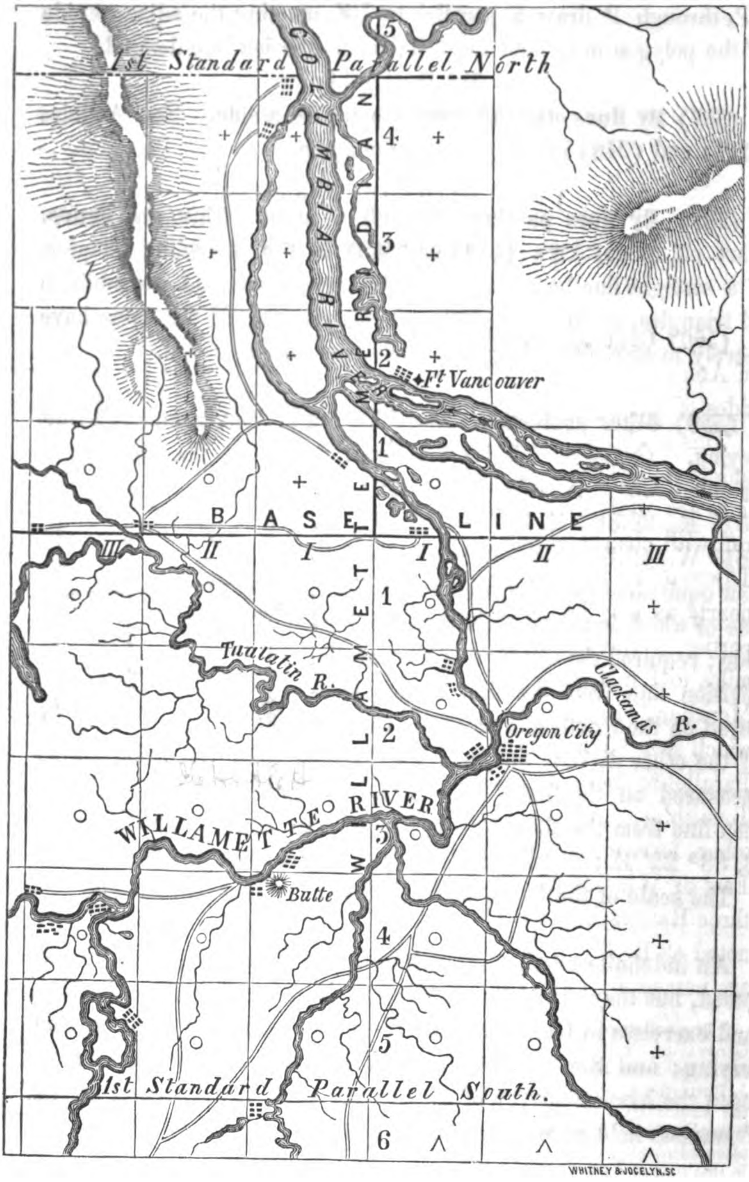
Fig. 395.



The scale of the figure is 40 chains to 1 inch = 1 : 31680.

An indefinite number of problems on this subject might be proposed, but they would be matters of curiosity rather than of utility, and exercises in Geometry and Trigonometry rather than in Surveying; and the youngest student will find his life too short for even the hastiest survey of merely the most fruitful parts of the boundless field of Mathematics.

Fig. 396.


1st Standard Parallel North
COLUMBIA RIVER
WILLAMETTE MERIDIAN
Ft. Vancouver
BASE LINE
Tualatin R.
Clackamas R.
Oregon City
WILLAMETTE RIVER
Butte
1st Standard Parallel South.
WHITNEY & JOCELYN.SC

PART XII.

THE PUBLIC LANDS OF THE UNITED STATES.*

(560) General system. The Public Lands of the United States of America are generally divided and laid out into squares, the sides of which run truly North and South, or East and West.

This is effected by means of Meridian lines and Parallels of Latitude, established six miles apart. The principal meridians and base lines are established astronomically, and the intermediate ones are run with chain and compass. The squares thus formed are called TOWNSHIPS. They contain 36 square miles, or 23040 acres, "as nearly as may be." The map on the opposite page represents a portion of the Territory of Oregon thus laid out. The scale is 10 miles to 1 inch = 1 : 633600. On it will be seen the "Willamette Meridian," running truly North and South, and a "Base line," which is a "Parallel of Latitude," running truly East and West. Parallel to these, and six miles from them, are other lines, forming Townships. All the Townships, situated North or South of each other, form a RANGE. The Ranges are named by their number East or West of the principal Meridian. In the figure are seen three Ranges East and West of the Willamette Meridian. They are noted as R. I. E., R. I. W., &c. The Townships in each Range are named by their number North or South of the Base line. In

* The substance of this Part is mainly taken from "Instructions to the Surveyor General of Oregon, being a Manual for Field Operations," prepared, in March, 1851, by John M. Moore, "Principal Clerk of Surveys," by direction of Hon. J. Butterfield, "Commissioner of the General Land Office," and communicated to the author by Hon. John Wilson, the present Commissioner. The aim of the "Instructions" is stated to be "simplicity, uniformity and permanency." They seem admirably adapted for these objects, and the lasting importance of the subject in this country has led the author to reproduce about half of them in this place. They were subsequently directed to be adopted for the Surveying service in Minnesota and California.

the figure along the principal Meridian are seen four North and five South of the Base line. They are noted as T. 1 N., T. 2 N., T. 1 S., &c.*

Each Township is divided into 36 SECTIONS, each 1 mile square, and therefore containing, "as nearly as may be," 640 acres. The sections in each Township are numbered, as in the margin, from 1 to 36, beginning at the North-east angle of the Township, and going West from 1 to 6, then East from 7 to 12, and so on alternately to Section 36, which will be in the South-east angle of the Township. The Sections are sub-divided into Quarter-sections, half-a-mile square, and containing 160 acres, and sometimes into half-quarter-sections of 80 acres, and quarter-quarter-sections of 40 acres.

N

	6	5	4	3	2	1	
	7	8	9	10	11	12	
	18	17	16	15	14	13	
W	19	20	21	22	23	24	E
	30	29	28	27	26	25	
	31	32	33	34	35	36	

S

By this beautiful system, the smallest subdivision of land can be at once designated; such as the North-east quarter of Section 31, in Township *two* South, in range *two* East of Willamette Meridian.

(561) Difficulty. "The law requires that the lines of the public surveys shall be governed by the true meridian, and that the townships shall be *six miles square*,—two things involving in connection a mathematical impossibility—for, strictly to conform to the meridian, necessarily throws the township out of square, by reason of *the convergency of meridians;* hence, adhering to the true meridian renders it necessary to depart from the strict requirements of law as respects the precise area of townships, and the subdivisional parts thereof, the township assuming something of a trapezoidal form, which inequality developes itself, more and more as such, the higher the latitude of the surveys. In view of these circumstances, the law provides that the sections of a mile square shall contain the quantity of 640 acres, *as nearly as may be;* and, moreover, provides that 'In all cases where the exterior lines of the townships, thus to be subdivided into sections or half-sections, shall exceed, or shall not extend, six miles, the excess or deficiency

* The marks ○, + and ∧, merely refer to the dates of the surveys. They are sometimes used to point out lands offered for sale, or reserved, &c.

shall be specially noted, and added to or deducted from the *western* or *northern* ranges of sections or half-sections in such township, according as the error may be in running the lines from east to west, or from south to north.' "

"In order to throw the excesses or deficiencies, as the case may be, on the *north* and on the *west* sides of a township, according to law, it is necessary to survey the *section* lines from *south* to north on a true meridian, leaving the result in the northern line of the township to be governed by the convexity of the earth and the convergency of meridians."

Thus, suppose the land to be surveyed lies between 46° and 47° of North Latitude. The length of a degree of Longitude in Lat. 46° N. is taken as 48.0705 statute miles, and in Lat. 47° N. as 47.1944. The difference, or convergency per square degree = 0.8761 = 70.08 chains. The convergency per Range (8 per degree of Longitude) equals one-eighth of this, or 8.76 chains; and per Township ($11\frac{1}{2}$ per degree of Latitude) equals the above divided by $11\frac{1}{2}$, i. e. 0.76 chain. We therefore know that the width of the Townships along their Northern line is 76 links less than on their Southern line. The townships North of the base line therefore become narrower and narrower than the six mile width with which they start, by that amount; and those South of it as much wider than six miles.

"STANDARD PARALLELS (usually called *correction lines*), are established at stated intervals (24 or 30 miles) to provide for or counteract the error that otherwise would result from the convergency of meridians; and, because the public surveys have to be governed by the true meridian, such lines serve also to arrest error arising from inaccuracies in measurements. Such lines, when lying north of the principal base, themselves constitute a *base* to the surveys on the north of them; and where lying south of the principal base, they constitute the base for the surveys south of them."

The convergency or divergency above noticed is taken up on these Correction lines, from which the townships start again with their proper widths. On these therefore there are found *Double Corners*, both for Townships and Sections, one set being the *Closing Corners* of the surveys ending there, and the other set being the *Standard Corners* for the surveys starting there.

(562) Running Township lines. "The principal meridian, the base line, and the standard parallels having been first astronomically run, measured, and marked, according to instructions, on true meridians, and true parallels of latitude, the process of running, measuring, and marking the exterior lines of townships will be as follows.

Townships situated NORTH *of the base line, and* WEST *of the principal meridian.** Commence at No. 1, being the southwest corner of T. 1 N.—R. 1 W., as established on the base line; thence run *north*, on a true meridian line, four hundred and eighty chains, establishing the mile and half-mile corners thereon, as per instructions, to No. 2, (the northwest corner of the same township), whereat establish the corner of Tps. 1 and 2 N.—Rs. 1 and 2 W.; thence *east*, on a random or trial line, setting *temporary* mile and half-mile stakes to No. 3, (the northeast corner of the same township), where measure and note the distance at which the line intersects the eastern boundary, north or south of the *true* or established corner. Run and measure *westward*, on the true line, (taking care to note all the land and water crossings, &c., as per instructions), to No. 4, which is identical with No. 2, establishing the mile and half-mile PERMANENT CORNERS on said line, the last half-mile of which will fall short of being forty chains, by about the amount of the calculated convergency per township, 76 links in the case above supposed. Should it ever happen, however, that such random line materially falls short, or overruns in length, or intersects the eastern boundary of the township at any considerable distance from the *true* corner thereon, (either of which would indicate an important error in the surveying), the lines must be *retraced*, even if found necessary to remeasure the meridional boundaries of the township (especially the western boundary), so as to discover and correct the error; in doing which, the *true corners* must be established and marked, and the *false ones* destroyed and obliterated, to prevent confusion in future; and *all the facts* must be distinctly set forth in the notes. Thence proceed in a similar manner north, from No. 4 to No. 5, (the N. W. corner of T. 2 N. —R. 1 W.), east from No. 5 to No. 6, (the N. E. corner of the same township), west from No. 6 to No. 7, (the same as No. 5), north from No. 7 to No. 8, (the N. W. corner of T. 3 N., R. 1 W.), east from No. 8 to No. 9, (the N. E. corner of same township), and thence west to No. 10, (the same as No. 8), or the southwest corner T. 4 N.—R. 1 W. Thence north, still on a true meridian line, establishing the mile and half-mile corners, until reaching the STANDARD PARALLEL or correction line, (which is here four town-

* The Surveyor should prepare a diagram of the townships, with the numbers here referred to, in their proper places, as here indicated

ships north of the base line); throwing the *excess* over, or *deficiency* under, *four hundred and eighty chains*, on the *last* half-mile, according to law, and at the intersection establishing the "CLOSING CORNER," the distance of which *from* the standard corner must be measured and noted as required by the instructions. But should it ever so happen that some impassable barrier will have prevented or delayed the extension of the standard parallel along and above the field of present survey, then the surveyor will plant, in place, the corner for the township, subject to correction thereafter, should such parallel be extended.

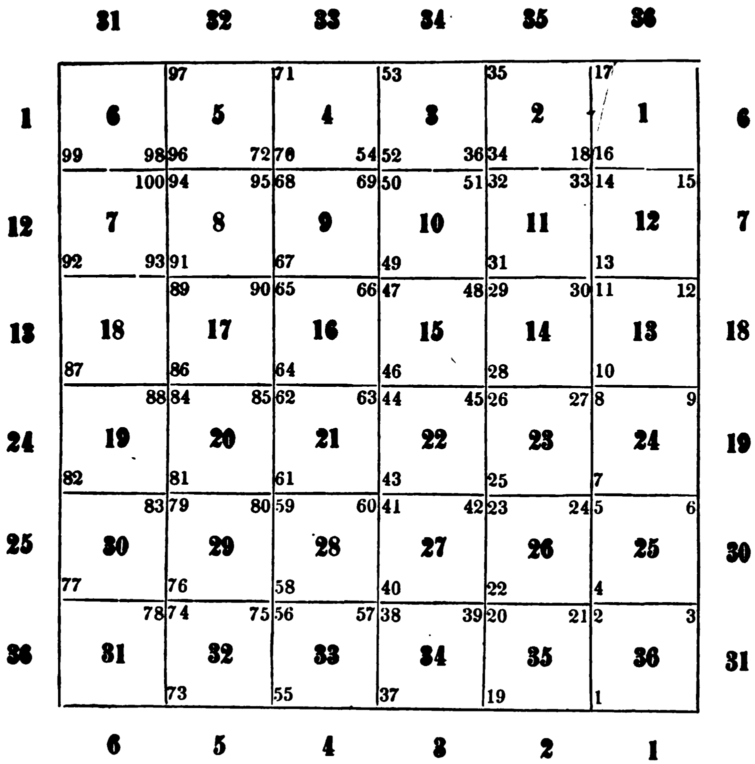
Townships situated NORTH *of the base line, and* EAST *of the principal meridian.* Commence at No. 1, being the *southeast* corner of T. 1 N.—R. 1 E., and proceed as with townships situated "north and west," except that the *random* or trial lines will be run and measured *west*, and the *true* lines, east, throwing the excess over or deficiency under four hundred and eighty chains on the *west end* of the line, as required by law; wherefore, the surveyor will commence his measurement with the length of the deficient or excessive half-section boundary on the west of the township, and thus the remaining measurements will all be *even* miles and half-miles.

Townships situated SOUTH *of the base line, and* WEST *of the principal meridian.* Commence at No. 1, the *northwest* corner of township 1 S., range 1 W., and proceed *due south* in running and measuring line, establishing and marking the mile, half-mile, and township corners thereon, precisely in the method prescribed for running NORTH and WEST, with the exception that, in order to throw the excess or deficiency (over or under four hundred and eighty chains) of the *western* boundaries of such of those townships as *close on the standard parallel* on the *south*, upon the most *northern half-mile* of the townships, according to law, the proceeding will be as follows.

The western (meridional) boundary line of every township, *closing on the standard parallel*, (being every *fifth* one in this case), will be carefully run *south*, on a true meridian, until it intersects the standard, planting stakes and making distinctive marks on line trees, in sufficient number to serve as *guides* in afterwards retracing the line *north* with ease and certainty. At the point of the line's intersection of the standard, the surveyor will establish the "*closing*" (southwest) corner of the township, noting in his field-book its distance and direction from the "standard corner." Then starting from such "closing corner," he will proceed *north* on the line identified by the guide stakes and marks, measuring such line, and establishing thereon the *mile* and *half-mile* stations, and noting, as he goes, all the land and water crossings, &c.

Townships situated SOUTH *of the base line, and* EAST *of the principal meridian.* Commence at No. 1, at the northeast corner of township 1 S., range 1 E., and proceed precisely as with the townships situated "south and west," except that the *random* lines will be run and measured *west*, and the *true* lines *east;* the deficiency or excess of the measurements being, as in all other cases, thrown upon the most western half-mile of line."

(563) Running Section lines. The interior or sectional lines of all townships, however situated in reference to the BASE and MERIDIAN lines, are laid off and surveyed as below.



In the above Diagram, the squares and large figures represent sections, and the small figures at their corners are those referred to in the following directions.

"Commence at No. 1, (see *small* figures on diagram), the corner established on the township boundary for sections 1, 2, 35, and

36; thence run *north* on a true meridian; at 40 chains setting the half-mile or quarter-section post, and at 80 chains (No. 2) establishing and marking the corner of sections 25, 26, 35, and 36. Thence *east*, on a *random* line, to No. 3, setting the temporary quarter-section post at 40 chains, noting the measurement to No. 3, and the measured distance of the random's intersection *north* or *south* of the true or established corner of sections 25, 36, 30, and 31, on the township boundary. Thence *correct*, *west*, on the *true* line to No. 4, setting the quarter-section post on this line exactly at the *equidistant point*, now known, between the section corners indicated by the small figures Nos. 3 and 4. Proceed, in like manner, from No. 4 to No. 5, 5 to 6, 6 to 7, and so on to No. 16, the corner to sections 1, 2, 11, and 12. Thence *north*, on a random line, to No. 17, setting a temporary quarter-section post at 40 chains, noting the length of the whole line, and the measured distance of the random's intersection *east* or *west* of the true corner of sections 1, 2, 35, and 36, established on the township boundary; thence *southwardly* from the latter, on a true line, noting the course and distance to No. 18, the established corner to sections 1, 2, 11, and 12, taking care to establish the quarter-section corner on the true line, at the distance of 40 chains from said section corner, so as to throw the *excess* or *deficiency* on the northern half-mile, according to law. Proceed in like manner through all the intervening tiers of sections to No. 73, the corner to sections 31, 32, 5, and 6; thence *north*, on a true meridian line, to No. 74, establishing the quarter-section corner at 40 chains, and at 80 chains the corner to sections 29, 30, 31, and 32; thence *east*, on a random line to No. 75, setting a temporary quarter-section post at 40 chains, noting the measurement to No. 75, and the distance of the random's intersection *north* or *south* of the established corner of sections 28, 29, 32, and 33; thence *west* from said corner, on the true line, setting the quarter-section post at the equidistant point, to No. 76, which is identical with 74; thence *west*, on a random line, to No. 77, setting a temporary quarter-section post at 40 chains, noting the measurement to No. 77, and the distance of the random's intersection with the western boundary, *north* or *south* of the established corner of sections 25, 36, 30, and 31; and from No. 77, correct, *eastward*, on the true line, *giving its course*, but establishing the quarter-section post, on this line, so as to retain the distance of 40 chains from the corner of sections 29, 30, 31, and 32; thereby throwing the *excess* or *deficiency* of measurement on the most *western* half-mile. Proceed *north*, in a similar manner, from No. 78 to 79, 79 to 80, 80 to 81, and so on to 96, the south-east corner of section 6, where having established the corner for sections, 5, 6, 7, and 8, run thence, successively, on random line *east* to 95, *north* to 97, and *west* to 99; and by reverse courses *correct on true lines back* to said *south-east* corner of section 6, establishing the quarter-section corners, and noting the courses, distances, &c., as before described.

In townships contiguous to standard parallels, the above method will be varied as follows. In every township SOUTH of the principal base line, which *closes* on a standard parallel, the surveyor will begin at the *south-east* corner of the township, and measure *west* on the standard, establishing thereon the *mile* and *half-mile* corners, and noting their distances from the pre-established corners. He then will proceed to subdivide, as directed under the above head.

In the townships NORTH of the principal base line, which *close* on the standard parallel, the sectional lines must be closed on the standard by true meridians, instead of by course lines, as directed under the above head for townships otherwise situated; and the connexions of the closing corners with the pre-established standard corners are to be ascertained and noted. Such procedure does away with any necessity for running the randoms. But in case he is unable to close the lines on account of the standard not having been run, from some inevitable necessity, as heretofore mentiond, he will plant a *temporary* stake, or mound, at the end of the *sixth* mile, thus leaving the lines and their connexions to be finished, and the *permanent* corners to be planted, at such time as the standard shall be extended."

(564) Exceptional methods. Departures from the general system of subdividing public lands have been authorized by law in certain cases, particularly on water-fronts.

Thus, an act of Congress, March 3, 1811, authorized the surveyors of Lousiana, "in surveying and dividing such of the public lands in the said territory, which are or may be authorized to be surveyed and divided, as are adjacent to any river, lake, creek, bayou, or water course, to lay out the same into tracts, as far as practicable, of fifty-eight poles in front, and four hundred and sixty-five poles in depth, of such shape, and bounded by such lines, as the nature of the country will render practicable and most convenient." Another act, of May 24, 1824, authorizes lands similarly situated "to be surveyed in tracts of two acres in width, fronting on any river, bayou, lake, or water course, and running back the depth of forty acres; which tracts of land, so surveyed, shall be offered for sale entire, instead of in half-quarter-sections."

The "Instructions" from which we have quoted say, "In those localities where it would best subserve the interests of the people to have fronts on the navigable streams, and to run back into the uplands for quantity and timber, the principles of the act of May 24th, 1824, may be adopted, and you are authorized to enlarge the quantity, so as to embrace four acres front by forty in depth, forming tracts of one hundred and sixty acres. But in so doing it is designed only to survey the lines *between every four lots*, (or 640 acres), but to establish the boundary posts, or mounds, *in front* and *in rear*, at the distances requisite to secure the quantity of 160 acres to each lot, either rectangularly, when practicable, or at oblique angles, when otherwise. The angle is not important, so that the principle be maintained, as far as practicable, of making the work to square in the rear with the regular sectioning.

The numbering of all anomalous lots will commence with No. 37, to avoid the possibility of conflict with the numbering of the regular sections."

The act of Sept. 27, 1850, authorized the Department, should it deem expedient, to cause the Oregon surveys to be executed according to the principles of what is called the "Geodetic Method."

The complete adoption of this has not been thought to be expedient; but "it was deemed useful to institute on the principal base and meridian lines of the public surveys in Oregon, ordered to be established by the act referred to, a system of triangulations from the recognized legal stations, to all prominent objects within the range of the theodolite; by means of which the relative distances of such objects, in respect to those main lines, and also to each other, might be observed, calculated, and protracted, with the view of contributing to the knowledge of the topography of the country in advance of the progressing linear surveys, and to obtain the elements for estimating areas of valleys intervening between the spurs of the mountains."

"Meandering" is a name given to the usual mode of surveying with the compass, particularly as applied to navigable streams. The "Instructions" for this are, in part, as follows.

"Both banks of *navigable* rivers are to be meandered by taking the courses and distances of their sinuosities, and the same are to be entered in the '*Meander* field-book.' At those points where either the township or section lines intersect the banks of a navigable stream, POSTS, or, where necessary, MOUNDS of *earth* or *stone*, (as noted in Art. (**566**,)) are to be established at the time of running these lines. These are called "meander corners;" and in meandering you are to commence at one of those corners on the township line, coursing the banks, and measuring the distance of each course from your commencing corner to the next 'meander corner,' upon the same or another boundary of the same township; carefully noting your intersection with all intermediate meander corners. By the same method you are to meander the opposite bank of the same river.

The crossing distance *between* the MEANDER CORNERS, on same line, is to be ascertained by triangulation, in order that the river may be protracted with entire accuracy. The particulars to be given in the field-notes.

The courses and distances on meandered navigable streams, govern the calculations wherefrom are ascertained the true areas of the tracts of land (sections, quarter sections, &c.) known to the law as *fractional*, and bounding on such streams."

You are also to meander, in manner aforesaid, all *lakes* and deep ponds of the area of twenty-five acres and upwards; also navigable bayous.

The precise relative position of islands, in a township made fractional by the river in which the same are situated, is to be determined trigonometrically. Sighting to a flag or other fixed object on the island, from a special and carefully measured base line, connected with the surveyed lines, on or near the river bank, you are to form connexion between the meander corners on the river to points corresponding thereto, in direct line, on the bank of the island, and there establish the proper meander corners, and calculate the distance across."

(565) Marking Lines. "All lines on which are to be established the legal corner boundaries, are to be marked after this method, viz: Those trees which may intercept your line, must have two chops or notches cut on each side of them without any other marks whatever. These are called '*sight trees*,' or '*line trees*.'

A sufficient number of other trees standing nearest to your line, on either side of it, are to be *blazed* on two sides, diagonally or quartering towards the line, in order to render the line conspicuous, and readily to be traced, the blazes to be opposite each other, coinciding in direction with the line where the trees stand very near it, and to approach nearer each other, the further the line passes from the blazed trees. Due care must ever be taken to have the lines so well marked as to be readily followed."

(566) Marking Corners. "After a true coursing, and most exact measurements, the corner boundary is the consummation of the work, for which all the previous pains and expenditure have been incurred. A boundary corner, in a timbered country, is to be a *tree*, if one be found at the precise spot; and if not, a *post* is to be planted thereat; and the position of the corner post is to be indicated by trees adjacent, (called Bearing trees) the angular bearings and distances of which from the corner are facts to be ascertained and registered in your field book.

In a region where stone abounds, the corner boundary will be a small *monument of stones* along side of a single marked stone, for a township corner—and *a single stone* for all other corners.

In a region where timber is not near, nor stone, the corner will be a *mound of earth*, of prescribed size, varying to suit the case.

Corners are to be fixed, for township boundaries at intervals of every six miles; for section boundaries at intervals of every mile, or 80 chains; and, for quarter section boundaries at intervals of every half mile, or 40 chains.

MEANDER CORNER POSTS are to be planted at all those points where the township or section lines intersect the banks of such rivers, lakes, or islands, as are by law directed to be meandered," as explained in Art. (564).

When *posts* are used, their length and size must be proportioned to the importance of the corner, whether township, section, or quarter-section, the first being at least 24 inches above ground, and 3 inches square.

Where a township post is a corner common to *four* townships, it is to be set in the earth *diagonally*, thus: W ◇ E (N above, S below), and the cardinal points of the compass are to be indicated thereon by a cross line, or wedge, (one-eighth of an inch deep at least), cut or sawed out of its top, as in the figure. On each surface of the post is to be marked the number of the particular township, and its range, which it *faces*. Thus, if the post be a common boundary to four townships, say *one* and *two*, south of the base line, of range *one*, west of the meridian; also to townships *one* and *two*, south of the base line, of range *two*, west of the meridian, it is to be marked thus:

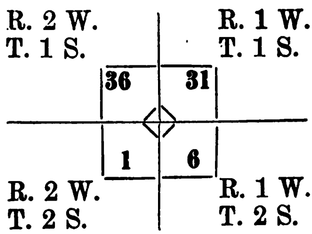
From N. to E. { R. 1 W. / T. 1 S. / S. 31 }

from N. to W. { 2 W. / 1 S. / 36 }

from E. to S. { 1 W. / 2 S. / 6 }

from W. to S. { 2 W. / 2 S. / 1 }

The position of the post which is here taken as an example, is shewn in the following diagram.



These marks are to be distinctly and neatly chiselled into the wood, at least the eighth of an inch deep; and to be also marked with *red chalk.* The *number* of the *sections* which they respectively *face*, will *also be marked* on the township post.

Section or mile posts, being corners of sections, when they are common to *four* sections, are to be set *diagonally* in the earth, (in the manner provided for township corner posts), and with a similar cross cut in the top, to indicate the cardinal points of the compass; and on each side of the squared surfaces is to be marked the appropriate *number* of the particular one of the *four sections*, respectively, which such side *faces;* also on one side thereof are to be marked the numbers of its *township* and *range;* and to make such marks yet more conspicuous, (in manner aforesaid), a streak of red chalk is to be applied.

In the case of an *isolated* township, subdivided into thirty-six sections, there are twenty-five interior sections, the south-west corner boundary of each of which will be *common* to *four* sections. On all the extreme sides of an isolated township, the outer tiers of sections have corners *common* only to *two* sections then surveyed. The posts, however, must be planted precisely like the former, but presenting two *vacant* surfaces to receive the appropriate marks when the adjacent survey may be made.

A quarter-section or half-mile post is to have no other mark on it than $\frac{1}{4}$ S., to indicate what it stands for.

Township corner posts are to be NOTCHED with *six* notches on each of the four angles of the squared part set to the cardinal points.

All mile posts *on township lines* must have as many notches on them, on two opposite *angles* thereof, as they are miles distant from the township corners, respectively. Each of the posts at the corners of sections in the *interior* of a township must indicate, by a number of notches on each of its four corners directed to the cardinal points, the corresponding number of miles that it stands from the *outlines* of the township. The four sides of the post will indicate the number of the section they respectively *face.* Should a tree be found at the place of any corner, it will be marked and notched, as aforesaid, and answer for the corner in lieu of a post; the kind of tree and its diameter being given in the field-notes.

The position of all corner posts, or corner trees of whatever description, which may be established, is to be perpetuated in the following manner, viz: From such post or tree the courses shall be taken, and the distances measured, to two or more adjacent trees, in opposite directions, as nearly as may be, which are called '*Bearing trees,*' and are to be blazed near the ground, with a large blaze facing the post, and having one notch in it, neatly and plainly made with an axe, square across, and a little below the middle of the blaze. The kind of tree and the diameter of each are facts to be distinctly set forth in the field-book.

On each bearing tree the letters B. T., must be distinctly cut into the wood, in the blaze, a little above the notch, or on the bark, with the number of the range, township, and section.

At all township corners, and at all section corners, on range or township lines, *four* bearing trees are to be marked in this manner, one in each of the adjoining sections.

At interior section corners *four* trees, one to stand within each of the four sections to which such corner is common, are to be marked in manner aforesaid, if such be found.

From quarter section and meander corners two bearing trees are to be marked, one within each of the adjoining sections.

Stones at township corners (a small monument of stones being alongside thereof) must have *six* notches cut with a pick or chisel on each edge or side towards the cardinal points; and where used as section corners on the range and township lines, or as section corners in the interior of a township, they will also be notched by a pick or chisel, to correspond with the directions given for notching posts similarly situated.

Stones, when used as quarter-section corners, will have ¼ cut on them; on the west side on north and south lines, and on the north side on east and west lines.

Whenever bearing trees are not found, MOUNDS of earth, or stone, are to be raised *around posts* on which the corners are to be marked in the manner aforesaid. Wherever a mound of earth is adopted, the same will present a conical shape; but at its base, on the earth's surface, a *quadrangular trench* will be dug; a *spade deep* of earth being thrown up from the four sides of the line, *outside* the trench, so as to form a *continuous elevation along its outer edge.* In mounds of earth, common to *four* townships or to *four* sections, they will present the *angles* of the quadrangular trench (*diagonally*) towards the cardinal points. In mounds common only to *two* townships or *two* sections, the *sides* of the quadrangular trench will *face* the cardinal points.

Prior to piling up the earth to construct a mound, in a *cavity* formed at the corner boundary point is to be deposited a *stone*, or a portion of *charcoal*, or a *charred stake* is to be driven twelve inches down into such centre point, to be a *witness* for the future.

The surveyor is farther specially enjoined to plant *midway* between each pit and the trench, seeds of some tree, those of fruit trees adapted to the climate being always to be preferred.

DOUBLE CORNERS are to be found nowhere except on the Standard Parallels or Correction lines, whereon are to appear both the corners which mark the intersections of the lines which close thereon, and those from which the surveys start in the opposite direction.

The corners which are established on the standard parallel, at the time of running it, are to be known as '*Standard Corners,*' and, in addition to all the *ordinary* marks, (as herein prescribed), they will be marked with the letters S. C. The '*closing corners*' will be marked C. C."

(567) Field Books. There should be several distinct and separate field-books; viz.:

"1. Field-notes of the MERIDIAN and BASE lines, showing the establishment of the *township*, *section* or mile, and *quarter-section* or half-mile, boundary corners thereon; with the crossings of streams, ravines, hills, and mountains; character of soil, timber, minerals, &c. These notes will be arranged, in series, by *mile stations*, from number *one* to number ——.

2. Field-notes of the 'STANDARD PARALLELS, or correction lines,' showing the establishment of the township, section, and quarter-section corners, besides exhibiting the topography of the country on line, as required on the base and meridian lines.

3. Field-notes of the EXTERIOR lines of TOWNSHIPS, showing the establishment of the corners on line, and the topography, as aforesaid.

4. Field notes of the SUBDIVISIONS of TOWNSHIPS into sections and quarter-sections; at the close whereof will follow the notes of the MEANDERS of navigable streams. These notes will also show, by *ocular* observation, the estimated rise and fall of the land on the line. A description of the timber, undergrowth, surface, soil, and minerals, upon each section line, is to follow the notes thereof, and not to be mixed up with them."

5. The "Geodetic Field-book," comprising all triangulations, angles of elevation and depression, levelling, &c.

The examples on the next two pages, taken from the "Instructions" which we have followed throughout, will shew what is required.

The ascents and descents are recorded in the right-hand columns.

FIELD NOTES OF

THE EXTERIOR LINES

OF AN ISOLATED TOWNSHIP.

Field notes of the Survey of township 25 *north, of range* 2 *west, of the Willamette meridian, in the Territory of* OREGON, *by Robert Acres, deputy surveyor, under his contract No.* 1, *bearing date the 2d day of January,* 1851.

	Chs. lks.		Feet.
Random tp. S. boundary.		TOWNSHIP LINES commenced January 20, 1851.	
		Southern boundary variation 18° 41′ E.	
	East.	On a *random* line on the south boundaries of sections 31, 32, 33, 34, 35, and 36. Set temporary mile and half-mile posts, and intersected the eastern boundary 2 chains 20 links north of the true corner 5 miles 74 chains 53 links.	
		Therefore the correction will be 5 chains 47 links W. 37.1 links S. per mile.	
Timbered corners.		TRUE SOUTHERN BOUNDARY variation 18° 41′ E.	
	West	On the southern boundary of sec. 36, Jan. 24, 1851.	
	40.00	Set qr. sec. post from which	*a* 10
		a beech 24 in. dia. bears N. 11 E. 38 lks. dist.	
		a do 9 do do S. 9 E. 17 do	
	62.50	a brook 8 l. wide, course NW.	*d* 10
	80.00	Set post cor. of secs. 35 & 36, 1 & 2, from which	*a* 5
		a beech 9 in. dia. bears S. 46 E. 8 l. dist.	
		a do 8 do do S. 62 W. 7 do	
		a W. oak 10 do do N. 19 W. 14 do	
		a B. oak 14 do do N. 29 E. 16 do	
		Land level, part wet and swampy; timber beech, oak, ash, hickory, &c.	
Deficient timbered corners.	West.	On the S. boundary of sec. 35—	
	40.00	Set qr. sec. post, with trench, from which	*a* 10
		a beech 6 in. dia. bears N. 80 E. 8 l. dist.	
		planted SW. a yellow locust seed.	
	65.00	To beginning of hill	*a* 5
	80.00	Set post, with trench, cor. of secs. 34 & 35, 2 & 3, from which	*a* 20
		a beech 10 in. dia. bears S. 51 E. 13 l. dist.	
		do 10 do do N. 56 W. 9 do	
		planted SW. a white oak acorn,	
		NE. a beech nut.	
		Land level, rich, and good for farming; timber same.	
Mound section corner.	West.	On the S. boundary of sec. 34—	
	40.00	Set qr. sec. post, with trench, from which	*a* 5
		a B. oak 10 in. dia. bears N. 2 E. 635 l. dist.	
		Planted SW. a beech nut.	
	80.00	To corner of sections 33, 34, 3 and 4, drove charred stake, raised mound with trench as per instructions, and Planted NE. a W. oak ac'n; NW. a yel. locust seed. SE. a butternut; SW. a beech nut.	*a* 10
		Land level, rich and good for farming, some scattering oak and walnut.	

&c., &c., &c.

FIELD NOTES OF THE

SUBDIVISIONAL OR SECTIONAL LINES,

AND MEANDERS.

Township 25 *N.*, *Range* 2 *W.*, *Willamette Mer.*

	Chs. lks.		Feet.
		SUBDIVISIONS. Commenced February 1, 1851.	
	North.	Between secs. 35 and 36—	
True line.	9.19	A beech 30 in. dia.	d 10
	29.97	A beech 30 in. dia.	d 5
	40.00	Set qr. sec. post, from which a beech 15 in. dia. bears S. 48 E. 12 l. dist. a do 8 do do N. 23 W. 45 do	d 5
	51.90	A beech 18 in. dia.	d 5
	76.73	A sugar 30 in. dia.	d 8
	80.00	Set a post cor. of secs. 25, 26, 35, 36, from which a beech 24 in. dia. bears N. 62 W. 17 l. dist. a poplar 36 do do S. 66 E. 34 do. a do 20 do do S. 70 W. 50 do. a beech 28 do do N. 60 E. 45 do. Land level, second rate; timber beech, poplar, sugar, and und'gr. spice, &c.	d 2
	East.	On random line between secs. 25 and 36—	
Random.	9.00	A brook 30 l. wide, course N.	d 10
	15.00	To foot of hill	d 10
	40.00	Set temporary qr. sec. post	a 60
	55.00	To opposite foot of hill	d 40
	72.00	A brook 15 l. wide, course N.	d 20
	80.00	Intersect E. boundary at post	a 10
		Land level, second rate; timber, beech, oak, ash, &c.	
		&c., &c., &c.	

MEANDERS OF CHICKEELES RIVER.

Beginning at a meander post in the northern township boundary, and thonce on the left bank down stream. ***Commenced February*** **11, 1851.**

Courses.	Dist. Chs. lks.	REMARKS.
S. 76 W.	18.46	In section 4 bearing to corner sec. 4 on right bank N. 70° W.
S. 61 W.	10.00	Bearing to cor. sec. 4 and 5, right bank N. 52° W.
S. 61 W.	8.18	To post in line between sections 4 and 5, breadth of river by triangulation 9 chains 51 links.
S. 54 W.	10.69	In section 5.
S. 40 W.	5.59	
S. 50 W.	8.46	
S. 37 W.	16.50	To upper corner of John Smith's claim, course E.
S. 44 W.	21.96	
S. 36 W.	27.53	To post in line between sections 5 and 8, breadth of river by triangulation 8 chains 78 links.
		&c., &c., &c.

APPENDIX.

APPENDIX A.

SYNOPSIS OF PLANE TRIGONOMETRY.*

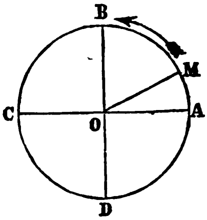
(1) Definition. Plane Trigonometry is that branch of Mathematical Science which treats of the relations between the sides and angles of plane triangles. It teaches how to find any three of these six parts, when the other three are given and one of them, at least, is a side.

(2) Angles and Arcs. The *angles* of a triangle are measured by the *arcs* described, with any radius, from the angular points as centres, and intercepted between the legs of the angles. These arcs are measured by comparing them with an entire circumference, described with the same radius. Every circumference is regarded as being divided into 360 equal parts, called *degrees*. Each degree is divided into 60 equal parts, called *minutes*, and each minute into 60 *seconds*. These divisions are indicated by the marks $^{\circ}\ '\ ''$. Thus 28 degrees, 17 minutes, and 49 seconds, are written $28^{\circ}\ 17'\ 49''$. Fractions of a second are best expressed decimally. An arc, including a quarter of a circumference and measuring a right angle, is therefore 90°. A semicircumference comprises 180°. It is often represented by π, which equals 3.14159, &c., or $3\frac{1}{7}$ approximately, the radius being unity.

The length of 1° in parts of radius $= 0.01745329$; that of $1' = 0.00029089$; and that of $1'' = 0.00000485$.

The length of the radius of a circle in degrees, or 360ths of the circumference $= 57^{\circ}.29578 = 57^{\circ}\ 17'\ 24''.8 = 3437'.747 = 206264''.8$.†

Fig. 397.



An arc may be regarded as generated by a point, M, moving from an origin, A, around a circle, in the direction of the arrow. The point may thus describe arcs of any lengths, such as AM; $AB = 90^{\circ} = \frac{1}{2}\pi$; $ABC = 180^{\circ} = \pi$; $ABCD = 270^{\circ} = \frac{3}{2}\pi$; $ABCDA = 360^{\circ} = 2\pi$.

The point may still continue its motion, and generate arcs greater than a circumference, or than two circumferences, or than three; or even infinite in length.

While the point, M, describes these arcs, the radius, OM, indefinitely produced, generates corresponding angles.

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* For merely solving triangles, only Articles (1), (2), (3), (5), (6), (10), (11), and (12), are needed.

† The number of seconds in any arc which is given in parts of radius, radius being unity, equals the length of the arc so given divided by the length of the arc of one second; or multiplied by the number of seconds in radius.

If the point, M, should move from the origin, A, in the contrary direction to its former movement, the arcs generated by it are regarded as *negative*, or *minus*; and so too, of necessity, the angles measured by the arcs.

Arcs and angles may therefore vary in length from 0 to $+\infty$ in one direction, and from 0 to $-\infty$ in the contrary direction.

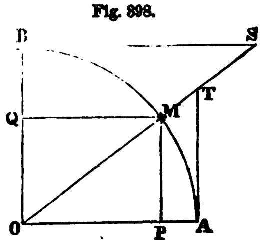
The *Complement* of an arc is the arc which would remain after subtracting the arc from a quarter of the circumference, or from 90°. If the arc be more than 90°, its complement is necessarily negative.

The *Supplement* of an arc is what would remain after subtracting it from half the circumference, or from 180°. If the arc be more than 180°, its supplement is necessarily negative.

(3) Trigonometrical Lines. The relations of the sides of a triangle to its angles are what is required; but it is more convenient to replace the angles by arcs; and, once more, to replace the arcs by certain straight lines depending upon them, and increasing and decreasing with them, or conversely, in such a way that the length of the lines can be found from that of the arcs, and vice versâ. It is with these lines that the sides of a triangle are compared.* These lines are called *Trigonometrical Lines*; or *Circular Functions*, because their length is a function of that of the circular arcs. The principal Trigonometrical lines are *Sines*, *Tangents*, and *Secants*. Chords and versed sines are also used.

The SINE of an arc, AM, is the perpendicular, MP, let fall, from one extremity of the arc, upon the diameter which passes through the other extremity.

Fig. 398.



The TANGENT of an arc, AM, is the distance, AT, intercepted, on the tangent drawn at one extremity of the arc, between that extremity and the prolongation of the radius which passes through the other extremity.

The SECANT of an arc, AM, is the part, OT, of the prolonged radius, comprised between the centre and the tangent.

The sine, tangent, and secant of the complement of an arc are called the CO-SINE, CO-TANGENT, and CO-SECANT of that arc. Thus, MQ is the cosine of AM, BS its cotangent, and OS its cosecant. The cosine MQ is equal to OP, the part of the radius comprised between the centre and the foot of the sine.

The *chord* of an arc is equal to twice the sine of half that arc.

The *versed-sine* of an arc, AM, is the distance, AP, comprised between the origin of the arc and the foot of the sine. It is consequently equal to the difference between the radius and the sine.

The Trigonometrical lines are usually written in an abbreviated form. Calling the arc $AM = a$, we write,

$MP = \sin.\ a.$	$AT = \tan.\ a.$	$OT = \sec.\ a.$
$MQ = \cos.\ a.$	$BS = \cot.\ a.$	$OS = \text{cosec.}\ a.$

The period after sin., tan., &c., indicating abbreviation, is frequently omitted.

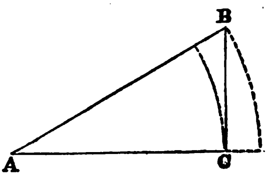
The arcs whose sines, tangents, &c., are equal to a line $=a$, are written,

$$\sin.^{-1} a, \text{ or arc } (\sin. = a);$$
$$\tan.^{-1} a, \text{ or arc } (\tan. = a); \ \&c.$$

* For the great value of this indirect mode of comparing the sides and angles of triangles, see Comte's "Philosophy of Mathematics," (Harpers', 1851,) page 225.

(4) The lines as ratios. The ratios between the trigonometrical lines and the radius are the same for the same angles, or number of degrees in an arc, whatever the length of the radius or arc. Consequently, radius being unity, these lines may be expressed as simple ratios. Thus, in the right-angled triangle A B C, we would have

Fig. 399.



$$\sin. A = \frac{BC}{AB} = \frac{\text{opposite side}}{\text{hypothenuse}}, \qquad \cos. A = \frac{AC}{AB} = \frac{\text{adjacent side}}{\text{hypothenuse}},$$

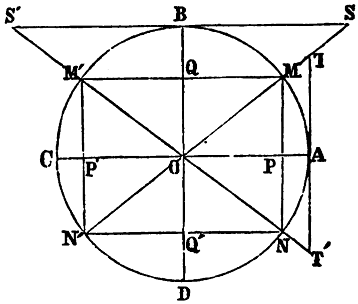
$$\tan. A = \frac{BC}{AC} = \frac{\text{opposite side}}{\text{adjacent side}}, \qquad \cot. A = \frac{AC}{BC} = \frac{\text{adjacent side}}{\text{opposite side}},$$

$$\sec. A = \frac{AB}{AC} = \frac{\text{hypothenuse}}{\text{adjacent side}}, \qquad \text{cosec}. A = \frac{AB}{BC} = \frac{\text{hypothenuse}}{\text{opposite side}}.$$

When the radius of the arcs which measure the angles is unity, these ratios may be used for the lines. If the radius be any other length, the results which have been obtained by the above supposition, must be modified by dividing each of the trigonometrical lines in the result by radius, and thus rendering the equations of the results "homogeneous." The same effect would be produced by multiplying each term in the expression by such a power of radius as would make it contain a number of *linear* factors equal to the greatest number in any term. The radius is usually represented by r, or R.

(5) Their variations in length. As the point M moves around the circle, and the arc thus increases, the sines, tangents, and secants, starting from zero, also increase; till, when the point M has arrived at B, and the arc has become 90°, the sine has become equal to radius, or unity, and the tangent and secant have become infinite. The complementary lines have decreased; the cosine being equal to radius or unity at starting and becoming zero, and the cotangent and cosecant passing from infinity to zero. When the point M has passed the first quadrant at B and is proceeding towards C, the sines, tangents, and secants begin to decrease, till, when the point has reached C, they have the same values as at A. They then begin to increase again, and so on. The Table on page 382 indicates these variations.

Fig. 400.



The sines and tangents of very small arcs may be regarded as sensibly proportional to the arcs themselves; so that for sin. a'', we may write a . sin. $1''$; and similarly, though less accurately, for sin. a', we may write a . sin. $1'$.

The sines and tangents of very small arcs may similarly be regarded as sensibly of the same length as the arcs themselves.*

* Consequently, the note on page 379 may read thus: The number of seconds in any very small arc given in parts of radius, radius being unity, is equal to the length of the arc so given divided by sin. $1''$.

a being the length of any arc expressed in parts of radius, the lengths of its sine and cosine may be obtained by the following series:

$$\sin. a = a - \frac{a^3}{2.3} + \frac{a^5}{2.3.4.5} - \frac{a^7}{2.3....7} +, \text{etc.}$$

$$\cos. a = 1 - \frac{a^2}{2} + \frac{a^4}{2.3.4} - \frac{a^6}{2....6} +, \text{etc.}$$

Let it be required to find cos. 30°, by the above series.

$$30° = \frac{180}{30}\pi = \frac{1}{6} \times 3.1416 = .5236.$$

Substituting this number for a, the series becomes, taking only three terms of it,

$$1 - \frac{(.5236)^2}{2} + \frac{(.5236)^4}{24} -, \text{etc.} = 1 - 0.137078 + 0.003130 = .866052;$$

which is the correct value of cos. 30° for the first four places of decimals.

The lengths of the other lines can be obtained from the mutual relations given in Art. (7.) Some particular values are given below.

sin. $30° = \frac{1}{2}$.	sin. $45° = \frac{1}{2}\sqrt{2}$.	sin. $60° = \frac{1}{2}\sqrt{3}$.
tan. $30° = \frac{1}{3}\sqrt{3}$.	tan. $45° = 1$.	tan. $60° = \sqrt{3}$.
sec. $30° = \frac{2}{3}\sqrt{3}$.	sec. $45° = \sqrt{2}$.	sec. $60° = 2$.

(6) Their changes of sign. Lines measured in contrary directions from a common origin, usually receive contrary algebraic signs. If then all the lines in the first quadrant are called positive, their signs will change in some of the other quadrants. Thus the *sines* in the first quadrant being all measured upward, when they are measured downward, as they are in the third and fourth quadrants, they will be negative. The *cosines* in the first quadrant are measured from left to right, and when they are measured from right to left, as in the second and third quadrants, they will be negative. The *tangents* and *secants* follow similar rules.

The variations in length and the changes of sign are all indicated in the following table, radius being unity. The terms "increasing" and "decreasing" apply to the lengths of the lines without any reference to their signs.

Lengths and Signs of the Trigonometrical Lines for Arcs from 0° *to* 360°.

Arcs.	0°	Between 0° and 90°.	90°	Between 90° and 180°.	180°
Sine . . .	0	+, and increasing,	+1	+, and decreasing,	0
Tangent . .	0	+, and increasing,	$\pm\infty$	−, and decreasing,	0
Secant . .	+1	+, and increasing,	$\pm\infty$	−, and decreasing,	−1
Cosine . .	+1	+, and decreasing,	0	−, and increasing,	−1
Cotangent .	$\pm\infty$	+, and decreasing,	0	−, and increasing,	$\mp\infty$
Cosecant . .	$\pm\infty$	+, and decreasing,	+1	+, and increasing,	$\pm\infty$

Arcs.	180°	Between 180° and 270°.	270°	Between 270° and 360°.	360°
Sine . . .	0	−, and increasing,	−1	−, and decreasing,	0
Tangent . .	0	+, and increasing,	$\pm\infty$	−, and decreasing,	0
Secant . .	−1	−, and increasing,	$\mp\infty$	+, and decreasing,	+1
Cosine. . .	−1	−, and decreasing,	0	+, and increasing,	+1
Cotangent .	$\mp\infty$	+, and decreasing,	0	−, and increasing,	$\mp\infty$
Cosecant . .	$\pm\infty$	−, and decreasing,	−1	−, and increasing,	$\mp\infty$

From this table, and Fig. 400, we see that *an arc and its supplement* have the same sine; and that their tangents, secants, cosines, and cotangents are of equal length but of contrary signs; while the cosecants are the same in both length and sign.

We also deduce from the figure the following consequences:

$$\text{sin.}\ (a^\circ + 180^\circ) = -\text{sin.}\ a^\circ. \qquad \text{cos.}\ (a^\circ + 180^\circ) = -\text{cos.}\ a^\circ.$$
$$\text{tan.}\ (a^\circ + 180^\circ) = \text{tan.}\ a^\circ. \qquad \text{cot.}\ (a^\circ + 180^\circ) = \text{cot.}\ a^\circ.$$
$$\text{sec.}\ (a^\circ + 180^\circ) = -\text{sec.}\ a^\circ. \qquad \text{cosec.}\ (a^\circ + 180^\circ) = -\text{cosec.}\ a^\circ.$$

$$\text{sin.}\ (-a^\circ) = -\text{sin.}\ a^\circ. \qquad \text{cos.}\ (-a^\circ) = \text{cos.}\ a^\circ.$$
$$\text{tan.}\ (-a^\circ) = -\text{tan.}\ a^\circ. \qquad \text{cot.}\ (-a^\circ) = -\text{cot.}\ a^\circ.$$
$$\text{sec.}\ (-a^\circ) = \text{sec.}\ a^\circ. \qquad \text{cosec.}\ (-a^\circ) = -\text{cosec.}\ a^\circ.$$

An infinite number of arcs have the same trigonometrical lines; for, an arc a, the same arc plus a circumference, the same arc plus two circumferences, and so on, would have the same sine, &c.

"To bring back to the first quadrant" the trigonometrical lines of any large arc, proceed thus: Let 1029° be an arc the sine of which is desired. Take from it as many times 360° as possible. The remainder will be 309°. Then we shall have

$$\text{sin.}309^\circ = \text{sin.}(180^\circ - 309^\circ) = \text{sin.} -129^\circ = -\text{sin.}129^\circ = -\text{sin.}(180^\circ - 129^\circ) = -\text{sin.}51^\circ.$$

(7) Their mutual relations. Radius being unity,

$$\text{tan.}\ a^\circ = \frac{\text{sin.}\ a^\circ}{\text{cos.}\ a^\circ}. \qquad \text{cot.}\ a^\circ = \frac{\text{cos.}\ a^\circ}{\text{sin.}\ a^\circ}.$$

$$\text{sec.}\ a^\circ = \frac{1}{\text{cos.}\ a^\circ}. \qquad \text{cosec.}\ a^\circ = \frac{1}{\text{sin.}\ a^\circ}.$$

$$\text{tan.}\ a^\circ \times \text{cot.}\ a^\circ = 1. \qquad (\text{sin.}\ a^\circ)^2 + (\text{cos.}\ a^\circ)^2 = 1.^*$$
$$1 + (\text{tan.}\ a^\circ)^2 = (\text{sec.}\ a^\circ)^2. \qquad 1 + (\text{cot.}\ a^\circ)^2 = (\text{cosec.}\ a^\circ)^2.$$

Hence, any one of the trigonometrical lines being given, the rest can be found from some of these equations.

(8) Two arcs. Let a and b represent any two arcs, a being the greater. Then the following formulas apply:

$$\text{sin.}\ (a + b) = \text{sin.}\ a \,.\, \text{cos.}\ b + \text{cos.}\ a \,.\, \text{sin.}\ b.$$
$$\text{sin.}\ (a - b) = \text{sin.}\ a \,.\, \text{cos.}\ b - \text{cos.}\ a \,.\, \text{sin.}\ b.$$
$$\text{cos.}\ (a + b) = \text{cos.}\ a \,.\, \text{cos.}\ b - \text{sin.}\ a \,.\, \text{sin.}\ b.$$
$$\text{cos.}\ (a - b) = \text{cos.}\ a \,.\, \text{cos.}\ b + \text{sin.}\ a \,.\, \text{sin.}\ b.$$
$$\text{tan.}\ (a + b) = \frac{\text{tan.}\ a + \text{tan.}\ b}{1 - \text{tan.}\ a \,.\, \text{tan.}\ b}.$$
$$\text{tan.}\ (a - b) = \frac{\text{tan.}\ a - \text{tan.}\ b}{1 + \text{tan.}\ a \,.\, \text{tan.}\ b}.$$
$$\text{cot.}\ (a + b) = \frac{\text{cot.}\ a \,.\, \text{cot.}\ b - 1}{\text{cot.}\ b + \text{cot.}\ a}.$$
$$\text{cot.}\ (a - b) = \frac{\text{cot.}\ a \,.\, \text{cot.}\ b + 1}{\text{cot.}\ b - \text{cot.}\ a}.$$

* The square, &c., of the sine, &c., of an arc, is often expressed by placing the exponent between the abbreviation of the name of the trigonometrical line and the number of the degrees in the arc; thus, $\text{sin.}^2\, a^\circ$, $\text{tan.}^3\, a^\circ$, &c. But the notation given above, places the index as used by Gauss, Delambre, Arbogast, &c., though the first two omit the parentheses.

$$\sin. a . \sin. b = \tfrac{1}{2} . \cos. (a - b) - \tfrac{1}{2} \cos. (a + b).$$
$$\cos. a . \cos. b = \tfrac{1}{2} . \cos. (a + b) + \tfrac{1}{2} \cos. (a - b).$$
$$\sin. a . \cos. b = \tfrac{1}{2} . \sin. (a + b) + \tfrac{1}{2} \sin. (a - b).$$
$$\cos. a . \sin. b = \tfrac{1}{2} . \sin. (a + b) - \tfrac{1}{2} \sin. (a - b).$$
$$\sin. a + \sin. b = 2 \sin. \tfrac{1}{2} (a + b) \cos. \tfrac{1}{2} (a - b).$$
$$\cos. a + \cos. b = 2 \cos. \tfrac{1}{2} (a + b) \cos. \tfrac{1}{2} (a - b).$$
$$\sin. a - \sin. b = 2 \sin. \tfrac{1}{2} (a - b) \cos. \tfrac{1}{2} (a + b).$$
$$\cos. b - \cos. a = 2 \sin. \tfrac{1}{2} (a - b) \sin. \tfrac{1}{2} (a + b).$$
$$\tan. a + \tan. b = \frac{\sin. (a + b)}{\cos. a . \cos. b}.$$
$$\tan. a - \tan. b = \frac{\sin. (a - b)}{\cos. a . \cos. b}.$$
$$\cot. b + \cot. a = \frac{\sin. (a + b)}{\sin. a . \sin. b}.$$
$$\cot. b - \cot. a = \frac{\sin. (a - b)}{\sin. a . \sin. b}.$$

(9) Double and half arcs. Letting a represent any arc, as before, we have the following formulas:

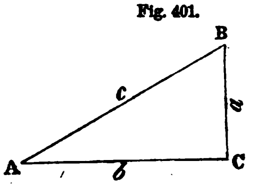
$$\sin. 2 a = 2 \sin. a . \cos. a.$$
$$\cos. 2 a = (\cos. a)^2 - (\sin. a)^2 = 2 (\cos. a)^2 - 1 = 1 - 2 (\sin. a)^2.$$
$$\tan. 2 a = \frac{2 \tan. a}{1 - (\tan. a)^2} = \frac{2 \cot. a}{(\cot. a)^2 - 1} = \frac{2}{\cot. a - \tan. a}.$$
$$\cot. 2 a = \frac{(\cot. a)^2 - 1}{2 \cot. a} = \tfrac{1}{2} (\cot. a - \tan. a).$$
$$\sin. \tfrac{1}{2} a = \surd [\tfrac{1}{2} (1 - \cos a)].$$
$$\cos. \tfrac{1}{2} a = \surd [\tfrac{1}{2} (1 + \cos. a)].$$
$$\tan. \tfrac{1}{2} a = \frac{\sin. a}{1 + \cos. a} = \frac{1 - \cos. a}{\sin. a} = \surd \left(\frac{1 - \cos. a}{1 + \cos. a}\right).$$
$$\cot. \tfrac{1}{2} a = \frac{1 + \cos. a}{\sin. a} = \frac{\sin. a}{1 - \cos. a} = \surd \left(\frac{1 + \cos. a}{1 - \cos. a}\right).$$

(10) Trigonometrical Tables. In the usual tables of the natural Trigonometrical lines, the degrees from 0° to 45° are found at the top of the table, and those from 45° to 90° at the bottom; the latter being complements of the former. Consequently, the columns which have *Sine* and *Tangent* at top have *Cosine* and *Cotangent* at bottom, since the cosine or cotangent of any arc is the same thing as the sine or tangent of its complement. The minutes to be added to the degrees are found in the left-hand column, when the number of degrees at the top of the page are used, and in the right-hand column for the degrees when at the bottom of the page. The lines for arcs intermediate between those in the tables are found by proportion. The lines are calculated for a radius equal unity. Hence, the values of the sines and cosines are decimal fractions, though the point is usually omitted. So too are the tangents from 0° to 45°, and the cotangents from 90° to 45°. Beyond those points they are integers and decimals.

The calculations, like all others involving large numbers, are shortened by the use of logarithms, which substitute addition and subtraction for multiplication and division; but the young student should avoid the frequent error of regarding logarithms as a necessary part of trigonometry.

SOLUTION OF TRIANGLES.

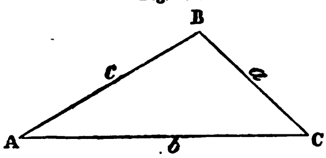
Fig. 401.



(11) Right-angled Triangles. Let ABC be any right-angled triangle. Denote the sides opposite the angles by the corresponding small letters. Then any one side and one acute angle, or any two sides being given, the other parts can be obtained by one of the following equations:

Given.	Required.	Formulas.
a, b	c, A, B	$c = \sqrt{(a^2 + b^2)}$; tan. $A = \frac{a}{b}$; cot. $B = \frac{a}{b}$.
a, c	b, A, B	$b = \sqrt{(c^2 - a^2)}$; sin. $A = \frac{a}{c}$; cos. $B = \frac{a}{c}$.
a, A	b, c, B	$b = a$. cot. A; $c = \frac{a}{\text{sin. A}}$; $B = 90° - A$.
b, A	a, c, B	$a = b$. tan. A; $c = \frac{b}{\text{cos. A}}$; $B = 90° - A$.
c, A	a, b, B	$a = c$. sin. A; $b = c$ cos. A; $B = 90° - A$.

Fig. 402.



(12) Oblique-angled Triangles. Let ABC be any oblique-angled triangle, the angles and sides being noted as in the figure. Then any three of its six parts being given, and one of them being a side, the other parts can be obtained by one of the following methods, which are founded on these three theorems.

THEOREM I.—*In every plane triangle, the sines of the angles are to each other as the opposite sides.*

THEOREM II.—*In every plane triangle, the sum of two sides is to their difference as the tangent of half the sum of the angles opposite those sides is to the tangent of half their difference.*

THEOREM III.—*In every plane triangle, the cosine of any angle is equal to a fraction whose numerator is the sum of the squares of the sides adjacent to the angle,* minus *the square of the side opposite to the angle, and whose denominator is twice the product of the sides adjacent to the angle.*

All the cases for solution which can occur, may be reduced to four.

CASE 1.—*Given a side and two angles.* The third angle is obtained by subtracting the sum of the two given angles from 180°. Then either unknown side can be obtained by Theorem I.

Calling the given side a, we have $b = a \cdot \frac{\text{sin. B}}{\text{sin. A}}$; and $c = a \frac{\text{sin. C}}{\text{sin. A}}$.

CASE 2.—*Given two sides and an angle opposite one of them.* The angle opposite the other given side is found by Theorem I. The third angle is obtained by subtracting the sum of the other two from 180°. The remaining side is then obtained by Theorem I.

Calling the given sides a and b, and the given angle A, we have $\sin. B = \sin. A . \frac{b}{a}$.

Since an angle and its supplement have the same sine, the result is ambiguous; for the angle B may have either of the two supplementary values indicated by the sine, if $b > a$, and A is an acute angle.

$$C = 180° - (A + B). \qquad c = \sin. C \frac{a}{\sin. A}.$$

CASE 3.—*Given two sides and their included angle.* Applying Theorem II. (obtaining the sum of the angles opposite the given sides by subtracting the given included angle from 180°), we obtain the difference of the unknown angles. Adding this to their sum we obtain the greater angle, and subtracting it from their sum we get the less. Then Theorem I. will give the remaining side.

Calling the given sides a and b, and the included angle C, we have $A + B = 180° - C$. Then

$$\tan. \tfrac{1}{2}(A - B) = \tan. \tfrac{1}{2}(A + B) . \frac{a - b}{a + b}.$$

$$\tfrac{1}{2}(A + B) + \tfrac{1}{2}(A - B) = A. \qquad \tfrac{1}{2}(A + B) - \tfrac{1}{2}(A - B) = B. \qquad c = a \frac{\sin. C}{\sin. A}.$$

In the first equation $\cot. \frac{1}{2} C$ may be used in the place of $\tan. \frac{1}{2}(A + B)$.

CASE 4.—*Given the three sides.* Let s represent half the sum of the three sides $= \frac{1}{2}(a + b + c)$. Then any angle, as A, may be obtained from either of the following formulas, founded on Theorem III.:

$$\sin. \tfrac{1}{2} A = \sqrt{\left[\frac{(s - b)(s - c)}{bc}\right]}.$$

$$\cos. \tfrac{1}{2} A = \sqrt{\left[\frac{s(s - a)}{bc}\right]}.$$

$$\tan. \tfrac{1}{2} A = \sqrt{\left[\frac{(s - b)(s - c)}{s(s - a)}\right]}.$$

$$\sin. A = 2\sqrt{\left[\frac{s(s - a)(s - b)(s - c)}{bc}\right]}.$$

$$\cos. A = \frac{b^2 + c^2 - a^2}{2bc}.$$

The first formula should be used when $A < 90°$, and the second when $A > 90°$. The third should not be used when A is nearly 180°; nor the fourth when A is nearly 90°; nor the fifth when A is very small. The third is the most convenient when all the angles are required.

APPENDIX B.

DEMONSTRATIONS OF PROBLEMS, ETC.

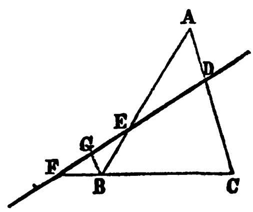
Many of the problems, &c., contained in the preceding pages, require Demonstrations. These will be given here, and will be designated by the same numbers as those of the Articles to which they refer.

As many of these Demonstrations involve the beautiful Theory of Transversals, &c., which has not yet found its way into our Geometries, a condensed summary of its principal Theorems will first be given.

TRANSVERSALS.

Theorem I.—*If a straight line be drawn so as to cut any two sides of a triangle, and the third side prolonged, thus dividing them into six parts (the prolonged side and its prolongation being two of the parts), then will the product of any three of those parts, whose extremities are not contiguous, equal the product of the other three parts.*

Fig. 403.



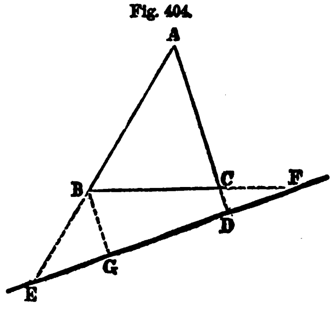
That is, in Fig. 403, ABC being the triangle, and DF the Transversal, $BE \times AD \times CF = EA \times DC \times BF$.

To prove this, from B draw BG, parallel to CA. From the similar triangles BEG and AED, we have $BG : BE :: AD : AE$. From the similar triangles BFG and CFD, we have $CD : CF :: BG : BF$. Multiplying these proportions together, we have $BG \times CD : BE \times CF :: AD \times BG : AE \times BF$. Multiplying extremes and means, and suppressing the common factor BG, we have $BE \times AD \times CF = EA \times DC \times BF$.

These six parts are sometimes said to be *in involution.*

If the Transversal passes entirely outside of the triangle, and cuts the prolongations of all three sides, as in Fig. 404, the theorem still holds good. The same demonstration applies without any change.*

Fig. 404.

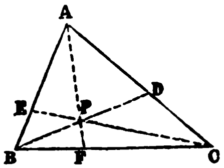


Theorem II.—Conversely: *If three points be taken on two sides of a triangle, and on the third side prolonged, or on the prolongations of the three sides, dividing them into six parts, such that the product of three non-consecutive parts equals the product of the other three parts; then will these three points lie in the same straight line.*

This Theorem is proved by a *Reductio ad absurdum.*

* This Theorem may be extended to polygons.

Fig. 405.

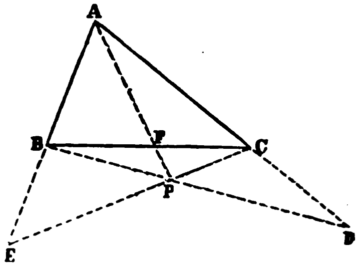


THEOREM III.—*If from the summits of a triangle, lines be drawn, to a point situated either within or without the triangle, and prolonged to meet the sides of the triangle, or their prolongations, thus dividing them into six parts; then will the product of any three non-consecutive parts be equal to the product of the other three parts.*

That is, in Fig. 405, or Fig. 406,

$$AE \times BF \times CD = EB \times FC \times DA.$$

Fig. 406.



For, the triangle ABF being cut by the transversal EC, gives the relation (Theorem I.),

$$AE \times BC \times FP = EB \times FC \times PA.$$

The triangle ACF, being cut by the transversal DB, gives

$$DC \times FB \times PA = AD \times CB \times FP.$$

Multiplying these equations together, and suppressing the common factors PA, CB, and FP, we have $AE \times BF \times CD = EB \times FC \times DA$.

THEOREM IV.—Conversely: *If three points are situated on the three sides of a triangle, or on their prolongations (either one, or three, of these points being on the sides), so that they divide these lines in such a way that the product of any three non-consecutive parts equals the product of the other three parts, then will lines drawn from these points to the opposite angles meet in the same point.*

This Theorem can be demonstrated by a *Reductio ad absurdum.*

COROLLARIES OF THE PRECEDING THEOREMS.

COR. 1.—*The* MEDIANS *of a triangle* (i. e., the lines drawn from its summits to the middles of the opposite sides) *meet in the same point.*

For, supposing, in Fig. 405, the points D, E, and F to be the middles of the sides, the products of the non-consecutive parts will be equal, i. e., $AE \times BF \times CD = DA \times EB \times FC$; since $AE = EB$, $BF = FC$, $CD = DA$. Then Theorem IV. applies.

COR. 2.—*The* BISSECTRICES *of a triangle* (i. e., the lines bisecting its angles) *meet in the same point.*

For, in Fig. 405, supposing the lines AF, BD, CE to be Bissectrices, we have (Legendre IV. 17):

$$\left.\begin{array}{l} BF : FC :: AB : AC, \\ CD : DA :: BC : BA, \\ AE : EB :: CA : CB, \end{array}\right\} \text{whence} \left\{\begin{array}{l} BF \times AC = FC \times AB, \\ CD \times BA = DA \times BC, \\ AE \times CB = EB \times CA. \end{array}\right.$$

Multiplying these equations together, and omitting the common factors, we have $BF \times CD \times AE = FC \times DA \times EB$. Then Theorem IV. applies.

COR. 3.—*The* ALTITUDES *of a triangle* (i. e., the lines drawn from its summits perpendicular to the opposite sides) *meet in the same point.*

For, in Fig. 405, supposing the lines AF, BD, and CE, to be Altitudes, we have three pairs of similar triangles, BCD and FCA, CAE and DAB, ABF and EBC, by comparing which we obtain relations from which it is easy to deduce $BF \times CD \times AE = EB \times FC \times DA$; and then Theorem IV. again applies.

COR. 4.—*If, in Fig.* 405, *or Fig.* 406, *the point* F *be taken in the middle of* BC, *then will the line* ED *be parallel to* BC.

For, since $BF = FC$, the equation of Theorem III. reduces to $AE \times CD = EB \times DA$; whence $AE : EB :: AD : DC$; consequently ED is parallel to BC.

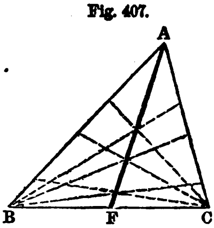
COR. 5.—Conversely: *If* ED *be parallel to* BC, *then is* $BF = FC$.

For, since $AE : EB :: AD : DC$, we have $AE \times DC = EB \times AD$; whence, in the equation of Theorem III., we must have $BF = FC$.

COR. 6.—From the preceding Corollary, we derive the following:

If two sides of a triangle are divided proportionally, starting from the same summit, as A, *and lines are drawn from the extremities of the third side to the points of division, the intersections of the corresponding lines will all lie in the same straight line joining the summit* A, *and the middle of the base.*

Fig. 407.



COR. 7.—A particular case of the preceding corollary is this:

In any trapezoid, the straight line which joins the intersection of the diagonals and the point of meeting of the non-parallel sides produced, passes through the middle of the two parallel bases.

COR. 8.—*If the three lines drawn through the corresponding summits of two triangles cut each other in the same point, then the three points in which the corresponding sides, produced if necessary, will meet, are situated in the same straight line.*

This corollary may be otherwise enunciated, thus:

If two triangles have their summits situated, two and two, on three lines which meet in the same point, then, &c.

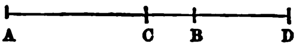
This is proved by obtaining by Theorem I. three equations, which, being multiplied together, and the six common factors cancelled, give an equation to which Theorem II. applies.

Triangles thus situated are called *homologic;* the common point of meeting of the lines passing through their summits is called the *centre of homology;* and the line on which the sides meet, the *axis of homology.*

HARMONIC DIVISION.

DEFINITIONS.—A straight line, AB, is said to be *harmonically divided* at the points C and D, when these points determine two additive segments, AC, BC, and two subtractive segments, AD, BD, proportional to one another; so that AC : BC :: AD : BD. It will be seen that AC must be more than BC, since AD is more than BD.*

Fig. 408.



This relation may be otherwise expressed, thus: the product of the whole line by the middle part equals the product of the extreme parts.

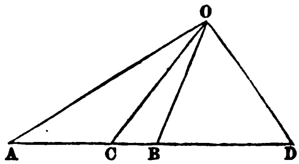
Reciprocally, the line DC is harmonically divided at the points B and A; since the preceding proportion may be written DB : CB :: DA : CA.

The four points, A, B, C, D, are called *harmonics*. The points C and D are called *harmonic conjugates*. So are the points A and B.

When a straight line, as AB, is divided harmonically, its half is a mean proportional between the distance from the middle of the line to the two points, C and D, which divide it harmonically.

If, from any point, O, lines be drawn so as to divide a line harmonically, these lines are called an *harmonic pencil*. The four lines which compose it, OA, OC, OB, OD, in the figure, are called its *radii*, and the pairs which pass through the conjugate points are called *conjugate radii*.

Fig. 409.



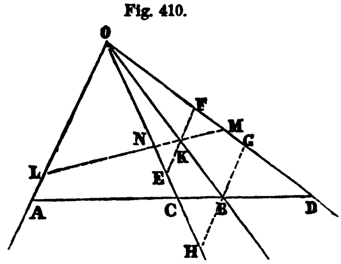
THEOREM V.—*In any harmonic pencil, a line drawn parallel to any one of the radii, is divided by the three other radii into two equal parts.*

Let EF be the line, drawn parallel to OA. Through B draw GH, also parallel to OA. We have,

GB : OA :: BD : AD; and
BH : OA :: BC : AC.

But, by hypothesis, AC : BC :: AD : BD. Hence, the first two proportions reduce to GB = BH; and consequently, EK = KF.

Fig. 410.



The *Reciprocal* is also true; i. e.,

If four lines radiating from a point are such that a line drawn parallel to one of them is divided into two equal parts by the other three, the four lines form an harmonic pencil.

* Three numbers, m, n, p, arranged in decreasing order of size, form an *harmonic proportion*, when the difference of the first and the second is to the difference of the second and the third, as the first is to the third. Such are the numbers 6, 4, and 3; or 6, 3, and 2; or 15, 12, and 10; &c. So, in Fig. 408, are the lines AD, AB, and AC, which thus give BD : CB :: AD : AC; or AC : CB :: AD : BD. The series of fractions, $\frac{1}{1}$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c., is called an *harmonic progression*, because any consecutive three of its terms form an *harmonic proportion*.

THEOREM VI.—*If any transversal to an harmonic pencil be drawn, it will be divided harmonically.*

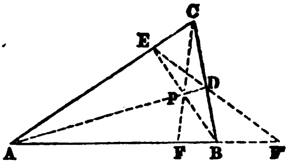
Let LM be the transversal. Through K, where LM intersects OB, draw EF parallel to OA. It is bisected at K by the preceding theorem; and the similar triangles, FMK and LMO, EKN and LNO, give the proportions

LM : KM : : OL : FK, and LN : NK : : OL : EK; whence,
since FK = EK, we have LN : NK : : LM : KM.

COROLLARY.—*The two sides of any angle, together with the bissectrices of the angle and of its supplement, form an harmonic pencil.*

THEOREM VII.—*If, from the summits of any triangle,* ABC, *through any point,* P, *there be drawn the transversals* AD, BE, CF, *and the transversal* ED *be drawn to meet* AB *prolonged, in* F′, *the points* F *and* F′ *will divide the base* AB *harmonically.*

Fig. 411.



This may be otherwise expressed, thus:

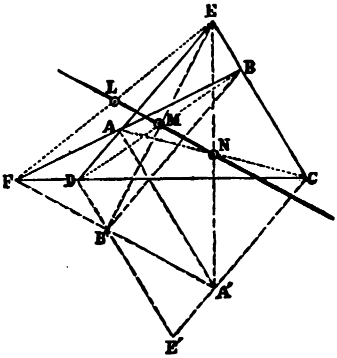
The line, CP, *which joins the intersection of the diagonals of any quadrilateral,* ABDE, *with the point of meeting,* C, *of two opposite sides prolonged, cuts the side* AB *in a point* F, *which is the harmonic conjugate of the point of meeting,* F′, *of the other two sides,* ED *and* AB, *prolonged.*

For, by Theorem I., AF′ × BD × CE = F′B × DC × EA; and
by Theorem III., AF × BD × CE = FB × DC × EA;
whence AF : FB : : AF′ : F′B.

THE COMPLETE QUADRILATERAL.

A *Complete Quadrilateral* is formed by drawing any four straight lines, so that each of them shall cut each of the other three, so as to give six different points of intersection. It is so called because in the figure thus formed are found three quadrilaterals; viz., in Fig. 412, ABCD, a common *convex* quadrilateral; EAFC, a *uni-concave* quadrilateral; and EBAFD, a *bi-concave* quadrilateral, composed of two opposite triangles.

Fig. 412.



The complete quadrilateral, AEBCDF, has three diagonals; viz., two interior, AC, BD; and one exterior, EF.

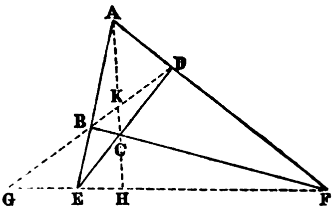
THEOREM VIII.—*In every* COMPLETE QUADRILATERAL *the middle points of its three diagonals lie in the same straight line.*

AEBCDF is the quadrilateral, and LMN the middle points of its three diagonals. From A and D draw parallels to BC, and from B and C draw parallels to

AD. The triangle EDC being cut by the transversal BF, we have (Theorem I.), DF × CB × EA = CF × EB × DA. From the equality of parallels between parallels, we have CB = E'B', EA = CA', EB = DB', DA = EA'. Hence, the above equation becomes DF × E'B' × CA' = CF × DB' × E'A'; therefore, by Theorem II., the points, F, B', A', lie in the same straight line. Now, since the diagonals of the parallelogram ECA'A bisect each other at N, and those of the parallelogram EBB'D at N, we have EN : NA' : : EM : MB'. Then MN is parallel to FA'; and we have EN : NA' : : EL : LF, or EL = LF, so that L is the middle of EF, and the same straight line passes through L, M, and N.

THEOREM IX.—*In every complete quadrilateral each of the three diagonals is divided harmonically by the two others.*

Fig. 413.



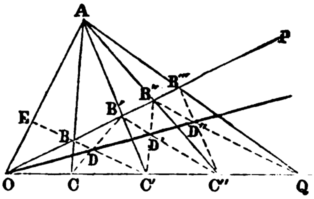
CEBADF is the complete quadrilateral. The diagonal EF is divided harmonically at G and H by DB and AC produced; since AH, DE, and FB are three transversals drawn from the summits of the triangle AEF through the same point C; and therefore, by Theorem VII., DBG and ACH divide EF harmonically.

So too, in the triangle ABD, CB, CA, CD, are the three transversals passing through C; and G and K therefore divide the diagonal BD harmonically.

So too, in the triangle, ABC, DA, DB, DC are the transversals, and H and K the points which divide the diagonal AC harmonically.

THEOREM X.—*If from a point,* A, *any number of lines be drawn, cutting the sides of an angle* POQ, *the intersections of the diagonals of the quadrilaterals thus formed will all lie in the same straight line passing through the summit of the angle.*

Fig. 414.



By the preceding Theorem, the diagonal BC' of the complete quadrilateral, BAB'C'CO, is divided harmonically at D and E. Hence, OA, OP, OD, and OQ, form an harmonic pencil. So do OA, OP, OD', and OQ. Therefore, the lines OD, OD' coincide. So for the other intersections.

If the point A moves on OA, the line OD is not displaced. If, on the contrary, OA is displaced, OD turns around the point O. Hence, the point A is said to be a *pole* with respect to the line OD, which is itself called the *polar* of the point A. Similarly, D is a pole of OA, which is the polar of D. OD is likewise the polar of any other point on the line OA; and this property is necessarily reciprocal for the two conjugate radii OA, OD, with respect to the lines OP, OQ, which are also conjugate radii. Hence; In every harmonic pencil, each of the radii is a *polar* with respect to each point of its conjugate; and each point of this latter line is a *pole* with respect to the former.

DEMONSTRATIONS.*

PART II.; CHAPTER V.

(140), (141) The equality of the triangles formed in these methods proves their correctness.

(143), (144) These methods depend on the principle of the square of the hypothenuse.

(145) CAD is an angle inscribed in a semicircle.

(146) Let fall a perpendicular from B to AC, meeting it at a point E, not marked in Fig. 91. Then (Legendre, IV. 12),

$$AB^2 = AC^2 + BC^2 - 2\ AC \,.\, CE;\ \text{whence}\ CE = \frac{AC^2 + BC^2 - AB^2}{2\ AC}.$$

When $AC = AB$, this becomes $CE = \dfrac{BC^2}{2\ AC}$. The similar triangles, BCE and DCA, give $EC : CB :: AC : CD$; whence

$$CD = \frac{CB \times AC}{CE} = CB \times AC \div \frac{BC^2}{2\ AC} = \frac{2\ AC^2}{BC}.$$

(147) Mark a point, G, in the middle of DF, and join GA. The triangle AGD will then be isosceles, since it is equal to the isosceles triangle ABC, having two sides and the included angle equal. Then $AG = GD = AB = GF$. The triangle AGF is then also isosceles. Now the angle $FAG = \frac{1}{2}\ AGD$; and $GAD = \frac{1}{2}\ FGA$. Therefore

$$FAG + GAD = FAD = \tfrac{1}{2}\,(AGD + FGA) = \tfrac{1}{2}\,(180^\circ) = 90^\circ.$$

(149) See Part VII., Art. (403).

(150) The proof follows from the equal triangles formed.

(151) The proof is found in the first half of the proof of Art. (146).

(153) ACP is an angle inscribed in a semicircle.

(154) Draw from C a perpendicular to the given line, meeting it at a point E. As in the proof of Art. (146), changing the letters suitably, we have $AE = \dfrac{AC^2}{2\ AB}$. The similar triangles AEC and ADP give

$$AC : AE :: AP : AD = \frac{AP}{AC} \times AE = \frac{AP}{AC} \times \frac{AC^2}{2\ AB} = \frac{AP \times AC}{2\ AB}.$$

(155) Similar triangles prove this.

(156) The equal triangles which are formed give $BP = CF$. Hence FP is parallel to BC, and consequently perpendicular to the given line DG.

(157) The proof of this is found in the "Theory of Transversals," corollary 3.

(158) The proof of this is the same as the last.

(161) The lines are parallel because of the equal angles formed.

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* Additional lines to the figures in the text will sometimes be employed. The student should draw them on the figures, as directed.

(**162**) The equal triangles give equal angles, and therefore parallels.

(**163**) AB is parallel to PF, since it cuts the sides of the triangle proportionally.

(**164**) The proof is found in corollary 4 of "Transversals."

(**165**) From the similar triangles, CAD and CEP, we have CE : CD : : CP : CA. From the similar triangles, CEF and CBD, we have CE : CD : : CF : CB. These two proportions give the following; CP : CA : : CF : CB. Therefore PF is parallel to AB.

(**166**) Draw PE. The similar triangles PCE and ACD give PE : CE : : AD : CD. The similar triangles CEF and CDB give EF : CE : : DB : CD. These proportions produce PE : EF : : AD : DB. Hence PEF is similar to ADB, and PF is parallel to AB.

(**173**) The equality of the symmetrical triangles which are formed, proves this method.

(**174**) ABP is a transversal to the triangle CDE. Then, by Theorem I. of "Transversals," $CA \times EB \times DP = AE \times BD \times CP$; whence we have

$$CP : DP :: CA \times EB : AE \times BD.$$

By "division," $CP - DP : DP :: CA \times EB - AE \times BD : AE \times BD$.

Hence, since $CP - DP = CD$, we obtain $DP = \dfrac{DC \times AE \times BD}{CA \times EB - AE \times BD}$.

The other formulas are simplified by the common factors obtained by making $AE = AC$, or $BE = BD$.

(**175**) By Theorem VII. "Harmonic Division," in the quadrilateral ABED, the line CF cuts DE in a point, L, which is the harmonic conjugate of the point at which AB and DE, produced, would meet. So too, in the quadrilateral DEHK, this same line, CG, produced, cuts DE in a point, L, which is the harmonic conjugate of the point at which DE and KH, produced, would meet. Consequently, AB, DE, and KH must meet in the same point. *Otherwise;* this problem may be regarded as the converse of Theorem X. of "Transversals," BCA being the angle, and P the point from which the radiating lines are drawn.

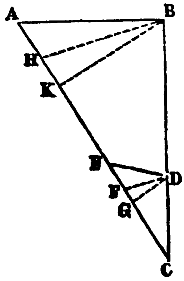
(**176**) EGCFDH is the "Complete Quadrilateral." Its three diagonals are FE, DC, and HG; and their middle points A, B, and P lie in the same straight line, by our Theorem VIII.

(**182**) This instrument depends on the optical principle of the equality of the angles of incidence and reflection.

(**184**) The first method given, Fig. 120, is another application of the Theory of Transversals. The second method in the article is proved by supposing the figure to be constructed, in which case we should have a triangle QZR, whose base, QR, and a parallel to it, BD, would be cut proportionally by the required line PSZ; so that $QR : BD :: QP : BS = \dfrac{BD \times QP}{QR}$.

(**189**) By "Transversals," Theorem I., we obtain, regarding CD as the transversal of the triangle ABE, $CB \times AF \times ED = AC \times FE \times DB$; and since $ED = DB$, this becomes $CB \times AF = AC \times FE$; whence the proportion $CB : AC :: FE : AF$. By "division," we have $CB - AC : AC :: FE - AF : AF$. Observing that $CB - AC = AB$, we obtain $AB = \dfrac{AC}{AF} \cdot (FE - AF)$.

Fig. 124, bis.



(**190**) Take $CH = CB$; and from B let fall a perpendicular, BK, to AC. Then, in the triangle CBH, we have (Legendre IV. 12),

$$HK = \frac{CH^2 + BH^2 - BC^2}{2\,CH} = \frac{BH^2}{2\,BC}; \qquad [1]$$

since $CH = BC$.

In the triangle ABH, we have (Leg. IV. 13)

$$AB^2 = AH^2 + BH^2 + 2\,AH \,.\, HK.$$

Substituting for HK, its value from [1], we get

$$AB^2 = AH^2 + BH^2\left(1 + \frac{AH}{BC}\right).$$

But $\qquad AH = AC - CH = AC - BC$

$$\therefore\ AB^2 = AH^2 + BH^2\left(1 + \frac{AC - BC}{BC}\right) = AH^2 + BH^2 \,.\, \frac{AC}{BC}. \qquad [2]$$

In the above expression for AB, BH is unknown. To find it, proceed thus. Take $CF = CD$. Then DF is parallel to BH; and we have $CD : CB :: DF : BH$; whence

$$BH^2 = DF^2 \,.\, \frac{CB^2}{CD^2}. \qquad [3]$$

In this equation DF^2 is unknown; but by proceeding as at the beginning of this investigation, we get an equation analogous to [2], giving $ED^2 = EF^2 + DF^2 \,.\, \frac{CE}{CD}$; whence $DF^2 = (DE^2 - EF^2) \,.\, \frac{CD}{CE}$.

Substituting this value of DF^2 in [3], we have

$$BH^2 = (DE^2 - EF^2)\frac{CB^2}{CD \times CE}.$$

Substituting this value of BH^2 in [2], we have

$$AB^2 = AH^2 + (DE^2 - EF^2) \,.\, \frac{AC \times BC}{CD \times CE} = (AC - BC)^2 + [DE^2 - (CE - CD)^2] \times \frac{AC \times BC}{CD \times CE}.$$

(**191**) Since BCD is a right angle, AC is a mean proportional between AB and AD.

(**192**) The proof follows from the similar triangles constructed.

(**193**) The similar triangles give $DE : AC :: DB : AB$; whence, by "division," $DE - AC : AC :: DB - AB : AB$; whence, since $DB - AB = AD$, we have $AB = \frac{AC \times AD}{DE - AC}$.

(**194**) From the similar triangles, we have $DE : CA :: EB : AB$; whence $DE - CA : CA :: EB - AB : AB$; whence, since $EB - AB = AE$, we get $AB = \frac{AC \times AE}{DE - AC}$.

(**195**) The triangles DEF and BAF, similar because of the parallelogram which is constructed, give $FE : ED :: AF : AB = \frac{ED \times AF}{FE} = \frac{AC \times AF}{FE}$.

The triangles DEF and BCD give similarly $FE : ED :: DC : CB = \frac{AC \times DC}{FE}$.

(196) The equality of the triangles formed proves this problem.

(197) The proof of this problem also depends on the equality of the triangles constructed. The details of the proof require attention.

(198) EB is the transversal of the triangle ACD. Consequently, $CB \times AF \times DE = AB \times FD \times CE$; or, since $CB = AB + AC$, $(AB + AC) \times AF \times DE = AB \times FD \times CE$; whence $AB = \frac{AC \times AF \times DE}{FD \times CE - AF \times DE}$.

Taking E, in the middle of CD, $CE = DE$, and those lines are cancelled. Taking F in the middle of AD, $AF = FD$, and those lines are cancelled.

(199) The line BE is harmonically divided at the points H and A, from Theorem IX., ECFBGD being a "Complete Quadrilateral." Consequently, $AE : EH :: AB : HB$. Hence, by "division," $AE - EH : AE :: AB - HB : AB$. We therefore have, since $AB - HB = AH$, $AB = \frac{AE \times AH}{AE - EH}$.

(200) For the same reasons as in the last article, CF is harmonically divided at H and D; and we have $CH : HF :: CD : DF$; whence $CH - HF : CH :: CD - DF : CD$. Hence, since $CD - DF = CF$, $CD = \frac{CH \times CF}{CH - HF}$.

The other two expressions come from writing CF as $CH + HF$, and HF as $CF - CH$.

(201) The equality of the triangles formed proves the equality of the corresponding sides KD and DE, &c.

(202) The similar triangles (made so by the measurement of CE) give $CD : DE :: CA : AB = \frac{AC \times DE}{CD}$.

(203) The similar triangles (made so by the parallel) give $CE : EA :: CD : AB = \frac{CD \times EA}{CE} = \frac{CD \times (AC - CE)}{CE}$.

(204) The similar triangles DFH and BCD give $HF : FD :: DC : BC = \frac{DF \times CD}{FH}$.

The similar triangles FGH and ABC give $FG : GH :: BC : AB = BC\frac{GH}{FG}$.

Substituting for BC, its above value, we have $AB = \frac{DF \times CD \times GH}{FH \times FG}$.

When $CD = CE$, $DF = CD$, whence the second formula.

(205) The equality of the symmetrical triangles which are formed, proves the equality of A′B′ to AB.

(206) The proof of this is similar to the preceding.

(207) Because the two triangles ABC and ADE have a common angle at A, we have $ADE : ABC :: AD \times AE : AB \times AC$; whence the expression for ABC.

(208) From B let fall a perpendicular to AC, meeting it at a point B′. Call this perpendicular $BB' = p$. From D let fall a perpendicular to AC, meeting it at a point D′. Call this perpendicular $DD' = q$.

The quadrilateral $ABCD = AC \times \frac{1}{2}(p+q)$.

The triangle $BCE = CE \times \frac{1}{2}p$; whence $p = \frac{2 . BCE}{CE}$.

The similar triangles EDD′ and BEB′ give $p : q :: BE : DE$, whence $q = p\frac{DE}{BE} = \frac{2 . BCE \times DE}{CE \times BE}$.

Then $\frac{1}{2}(p+q) = \frac{BCE}{CE} + \frac{BCE \times DE}{CE \times BE} = BCE \times \frac{BE + DE}{CE \times BE} = BCE \times \frac{BD}{CE \times BE}$.

Lastly, $ABCD = AC \times BCE \times \frac{BD}{CE \times BE} = BCE \times \frac{AC \times BD}{CE \times BE}$.

DEMONSTRATIONS FOR PART V.

(**382**) Let $B =$ the measured inclined length, $b =$ this length reduced to a horizontal plane, and $A =$ the angle which the measured base makes with the horizon. Then $b = B . \cos. A$; and the excess of B over b, i. e., $B - b = B\,(1 - \cos. A)$. Since $1 - \cos. A = 2\,(\sin. \frac{1}{2} A)^2$ [Trigonometry, Art. (9)], we have $B - b = 2\,B\,(\sin. \frac{1}{2} A)^2$. Substituting for $\sin. \frac{1}{2} . A$, its approximate equivalent, $\frac{1}{2} A \times \sin. 1'$ [Trigonometry, Art. (5)], we obtain $B - b = 2\,B\,(\frac{1}{2}\,A \times \sin. 1')^2 = \frac{1}{2}\,(\sin. 1')^2 . A^2 . B, = 0.000000004231\ A^2\,B.$ By logarithms, $\log. (B - b) = 2.626422 + 2 \log. A + \log. B$. The greater precision of this calculation than that of $b = B . \cos. A$, arises from the slowness with which the cosines of very small angles increase or decrease in length.

(**386**) The exterior angle $LER = LCR + CLD$. Also, $LER = LDR + CRD$.
$\therefore LCR + CLD = LDR + CRD$, and $LCR = LDR + CRD - CLD$.

From the triangle CRD we get $\sin. CRD = \sin. CDR \times \frac{CD}{CR}$.

From the triangle CLD we get $\sin. CLD = \sin. LCD \times \frac{CD}{CL}$.

As the angles CRD and CLD are very small, these values of the sines may be called the values of the arcs which measure the angles, and we shall have

$$LCR = LDR + \sin. CDR \times \frac{CD}{CR} - \sin. LCD \times \frac{CD}{CL}.$$

The last two terms are expressed in parts of radius, and to have them in seconds, they must be divided by sin. 1″ [Trigonometry, Art. (5), Note], which gives the formula in the text. Otherwise, the correction being in parts of radius, may be brought into seconds by multiplying it by the length of the radius in seconds; i. e., $\frac{180° \times 60 \times 60}{3.14159, \&c.} = 206264''.80625$ [Trigonometry, Art. (2)].

(**391**) The triangles AOB, BOC, COD, &c., give the following proportions [Trigonometry, Art. (12), Theorem I.]; AO : OB : : sin. (2) : sin. (1); OB : OC : : sin. (4) : sin. (3); OC : OD : : sin. (6) : sin. (5); and so on around the polygon. Multiplying together the corresponding terms of all the proportions, the sides will all be cancelled, and there will result

$$1 : 1 :: \sin. (2) \times \sin. (4) \times \sin. (6) \times \sin. (8) \times \sin. (10) \times \sin. (12) \times \sin. (14) : \sin. (1) \times \sin. (3) \times \sin. (5) \times \sin. (7) \times \sin. (9) \times \sin. (11) \times \sin. (13).$$

Hence the equality of the last two terms of the proportion.

DEMONSTRATION FOR PART VI.

(399) In the triangle ABS, we have

$$\sin.\ \mathrm{ASB} : \sin.\ \mathrm{BAS} :: \mathrm{AB} : \mathrm{SB} = \frac{\mathrm{AB} . \sin.\ \mathrm{BAS}}{\sin.\ \mathrm{ASB}} = \frac{c . \sin.\ \mathrm{U}}{\sin.\ \mathrm{S}}. \qquad [1]$$

In the triangle CBS, we have

$$\sin.\ \mathrm{BSC} : \sin.\ \mathrm{BCS} :: \mathrm{BC} : \mathrm{SB} = \frac{\mathrm{BC} . \sin.\ \mathrm{BCS}}{\sin.\ \mathrm{BSC}} = \frac{a . \sin.\ \mathrm{V}}{\sin.\ \mathrm{S}'}. \qquad [2]$$

Hence, $\dfrac{c . \sin.\ \mathrm{U}}{\sin.\ \mathrm{S}} = \dfrac{a . \sin.\ \mathrm{V}}{\sin.\ \mathrm{S}'}$; whence, $c . \sin.\ \mathrm{S}' . \sin.\ \mathrm{U} - a . \sin.\ \mathrm{S} . \sin.\ \mathrm{V} = 0$. [3]

In the quadrilateral ABCS, we have

$\mathrm{BCS} = 360° - \mathrm{ASB} - \mathrm{BSC} - \mathrm{ABC} - \mathrm{BAS}$; or $\mathrm{V} = 360° - \mathrm{S} - \mathrm{S}' - \mathrm{B} - \mathrm{U}$.

Let $\mathrm{T} = 360° - \mathrm{S} - \mathrm{S}' - \mathrm{B}$, and we have $\mathrm{V} = \mathrm{T} - \mathrm{U}$. [4]

Substituting this value of V, in equation [3], we get [Trig., Art. (8)],

$$c . \sin\ \mathrm{S}'\ \sin.\ \mathrm{U} - a . \sin.\ \mathrm{S}\ (\sin.\ \mathrm{T} . \cos.\ \mathrm{U} - \cos.\ \mathrm{T} . \sin.\ \mathrm{U}) = 0.$$

Dividing by sin. U, we get

$$c . \sin.\ \mathrm{S}' - a . \sin.\ \mathrm{S}\ \left(\sin.\ \mathrm{T} . \frac{\cos.\ \mathrm{U}}{\sin.\ \mathrm{U}} - \cos.\ \mathrm{T}\right) = 0.$$

Whence we have

$$\frac{\cos.\ \mathrm{U}}{\sin.\ \mathrm{U}} = \cot.\ \mathrm{U} = \frac{c . \sin.\ \mathrm{S}' + a . \sin.\ \mathrm{S} . \cos.\ \mathrm{T}}{a . \sin.\ \mathrm{S} . \sin.\ \mathrm{T}}.$$

Separating this expression into two parts, and cancelling, we get

$$\cot.\ \mathrm{U} = \frac{c . \sin.\ \mathrm{S}'}{a . \sin.\ \mathrm{S} . \sin.\ \mathrm{T}} + \frac{\cos.\ \mathrm{T}}{\sin.\ \mathrm{T}}.$$

Separating the second member into factors, we get

$$\cot.\ \mathrm{U} = \frac{\cos.\ \mathrm{T}}{\sin.\ \mathrm{T}} \left(\frac{c . \sin.\ \mathrm{S}'}{a . \sin.\ \mathrm{S} . \cos.\ \mathrm{T}} + 1\right); \text{ or}$$

$$\cot.\ \mathrm{U} = \cot.\ \mathrm{T} \left(\frac{c . \sin.\ \mathrm{S}'}{a . \sin.\ \mathrm{S} . \cos.\ \mathrm{T}} + 1\right).$$

Having found U, equation [4] gives V; and either [1] or [2] gives SB; and SA and SC are then given by the familiar "Sine proportion" [Trig., Art. (12)].

DEMONSTRATIONS FOR PART VII.

(403) If APC be a right angle, $\frac{\text{CP}}{\text{CA}} = \cos.\ \text{CAB}$ [Trigonometry, Art. (4)].

(405) $\text{AC} = \text{PC} . \tan.\ \text{APC}$; and $\text{CB} = \text{PC} . \tan.\ \text{BPC}$ [Trigonometry, Art. (4)]. Hence

$$\text{AC} : \text{CB} :: \tan.\ \text{APC} : \tan.\ \text{BPC}; \text{ and}$$

$$\text{AC} : \text{AC} + \text{CB} :: \tan.\ \text{APC} : \tan.\ \text{APC} + \tan.\ \text{BPC}.$$

Consequently, since $\text{AC} + \text{CB} = \text{AB}$, $\quad \text{AC} = \text{AB} \cdot \frac{\tan.\ \text{APC}}{\tan.\ \text{APC} + \tan.\ \text{BPC}}$.

(414) The equal and supplementary angles formed prove the operation.

(421) In Fig. 285, $\text{CA} : \text{EG} :: \text{AB} : \text{GB}$. By "division," $\text{CA} - \text{EG} : \text{EG} :: \text{AB} - \text{GB} : \text{GB}$. Hence, observing that $\text{AB} - \text{GB} = \text{AG}$, we shall have $\text{AG} = \frac{\text{GB}\,(\text{CA} - \text{EG})}{\text{EG}}$.

(423) Art. (12), Theorem III., [Trigonometry, Appendix A,] gives; $\cos.\ \text{C} = \frac{a^2 + b^2 - c^2}{2\,ab}$; or $c^2 = a^2 + b^2 - 2\,ab . \cos.\ \text{C}$. This becomes [Trig., Art. (6)], K being the supplement of C, $c^2 = a^2 + b^2 + 2\,ab . \cos.\ \text{K}$. The series [Trig., Art. (5)] for the length of a cosine, gives, taking only its first two terms, since K is very small, $\cos.\ \text{K} = 1 - \frac{1}{2}\,\text{K}^2$. Hence,

$$c^2 = a^2 + b^2 + 2\,ab - ab\,\text{K}^2 = (a+b)^2 - ab\,\text{K}^2 = (a+b)^2\left(1 - \frac{ab\,\text{K}^2}{(a+b)^2}\right);$$

whence,

$$c = (a+b)\sqrt{\left(1 - \frac{ab\,\text{K}^2}{(a+b)^2}\right)}.$$

Developing the quantity under the radical sign by the binomial theorem, and neglecting the terms after the second, it becomes

$$1 - \frac{1}{2} \cdot \frac{ab\,\text{K}^2}{(a+b)^2} +, \text{\&c.}$$

Substituting for K minutes, K . sin. 1′ [Trig., Art. (5)], and performing the multiplication by $a + b$, we obtain

$$c = a + b - \frac{ab\,\text{K}^2 . (\sin.\ 1')^2}{2\,(a+b)}. \quad \text{Now } (\sin.\ 1')^2 = 0.0000000423079;$$

whence the formula in the text, $c = a + b - 0.000000042308 \times \frac{ab\,\text{K}^2}{a+b}$.

(430) In the triangle ABC, designate the angles as A, B, C; and the sides opposite to them as a, b, c. Let $\text{CD} = d$. The triangle BCD gives [Trig., Art. (12), Theorem I], $a = d\,\frac{\sin.\ \text{BDC}}{\sin.\ \text{CBD}}$. The triangle ACD similarly gives $b = d \cdot \frac{\sin.\ \text{ADC}}{\sin.\ \text{CAD}}$.

In the triangle ABC, we have [Trig., Art. (12), Theorem II.],

$$\tan.\ \tfrac{1}{2}\,(\text{A} - \text{B}) : \cot.\ \tfrac{1}{2}\,\text{C} :: a - b : a + b;$$

whence

$$\tan.\ \tfrac{1}{2}\,(\text{A} - \text{B}) = \frac{a-b}{a+b} \cdot \cot.\ \tfrac{1}{2}\,\text{C}. \qquad [1]$$

Let K be an auxiliary angle, such that $b = a . \tan.\ \text{K}$; whence $\tan.\ \text{K} = \frac{b}{a}$.

Dividing the second member of equation [1], above and below, by a, and substituting tan. K for $\frac{b}{a}$, we get $\tan.\ \frac{1}{2}(A - B) = \frac{1 - \tan.\ K}{1 + \tan.\ K} \cdot \cot.\ \frac{1}{2} C.$

Since $\tan.\ 45° = 1$, we may substitute it for 1 in the preceding equation, and we get $\tan.\ \frac{1}{2}(A - B) = \frac{\tan.\ 45° - \tan.\ K}{\tan.\ 45° + \tan.\ K} \cdot \cot.\ \frac{1}{2} C.$

From the expression for the tangent of the difference of two arcs [Trig., Art. (8)], the preceding fraction reduces to tan. $(45° - K)$; and the equation becomes

$$\tan.\ \tfrac{1}{2}(A - B) = \tan.\ (45° - K) . \cot.\ \tfrac{1}{2} C. \qquad [2]$$

In the equation $\tan.\ K = \frac{b}{a}$, substitute the values of b and a from the formulas at the beginning of this investigation. This gives

$$\tan.\ K = d \cdot \frac{\sin.\ ADC}{\sin.\ CAD} \div d \cdot \frac{\sin.\ BDC}{\sin.\ CBD} = \frac{\sin.\ ADC . \sin.\ CBD}{\sin.\ CAD . \sin.\ BDC}.$$

$(A - B)$ is then obtained by equation [2]; $(A + B)$ is the supplement of C; therefore the angle A is known.

Then
$$c = AB = \frac{a . \sin.\ C}{\sin.\ A} = \frac{d . \sin.\ BDC . \sin.\ ACB}{\sin.\ CBD . \sin.\ CAB}.$$

The use of the auxiliary angle K, avoids the calculation of the sides a and b.

(**434**) In the figure on page 292, produce AD to some point F. The exterior angles, $EBC = A + P$; $ECD = A + Q$; $EDF = A + R$. The triangle ABE gives $\frac{BE}{a} = \frac{\sin.\ A}{\sin.\ P}$. The triangle ACE gives $\frac{CE}{a + x} = \frac{\sin.\ A}{\sin.\ Q}$. Dividing member by member, we get $\frac{BE}{CE} = \frac{a . \sin.\ Q}{(a + x)\ \sin.\ P}$.

In the same way the triangles BED and CED give $\frac{BE}{b + x} = \frac{\sin.\ (A + R)}{\sin.\ (R - P)}$; and $\frac{CE}{b} = \frac{\sin.\ (A + R)}{\sin.\ (R - Q)}$. Whence as before, $\frac{BE}{CE} = \frac{(b + x)\ \sin.\ (R - Q)}{b . \sin.\ (R - P)}$.

Equating these two values of the same ratio, we get

$$\frac{a . \sin.\ Q}{(a + x)\ \sin.\ P} = \frac{(b + x)\ \sin.\ (R - Q)}{b . \sin.\ (R - P)}; \text{ and thence}$$

$$\frac{ab . \sin.\ Q . \sin.\ (R - P)}{\sin.\ P . \sin.\ (R - Q)} = (a + x)(b + x) = ab + (a + b)x + x^2.$$

To solve this equation of the 2d degree, with reference to x, make

$$\tan.^2 K = \frac{4\ ab}{(a - b)^2} \cdot \frac{\sin.\ Q\ (\sin.\ R - P)}{\sin.\ P\ (\sin.\ R - Q)}.$$

Then the first member of the preceding equation $= \frac{1}{4} \cdot (a - b)^2 \times \tan.^2 K$; and we get
$$x^2 + (a + b)x = \tfrac{1}{4}(a - b)^2 \cdot \tan.^2 K - ab,$$
and
$$x = -\tfrac{1}{2}(a + b) \pm \sqrt{[\tfrac{1}{4}(a - b)^2 \cdot \tan.^2 K - ab + \tfrac{1}{4}(a + b)^2]}$$
$$= -\tfrac{1}{2}(a + b) \pm \sqrt{[\tfrac{1}{4}(a - b)^2 \cdot \tan.^2 K + \tfrac{1}{4}(a - b)^2]}$$
$$= -\tfrac{1}{2}(a + b) \pm \tfrac{1}{2}(a - b)\sqrt{(\tan.^2 K + 1)}.$$

Or, since $\sqrt{(\tan.^2 K + 1)} = \text{secant } K = \frac{1}{\cos.\ K}$, we have $x = -\frac{a + b}{2} \pm \frac{a - b}{2 . \cos.\ K}$.

DEMONSTRATIONS FOR PART XI.

(493) The content being given, and the length to be n times the breadth; Breadth $\times$ n times breadth $=$ content; whence, Breadth $= \sqrt{\left(\frac{\text{content}}{n}\right)}$.

Given the content $= c$, and the difference of the length and breadth $= d$; to find the length l, and the breadth b. We have $l \times b = c$; and $l - b = d$. From these two equations we get $l = \frac{1}{2} d + \frac{1}{2} \sqrt{(d^2 + 4c)}$.

Given the content $= c$, and the sum of the length and breadth $= s$; to find l and b. We have $l \times b = c$; and $l + b = s$; whence we get $l = \frac{1}{2} s + \frac{1}{2} \sqrt{(s^2 - 4c)}$.

(494) The first rule is a consequence of the area of a triangle being the product of its height by half its base.

To get the second rule, call the height h; then the base $= mh$; and the area $= \frac{1}{2} h \times mh$; whence $h = \sqrt{\left(\frac{2 \times \text{area}}{m}\right)}$.

For the equilateral triangle, calling its side e, the formula for the area of a triangle $\sqrt{[(\frac{1}{2} s)(\frac{1}{2} s - a)(\frac{1}{2} s - b)(\frac{1}{2} s - c)]}$ reduces to $\frac{1}{4} e^2 \sqrt{3}$. Hence $e = 2\sqrt{\left(\frac{\text{area}}{\sqrt{3}}\right)}$ $= 1.5197 \sqrt{\text{area}}$.

(495) By Art. (65), Note, $\frac{1}{2}$. AB $\times$ BC $\times$ sin. B $=$ content of ABC; whence, $\text{BC} = \frac{2 \times \text{ABC}}{\text{AB} \cdot \sin. \text{B}}$.

(496) The area of a circle $=$ radius$^2 \times \frac{22}{7}$; whence radius $= \sqrt{\left(\frac{7 \times \text{area}}{22}\right)}$.

(497) The blocks, including half of the streets and avenues around them, are $900 \times 260 = 234000$ square feet. This area gives 64 lots; then an acre, or 43560 feet, would give not quite 12 lots.

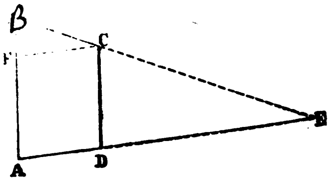
(502) The parallelogram ABDC being double the triangle ABC, the proof for Art. (495), slightly modified, applies here.

(504) Produce BC and AD to meet in E. By similar triangles,

$$\text{ABE} : \text{DCE} :: \text{AB}^2 : \text{DC}^2.$$

$$\text{ABE} - \text{DCE} : \text{ABE} :: \text{AB}^2 - \text{DC}^2 : \text{AB}^2.$$

Fig. 346, bis.



Now ABE $-$ DCE $=$ ABCD; also, by Art. (65), Note,

$$\text{ABE} = \text{AB}^2 . \frac{\sin. \text{A} . \sin. \text{B}}{2 . \sin. (\text{A}+\text{B})}.$$

The above proportion therefore becomes

$$\text{ABCD} : \text{AB}^2 . \frac{\sin. \text{A} . \sin. \text{B}}{2 . \sin. (\text{A} + \text{B})} :: \text{AB}^2 - \text{CD}^2 : \text{AB}^2.$$

Multiplying extremes and means, cancelling, transposing, and extracting the square root, we get $\text{CD} = \sqrt{\left[\text{AB}^2 - \frac{2 . \text{ABCD} . \sin. (\text{A} + \text{B})}{\sin. \text{A} . \sin. \text{B}}\right]}$.

When $A + B > 180°$, sin. $(A + B)$ is negative, and therefore the fraction in which it occurs becomes positive.

OF being drawn parallel to DA, we have

$$AD = FO = FB \,.\, \frac{\sin. B}{\sin. BCF} = FB \,.\, \frac{\sin. B}{\sin. (180° - A - B)} = (AB - CD) \frac{\sin. B}{\sin. (A + B)}$$

$$BC = (AB - CD) \frac{\sin. A}{\sin. (A + B)}.$$

(**505**) Since similar triangles are as the squares of their homologous sides, $BDE : BFG :: BD^2 : BF^2$; whence $BF = BD\sqrt{\left(\frac{BFG}{BDE}\right)}$.

(**506**) $BFG = \frac{1}{2} \,.\, BF \times FG = \frac{1}{2} \,.\, BF \times BF \,.\, \tan. B$;

whence, $$BF = \sqrt{\left(\frac{2 \,.\, BFG}{\tan. B}\right)}.$$

(**510**) By Art. (65), Note, $BFG = BF^2 \,.\, \frac{\sin. B \,.\, \sin. F}{2 \,.\, \sin. (B + F)}$;

whence, $$BF = \sqrt{\left(\frac{2 \,.\, \sin. (B + F) \,.\, BFG}{\sin. B \,.\, \sin. F}\right)}.$$

(**511**) The final formula results from the proportion

$$FAE : CDE :: AE^2 : ED^2.$$

(**512**) Since triangles which have an angle in each equal, are as the products of the sides about the equal angles, we have

$$ABE : CDE :: AE \times BE : CE \times DE.$$

$$ABE = \tfrac{1}{2} \,.\, AB^2 \,.\, \frac{\sin. A \,.\, \sin. B}{\sin. (A + B)}. \qquad AE = AB \,.\, \frac{\sin. B}{\sin. E}.$$

$$BE = AB \,.\, \frac{\sin. A}{\sin. E}. \qquad CE = DE \,.\, \frac{\sin. CDE}{\sin. DCE}.$$

Substituting these values in the preceding proportion, cancelling the common factors, observing that $\sin. (A + B) = \sin. E$, multiplying extremes and means, and dividing, we get $$DE = \sqrt{\left(\frac{2 \,.\, CDE \,.\, \sin. DCE}{\sin. E \,.\, \sin. CDE}\right)}.$$

(**515**) The first formula is a consequence of the expression for the area of a triangle, given in the first paragraph of the Note to Art. (65).

(**517**) The reasons for the operations in this article (which are of very frequent occurrence), are self-evident.

(**518**) The expression for DZ follows from Art. (65), Note. The proportion in the next paragraph exists because triangles having the same altitude are as their bases.

(**519**) By construction, GPC = the required content. Now, GPC = GDC, since they have the same base and equal altitudes. We have now to prove that LMC = GDC. These two triangles have a common angle at C. Hence, they are to each other as the rectangles of the adjacent sides; i. e.,

$$GDC : LMC :: GC \times CD :: LC \times CM.$$

Here CM is unknown, and must be eliminated. We obtain an expression for it by means of the similar triangles LCM and LEP, which give

$$LE : LC :: EP = CD : CM.$$

Hence, $CM = \frac{CD \times LC}{LE}$. Substituting this value of CM in the first proportion, and cancelling CD in the last two terms, we get

$$GDC : LMC :: GC : \frac{LC^2}{LE}; \text{ or } GDC : LMC :: GC \times LE : LC^2.$$

$$LC^2 = (LH + HC)^2 = LH^2 + 2\,LH \times HC + HC^2.$$

But, by construction,

$$LH^2 = HK^2 = HE^2 - EK^2 = HE^2 - EC^2 = (HE + EC)\,(HE - EC) = HC\,(HE - EC).$$

Also, $$GC = 2\,HC;\ \text{and } LE = LH + HE.$$

Substituting these values in the last proportion, it becomes

$$\begin{aligned} GDC : LMC :: 2\,.\,HC\,(LH + HE) &: HC\,(HE - EC) + 2\,LH \times HC + HC^2. \\ :: 2\,LH + 2\,HE \quad &: HE - EC + 2\,LH + HC. \\ &: HE - EC + 2\,LH + HE + EC. \\ &: 2\,HE + 2\,LH. \end{aligned}$$

The last two terms of this proportion are thus proved to be equal. Therefore, the first two terms are also equal; i. e., LMC = GDC = the required content.

Since $HK = \sqrt{(HE^2 - EK^2)}$, it will have a negative as well as a positive value. It may therefore be set off in the contrary direction from L, i. e., to L′. The line drawn from L′ through P, and meeting CB produced beyond B, will part off *another* triangle of the required content.

(520) Suppose the line LM drawn. Then, by Art. (65), Note, the required content, $c = \frac{1}{2} \cdot CL \times CM\,.\,\sin.\,LCM$. This content will also equal the sum of the two triangles LCP and MCP; i. e., $c = \frac{1}{2} \cdot CL \times p + \frac{1}{2} \cdot CM \times q$. The first of these equations gives $CM = \frac{2\,c}{CL\,.\,\sin.\,LCM}$. Substituting this in the second equation, we have

$$c = \tfrac{1}{2}\,.\,CL \times p + \frac{cq}{CL\,.\,\sin.\,LCM}.$$

Whence, $$\tfrac{1}{2}\,p\,.\,CL^2\,.\,\sin.\,LCM + cq = c\,.\,CL\,.\,\sin.\,LCM.$$

Transposing and dividing by the coefficient of CL^2, we get

$$CL^2 - \frac{2\,c}{p}\cdot CL = -\frac{2\,cq}{p\,.\,\sin.\,CLM}.$$

$$CL = \frac{c}{p} \pm \sqrt{\left(\frac{c^2}{p^2} - \frac{2\,cq}{p\,.\,\sin.\,LCM}\right)}.$$

If the given point is *outside* of the lines CL and CM, conceive the desired line to be drawn from it, and another line to join the given point to the corner of the field. Then, as above, get expressions for the two triangles thus formed, and put their sum equal to the expression for the triangle which comprehends them both, and thence deduce the desired distance, nearly as above.

(522) The difference d, between the areas parted off by the guess line AB, and the required line CD, is equal to the difference between the triangles APC and BPD

By Art. (65), Note, the triangle $APC = \frac{1}{2} \cdot AP^2 . \frac{\sin.\,A\,.\,\sin.\,P}{\sin.\,(A + P)}$.

Similarly, the triangle $BPD = \frac{1}{2} \cdot BP^2 \frac{\sin.\,B\,.\,\sin.\,P}{\sin.\,(B + P)}$.

$$\therefore d = \tfrac{1}{2} \cdot AP^2 \frac{\sin.\,A\,.\,\sin.\,P}{\sin.\,(A + P)} - \tfrac{1}{2}\,BP^2 \cdot \frac{\sin.\,B\,.\,\sin.\,P}{\sin.\,(B + P)}.$$

By the expression for sin. $(a+b)$ [Trigonometry, Art. (8)], we have

$$d = \tfrac{1}{2}\,AP^2 \cdot \frac{\sin. A \,.\, \sin. P}{\sin. A \,.\, \cos. P + \sin. P \,.\, \cos. A} - \tfrac{1}{2}\,BP^2 \cdot \frac{\sin. B \,.\, \sin. P}{\sin. B \,.\, \cos. P + \sin. P \,.\, \cos. B}.$$

Dividing each fraction by its numerator, and remembering that $\frac{\cos. a}{\sin. a} = \cot. a$, we have

$$d = \frac{\tfrac{1}{2}\,AP^2}{\cot. P + \cot. A} - \frac{\tfrac{1}{2}\,BP^2}{\cot. P + \cot. B}.$$

For convenience, let $p = \cot. P$; $a = \cot. A$; and $b = \cot. B$. The above equation will then read, multiplying both sides by 2,

$$2\,d = \frac{AP^2}{p+a} - \frac{BP^2}{p+b}.$$

Clearing of fractions, we have

$$2\,dp^2 + 2\,dap + 2\,dbp + 2\,dab = p \,.\, AP^2 + b \,.\, AP^2 - p \,.\, BP^2 - a \,.\, BP^2.$$

Transposing, dividing through by $2\,d$, and separating into factors, we get

$$p^2 + \left(a + b - \frac{AP^2 - BP^2}{2\,d}\right)p = \frac{b \,.\, AP^2 - a \,.\, BP^2}{2\,d} - ab.$$

$$\therefore p = -\tfrac{1}{2}\left(a+b-\frac{AP^2-BP^2}{2\,d}\right) \pm \sqrt{\left[\frac{b \,.\, AP^2 - a \,.\, BP^2}{2\,d} - ab + \tfrac{1}{4}\left(a+b-\frac{AP^2-BP^2}{2d}\right)^2\right]}.$$

If $A = 90°$, $\cot. A = a = 0$; and the expression reduces to the simpler form given in the article.

(523) Conceive a perpendicular, BF, to be let fall from B to the required line DE. Let B represent the angle DBE, and β the unknown angle DBF. The angle $BDF = 90° - \beta$; and the angle $BEF = 90° - (B - \beta) = 90° - B + \beta$. By Art. (65), Note, the area of the triangle $DBE = \tfrac{1}{2}\ DE^2 \cdot \frac{\sin. BDE \,.\, \sin. BED}{\sin. (BDE + BED)} = \tfrac{1}{2} \cdot DE^2 \cdot \frac{\sin. (90° - \beta) \sin. (90° - B + \beta)}{\sin. B}$.

$$\text{Hence,}\quad DE^2 = \frac{2 \times DBE \times \sin. B}{\sin. (90° - \beta) \,.\, \sin. (90° - B + \beta)} = \frac{2 \times DBE \times \sin. B}{\cos. \beta \,.\, \cos. (B - \beta)}.$$

Now in order that DE may be the least possible, the denominator of the last fraction must be the greatest possible. It may be transformed, by the formula, $\cos. a \,.\, \cos. b = \tfrac{1}{2} \cos. (a+b) + \tfrac{1}{2} \,.\, \cos. (a-b)$ [Trigonometry, Art. (8)], into $\tfrac{1}{2} \cos. B + \tfrac{1}{2} \,.\, \cos. (B - 2\,\beta)$. Since B is constant, the value of this expression depends on its second term, and that will be the greatest possible when $B - 2\,\beta = 0$, in which case $\beta = \tfrac{1}{2}\,B$.

It hence appears that the required line DE is perpendicular to the line, BF, which bisects the given angle B. This gives the *direction* in which DE is to be run.

Its starting point, D or E, is found thus. The area of the triangle $DBE = \tfrac{1}{2}\,BD \,.\, BE \,.\, \sin. B$. Since the triangle is isosceles, this becomes

$$DBE = \tfrac{1}{2}\,BD^2 \,.\, \sin. B;\ \text{whence}\ BD = \sqrt{\left(\frac{2\ DBE}{\sin. B}\right)}.$$

DE is obtained from the expression for DE^2, which becomes, making $\beta = \tfrac{1}{2}B$,

$$DE^2 = \frac{2 \times DBE \times \sin. B}{\cos. \tfrac{1}{2}\,B \,.\, \cos. \tfrac{1}{2}\,B};\ \text{whence,}\ DE = \frac{\sqrt{(2 \,.\, DBE \,.\, \sin. B)}}{\cos. \tfrac{1}{2}\,B}.$$

(524) Let a = value per acre of one portion of the land, and b that of the other portion. Let x = the width required, BC or AD. Then the value of $BCFE = a \times \frac{x \times BE}{10}$, and the value of $ADFE = b \times \frac{x \times AE}{10}$.

Putting the sum of these equal to the value required to be parted off, we obtain $x = \frac{\text{value required} \times 10}{a \times BE + b \times AE}$.

(525) All the constructions of this article depend on the equivalency of triangles which have equal bases, and lie between parallels. The length of AD is derived from the area of a triangle being equal to its base by half its altitude.

(527) Since similar triangles are to each other as the squares of their homologous sides,

$$ABC : DBE :: AB^2 : BD^2;\ \text{whence}\ BD = AB\sqrt{\frac{DBE}{ABC}} = AB\sqrt{\frac{m}{m+n}}.$$

The construction of Fig. 363 is founded on the proportion

$$BF : BG :: BG : BA;\ \text{when}\ BD = BG = \sqrt{(BA \times BF)} = BA\sqrt{\frac{m}{m+n}}.$$

(528) By hypothesis, AEF : EFBC :: m : n; whence AEF : ABC :: m : $m+n$; and $AEF = ABC\frac{m}{m+n} = \frac{AC \times DB}{2} \cdot \frac{m}{m+n}$. Also, $AEF = \frac{1}{2} \cdot AE \times EF$. The similar triangles AEF and ABD give $AD : DB :: AE : EF = \frac{DB \times AE}{AD}$. The second expression for AEF then becomes $AEF = \frac{1}{2} AE \cdot \frac{DB \times AE}{AD}$. Equating this with the other value of AEF, we have

$$\frac{AC \times DB}{2} \cdot \frac{m}{m+n} = \frac{AE^2 \times DB}{2 . AD};\ \text{whence}\ AE = \sqrt{\left(AC \times AD \times \frac{m}{m+n}\right)}.$$

(530) In Fig. 366, the triangles ABD, DBC, having the same altitude, are to each other as their bases.

In the next paragraph, we have ABD : DBC :: AD : DC :: m : n; whence AD : AC :: m : $m+n$; and AC : DC :: $m+n$: n; whence the expressions for AD and DC.

In Fig. 367, the expression for AD is given by the proportion AD : AC :: m : $m+n$. Similarly for DE, and EC.

(531) In Fig. 368, conceive the line EB to be drawn. The triangle $AEB = \frac{1}{2}$ ABC, having the same altitude and half the base; and AFD = AEB, because of the equivalency of the triangles EFD and EFB, which, with AEF, make up AFD and AEB.

The point F is fixed by the similar triangles ADB and AEF

The expression for AF, in the last paragraph, is given by the proportion,

$$ABC : ADF :: AB \times AC : AD \times AF;$$

whence,
$$AF = \frac{AB \times AC}{AD} \cdot \frac{ADF}{ABC} = \frac{AB \times AC}{AD} \cdot \frac{m}{m+n}.$$

(532) The areas of triangles being equal to the product of their altitudes by half their bases, the constructions in Fig. 369 and Fig. 370 follow therefrom.

(**533**) In Fig. 371, conceive the line BL to be drawn. The triangle ABL will be a third of ABC, having the same altitude and one-third the base; and AED is equivalent to ABL, because ELB = ELD, and AEL is common to both. A similar proof applies to DCG.

(**534**) In Fig. 372, the four smaller triangles are mutually equivalent, because of their equal bases and altitudes, two pairs of them lying between parallels.

(**535**) In Fig. 373, conceive AE to be drawn. The triangle AEC $= \frac{1}{3}$. ABC, having the same altitude and half the base; and EDFC = AEC, because of the common part FEC and the equivalency of FED and FEA.

(**536**) In Fig. 374, in addition to the lines used in the problem, draw BF and DG. The triangle BFC $= \frac{1}{2}$ ABC, having the same altitude and half the base. Also, the triangle DFG = DFB, because of the parallels DF and BG. Adding DFC to each of these triangles, we have DCG = BFC $= \frac{1}{2}$ ABC. We have then to prove LMC = DCG. This is done precisely as in the demonstration of Art. (519), page 402.

(**537**) Let AE $= x$, ED $= y$, AH $= x'$, HF $= y'$, AK $= a$, KB $= b$.

The quadrilateral AFDE, equivalent to $\frac{1}{3}$ ABC, but which we will represent, generally, by m^2, is made up of the triangle AFH and the trapezoid FHED.

$$\text{AFH} = \tfrac{1}{2} \,.\, x'y'. \qquad \text{FHED} = \tfrac{1}{2}(x - x')(y + y').$$

$$\therefore \text{AFDE} = m^2 = \tfrac{1}{2} \,.\, x'y' + \tfrac{1}{2}(x - x')(y + y') = \tfrac{1}{2}\, x\,(y + y') - \tfrac{1}{2}\, x'y.$$

The similar triangles, AHF and AKB, give

$$a : b :: x' : y' = \frac{bx'}{a}.$$

Substituting this value of y' in the expression for m^2, we have

$$m^2 = \tfrac{1}{2}\, x \left(y + \frac{bx'}{a}\right) - \tfrac{1}{2}\, x'y;$$

whence, $$x' = \frac{a\,(2\,m^2 - xy)}{bx - ay} = \frac{\text{AK}\,(\tfrac{2}{3}\,\text{ABC} - \text{AE} \times \text{ED})}{\text{KB} \times \text{AE} - \text{AK} \times \text{ED}}.$$

The formula is general, whatever may be the ratio of the area m^2 to that of the triangle ABC.

(**538**) In Fig. 376, FD is a line of division, because BF = the triangle BDF divided by half its altitude, which gives its base. So for the other triangles.

(**539**) In Fig. 377, DG is a second line of division, because, drawing BL, the triangle BLC $= \frac{1}{3}$ ABC; and BDGC is equivalent to BLC, because of the common part BCLD, and the equivalency of the triangles DLG and DLB.

To prove that DF is a third line of division, join MD and MA. Then BMA $= \frac{1}{2}$ BGA. From BMA take MFA and add its equivalent MFD, and we have

$$\text{MDFB} = \tfrac{1}{2}\,\text{BGA} = \tfrac{1}{2}\,(\text{ABDG} - \text{BDG}) = \tfrac{1}{2}\,(\tfrac{2}{3}\,\text{ABC} - \text{BDG}) = \tfrac{1}{3}\,\text{ABC} - \tfrac{1}{2}\,\text{BDG}.$$

To MDFB add MDB, and add its equivalent, $\frac{1}{2}$ BDG, to the other side of the equation, and we have

$$\text{MDFB} + \text{MDB} = \tfrac{1}{3}\,\text{ABC} - \tfrac{1}{2}\,\text{BDG} + \tfrac{1}{2}\,\text{BDG}; \text{ or, BDF} = \tfrac{1}{3}\,\text{ABC}.$$

(**540**) In Fig. 378, the triangle AFC $= \frac{1}{3}$ ABC, having the same base and one-third the altitude. The triangles AFB and BFC are equivalent to each other, each being composed of two triangles of equal bases and altitudes; and each is therefore one-third of ABC.

In Fig. 379, AFC : ABC :: AD : AB; since these two triangles have the common base AC, and their altitudes are in the above ratio. So too, BFC : ABC :: BE : BA. Hence, the remaining triangle AFB : ABC :: DE : AB.

(541) By Art. (65), Note, $ABC = \frac{1}{2} AC \times CB \times \sin. ACB$. But the angle $ACB = ACD + DCB = \frac{1}{2}(180° - ADC) + \frac{1}{2}(180° - CDB) = 180° - \frac{1}{2}(ADC + CDB)$. Hence, $ABC = \frac{1}{2} AC \times CB \times \sin. \frac{1}{2}(ADC + CDB) = \frac{1}{2} AC \times CB \times \sin. \frac{1}{2} ADB$.

Let $r = DA = DB = DC$. Since AB is the chord of ADB to the radius r, and therefore equal to twice the sine of half that angle, we have

$\sin. \frac{1}{2}. ADB = \frac{AB}{2r}$; whence, $ABC = \frac{1}{2} AC \times CB \times \frac{AB}{2r}$; and $r = \frac{AB \times BC \times CA}{4 . ABC}$.

Also, since the area of each of the three small triangles equals half the product of one of the equal sides ($= r$), by the sine of the included angle at D, these triangles will be to each other as the sines of those angles. These angles are found thus:

$$\sin. \tfrac{1}{2} ADB = \frac{AB}{2r};\ \sin. \tfrac{1}{2} BDC = \frac{BC}{2r};\ \sin. \tfrac{1}{2} ADC = \frac{AC}{2r}.$$

(542) The formulas in this article are obtained by substituting, in those of Art. (523), for the triangle DBE, its equivalent $\frac{m}{m+n} \times \frac{1}{2} AB \times BC \times \sin. B$.

BD thus becomes $= \sqrt{\left(\frac{m}{m+n} \cdot \frac{AB \times BC \times \sin. B}{\sin. B}\right)} = \sqrt{\left(\frac{m}{m+n} \times AB \times BC\right)}$;

and $DE = \frac{\sqrt{\left(\frac{m}{m+n} \times AB \times BC \times \sin.^2 B\right)}}{\cos. \frac{1}{2} B} = \frac{\sin. B}{\cos. \frac{1}{2} B} \cdot \sqrt{\left(\frac{m}{m+n} \times AB \times BC\right)}$.

(543) The rule and example prove themselves.

(544) In Fig. 383, conceive the sides AB and DC, produced, to meet in some point P. Then, by reason of the similar triangles, $ADP : BCP :: AD^2 : BC^2$; whence, by "division," $ADP - BCP = ABCD : BCP :: AD^2 - BC^2 : BC^2$.

In like manner, comparing EFP and BCP, we get $EBCF : BCP :: EF^2 - BC^2 : BC^2$. Combining these two proportions, we have

$$ABCD : EBCF :: AD^2 - BC^2 : EF^2 - BC^2;$$

or, $\quad m + n : m :: AD^2 - BC^2 : EF^2 - BC^2$.

Whence, $\quad (m+n) EF^2 - m . BC^2 - n\, BC^2 = m . AD^2 - m . BC^2$;

$$\therefore EF = \sqrt{\left(\frac{m \times AD^2 + n \times BC^2}{m+n}\right)}.$$

Also, from the similar triangles formed by drawing BL parallel to CD, we have

$$AL : EK :: BA : BE = \frac{BA \times EK}{AL} = \frac{AB\,(EF - BC)}{AD - BC}.$$

(545) Let $BEFC = \frac{m}{m+n} \cdot ABCD = a$; let $BC = b$; $BH = h$; and $AD - BC = c$. Also let $BG = x$; and $EF = y$. Draw BL parallel to CD. By similar triangles, $AL : EK :: BA : BE :: BH : BG$; or, $AD - BC : EF - BC :: BH : BG$; i. e., $c : y - b :: h : x$; whence $x = \frac{h\,(y - b)}{c}$.

Also, the area $BEFC = a = \frac{1}{2} . BG\,(EF + BC) = \frac{1}{2} x\,(y + b)$; whence $y = \frac{2a}{x} - b$.

Substituting this value of y in the expression for x, and reducing, we obtain

$$x^2 + \frac{2\,bh}{c}\,x = \frac{2\,ah}{c};\ \text{whence we have } x = -\frac{bh}{c} \pm \sqrt{\left(\frac{2\,ah}{c} + \frac{b^2h^2}{c^2}\right)}.$$

The second proportion above gives $y - b = \frac{cx}{h}$; whence $y = b + \frac{c}{h} \cdot x$.

Replacing the symbols by their lines, we get the formulas in the text.

(**546**) ABEF $= \frac{1}{2}$ ABCD. But ABRP $=$ ABEF, because of the common part ABRF, and the triangles FRP and FRE, which make up the two figures, and which are equivalent because of the parallels FR and PE. So for the other parts.

(**547**) The truth of the foot-note is evident, since the first line bisects the trapezoid, and any other line drawn through its middle, and meeting the parallel sides, adds one triangle to each half, and takes away an equal triangle; and thus does not disturb the equivalency.

(**548**) In Fig. 385, since EF is parallel to AD, we have ADG : EGF :: $\text{GH}^2 : \text{GK}^2$. EGF is made up of the triangle BCG $= a'$, and the quadrilateral BEFC $= \frac{m}{m+n} \cdot$ ABCD $= \frac{m}{m+n} \cdot (a - a')$. Hence the above proportion becomes

$$a : a' + \frac{m}{m+n}\,(a - a') :: \text{GH}^2 : \text{GK}^2;\ \text{or,}$$

$$(m+n)\,a : ma + na' :: \text{GH}^2 : \text{GK}^2;\ \text{whence GK} = \text{GH}\sqrt{\left(\frac{ma+na'}{(m+n)\,a}\right)}.$$

GE is given by the proportion GH : GK :: GA : GE $=$ GA $\cdot \frac{\text{GK}}{\text{GH}}$.

In Fig. 386, the division into p parts is founded on the same principle. The triangle EFG $=$ GBC $+$ EFCB $= a' + \frac{\text{Q}}{p}$. Now ADG : EFG :: $\text{AG}^2 : \text{EG}^2$;

$$\text{or,}\qquad a' + \text{Q} : a' + \frac{\text{Q}}{p} :: \text{AG}^2 : \text{EG}^2;\ \text{whence GE} = \text{AG}\sqrt{\left(\frac{a' + \frac{\text{Q}}{p}}{a' + \text{Q}}\right)}.$$

GL is obtained by taking the triangle LMG $= a' + \frac{2\,\text{Q}}{p}$; and so for the rest.

(**552**) In Fig. 390, join FC and GC. Because of the parallels CA and BF, the triangle FCD will be equivalent to the quadrilateral ABCD, of which GCD will therefore be one half; and because of the parallels GE and CH, EHDC will be equivalent to GCD.

(**553**) In Fig. 391, by drawing certain lines, the quadrilateral can be divided into three equivalent parts, each composed of an equivalent trapezoid and an equivalent triangle. These three equivalent parts can then be transformed, by means of the parallels, into the three equivalent quadrilaterals shown in the figure. The full development of the proof is left as an exercise for the student.

In Fig. 392, draw CG. Then CBG $= \frac{1}{2}$ ABCD. But CKQ $=$ CGQ. Therefore CKQB $= \frac{1}{2}$ ABCD. So for the other division line.

(**556**) The division of the base of the equivalent triangle, divides the polygon similarly. The point Q results from the equivalency of the triangles ZBP and ZBQ, PQ being parallel to BZ.

APPENDIX C.

INTRODUCTION TO LEVELLING.

(1) The Principles. LEVELLING is the art of finding how much one point is higher or lower than another; i. e., how much one of the points is above or below a level line or surface which passes through the other point.

A *level or horizontal line* is one which is perpendicular to the direction of gravity, as indicated by a plumb-line or similar means. It is therefore parallel to the surface of standing water.

A *level or horizontal surface* is defined in the same way. It will be determined by two level lines which intersect each other.*

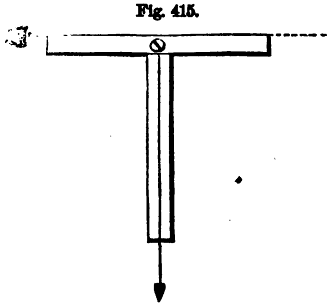
Levelling may be named VERTICAL SURVEYING, or *Up-and-down Surveying;* the subject of the preceding pages being *Horizontal Surveying*, or *Right-and-left* and *Fore-and-aft Surveying.*

All the methods of Horizontal Surveying may be used in Vertical Surveying. The one which will be briefly sketched here corresponds precisely to the method of "Surveying by offsets," founded on the Second Method, Art. (6), "Rectangular Co-ordinates," and fully explained in Arts. (114), &c.

The operations of levelling by this method consist, firstly, in obtaining a level line or plane; and, secondly, in measuring how far below it or above it (usually the former) are the two points whose relative heights are required.

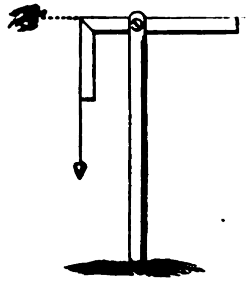
(2) The Instruments. A level line may be obtained by the following simple instrument, called a "*Plumb-line level.*" Fasten together two pieces of wood at right angles to each other, so as to make a T, and draw a line on the upright one so as to be exactly perpendicular to the top edge of the other. Suspend a plumb-line as in the figure. Fix the T against a staff stuck in the ground, by a screw through the middle of the cross-piece. Turn the T till the plumb-line exactly covers the line which was drawn. Then will the upper edge of the cross-piece be a level line, and the eye can sight across it, and note how far above or below any other point this level line, prolonged, would strike. It will be easier to look across sights fixed on each end of the cross-piece, making them of horsehair stretched across a piece of wire, bent into three sides of a square, and stuck into each end of the cross-piece; taking care that the hairs are at exactly equal heights above the upper edge of the cross-piece.

Fig. 415.



* Certain small corrections, to be hereafter explained, will be ignored for the present, and we will consider level lines as straight lines, and level surfaces as planes.

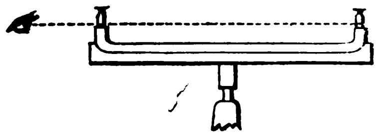
Fig. 416.



A modification of this is to fasten a common carpenter's square in a slit in the top of a staff, by means of a screw, and then tie a plumb-line at the angle so that it may hang beside one arm. When it has been brought to do so, by turning the square, then the other arm will be level.

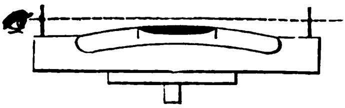
Another simple instrument depends upon the principle that "water always finds its level," corresponding to the second part of our definition of a level line. If a tube be bent up at each end, and nearly filled with water, the surface of the water in one end will always be at the same height as that in the other, however the position of the tube may vary. On this truth depends the "*Water-level.*" It may be easily constructed with a tube of tin, lead, copper, &c., by bending up, at right angles, an inch or two of each end, and supporting the tube, if too flexible, on a wooden bar. In these ends cement (with putty, twine dipped in white-lead, &c.), thin phials, with their bottoms broken off, so as to leave a free communication between them. Fill the tube and the phials, nearly to their top, with colored water. Blue vitriol, or cochineal, may be used for coloring it. Cork their mouths, and fit the instrument, by a steady but flexible joint, to a tripod. Figures of joints are given on page 184, and of tripods on page 183.

Fig. 417.



To use it, set it in the desired spot, place the tube by eye nearly level, remove the corks, and the surfaces of the water in the two phials will come to the same level. Stand about a yard behind the nearest phial, and let one eye, the other being closed, glance along the right-hand side of one phial and the left-hand side of the other. Raise or lower the head till the two surfaces seem to coincide, and this line of sight, prolonged, will give the level line desired. Sights of equal height, floating on the water, and rising above the tops of the phials, would give a better-defined line.

Fig. 418.

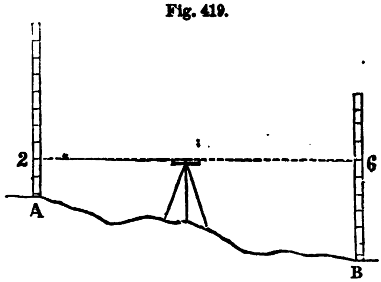


The "*Spirit-level*" consists essentially of a curved glass tube nearly filled with alcohol, but with a bubble of air left within, which always seeks the highest spot in the tube, and will therefore by its movements indicate any change in the position of the tube. Whenever the bubble, by raising or lowering one end, has been brought to stand between two marks on the tube, or, in case of expansion or contraction, to extend an equal distance on either side of them, the bottom of the block (if the tube be in one), or sights at each end of the tube, previously properly adjusted, will be on the same level line. It may be placed on a board fixed to the top of a staff or tripod.

When, instead of the sights, a telescope is made parallel to the level, and various contrivances to increase its delicacy and accuracy are added, the instrument becomes the Engineer's spirit-level.

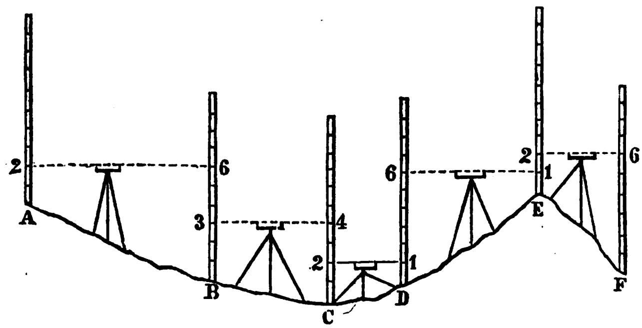
(8) The Practice. By whichever of these various means a level line has been obtained, the subsequent operations in making use of it are identical. Since the "water-level" is easily made and tolerably accurate, we will suppose it to be employed. Let A and B, Fig. 419, represent the two points, the difference of the heights of which is required. Set the instrument on any spot from which both the points can be seen, and at such a height that the level line will pass above the highest one. At A let an assistant hold a rod graduated into feet, tenths, &c. Turn the instrument towards the staff, sight along the level line, and note what division on the staff it strikes. Then send the staff to B, direct the instrument to it, and note the height observed at that point. If the level line, prolonged by the eye, passes 2 feet above A and 6 feet above B, the difference of their heights is 4 feet. The *absolute* height of the level line itself is a matter of indifference. The rod may carry a target or plate of iron, clasped to it so as to slide up and down, and be fixed, at will. This target may be variously painted, most simply with its upper half red and its lower half white. The horizontal line dividing the colors is the line sighted to, the target being moved up or down till the line of sight strikes it. A hole in the middle of the target shows what division on the rod coincides with the horizontal line, when it has been brought to the right height.

Fig. 419.



If the height of another point, C, Fig. 420, not visible from the first station, be required, set the instrument so as to see B and C, and proceed exactly as with A and B. If C be 1 foot below B, as in the figure, it will be 5 feet below A. If it were found to be 7 feet above B, it would be 3 feet above A. The comparative height of a series of any number of points, can thus be found in reference to any one of them.

Fig. 420.



The beginner in the practice of levelling may advantageously make in his note-book a sketch of the heights noted, and of the distances, putting down each as it is observed, and imitating, as nearly as his accuracy of eye will permit, their proportional dimensions.* But when the observations are numerous, they should be kept in a tabular form, such as that which is given below. The names of the points, or "Stations," whose heights are demanded, are placed in the first column; and their heights, as finally ascertained, in reference to the first point, in the last column. The heights above the starting point are marked +, and those below it are marked —. The back-sight to any station is placed on the line below the point to which it refers. When a back-sight exceeds a fore-sight, their difference is placed in the column of "Rise;" when it is less, their difference is a "Fall." The following table represents the same observations as the last figure, and their careful comparison will explain any obscurities in either.

Stations.	Distances.	Back-sights.	Fore-sights.	Rise.	Fall.	Total Heights.
A						0.00
B	100	2.00	6.00		— 4.00	— 4.00
C	60	3.00	4.00		— 1.00	— 5.00
D	40	2.00	1.00	+ 1.00		— 4.00
E	70	6.00	1.00	+ 5.00		+ 1.00
F	50	2.00	6.00		— 4.00	— 3.00
		15.00	18.00		— 3.00	

The above table shows that B is 4 feet below A; that C is 5 feet below A; that E is 1 foot above A; and so on. To test the calculations, add up the back-sights and fore-sights. The difference of the sums should equal the last "total height."

Another form of the levelling field-book is presented below. It refers to the same stations and levels, noted in the previous form, and shown in Fig. 420.

Stations.	Distances.	Back-sights.	Ht. Inst. above Datum.	Fore-sights.	Total Heights.
A					0.00
B	100	2.00	+ 2.00	6.00	— 4.00
C	60	3.00	— 1.00	4.00	— 5.00
D	40	2.00	— 3.00	1.00	— 4.00
E	70	6.00	+ 2.00	1.00	+ 1.00
F	50	2.00	+ 3.00	6.00	— 3.00
		15.00		18.00	— 3.00

In the above form it will be seen that a new column is introduced, containing the Height of the Instrument (i. e., of its line of sight), not above the ground where it stands, but above the *Datum*, or starting-point, of the levels. The former columns of "Rise" and "Fall" are omitted. The above notes are taken thus: The height of the starting-point or "Datum," at A, is 0.00. The instrument being set up and levelled, the rod is held at A. The back-sight upon it is 2.00; therefore the height of the instrument is also 2.00. The rod is next held at B. The fore-sight to it is 6.00. That point is therefore 6.00 below the instrument, or $2.00 - 6.00 = -4.00$ below the datum. The instrument is now moved, and again set up, and the back-sight to B, being 3.00, the Ht. Inst. is $-4.00 + 3.00 = -1.00$;

* In the figure, the limits of the page have made it necessary to contract the horizontal distances to one-tenth of their proper proportional size.

and so on: the Ht. Inst. being always obtained by adding the back-sight to the height of the peg on which the rod is held, and the height of the next peg being obtained by subtracting the fore-sight to the rod held on that peg, from the Ht. Inst.

The level lines given by these instruments are all lines of *apparent* level, and not of *true* level, which should curve with the surface of the earth. These level lines strike too high; but the difference is very small in sights of ordinary length, being only one-eighth of an inch for a sight of one-eighth of a mile, and diminishing as the square of the distance; and it may be completely compensated by setting the instrument midway between the points whose difference of level is desired; a precaution which should always be taken, when possible.

It may be required to show on paper the ups and downs of the line which has been levelled; and to represent, to any desired scale, the heights and distances of the various points of a line, its ascents and descents, as seen in a side-view. This is called a "*Profile.*" It is made thus. Any point on the paper being assumed for the first station, a horizontal line is drawn through it; the distance to the next station is measured along it, to the required scale; at the termination of this distance a vertical line is drawn; and the given height of the second station above or below the first is set off on this vertical line. The point thus fixed determines the second station, and a line joining it to the first station represents the slope of the ground between the two. The process is repeated for the next station, &c.

But the rises and falls of a line are always very small in proportion to the distances passed over; even mountains being merely as the roughnesses of the rind of an orange. If the distances and the heights were represented on a profile to the same scale, the latter would be hardly visible. To make them more apparent it is usual to "exaggerate the vertical scale" ten-fold, or more; i. e., to make the representation of a foot of height ten times as great as that of a foot of length, as in Fig. 420, in which one inch represents one hundred feet for the distances, and ten feet for the heights.

The preceding Introduction to Levelling has been made as brief as possible; but by any of the simple instruments described in it, and either of its tabular forms, any person can determine with sufficient precision whether a distant spring is higher or lower than his house, and how much; as well as how deep it would be necessary to cut into any intervening hill to bring the water. He may in like manner ascertain whether a swamp can be drained into a neighboring brook; and can cut the necessary ditches at any given slope of so many inches to the rod, &c., having thus found a level line; or he can obtain any other desired information which depends on the relative heights of two points.

To explain the peculiarities of the more elaborate levelling instruments, the precautions necessary in their use, the prevention and correction of errors, the overcoming of difficulties, and the various complicated details of their applications, would require a great number of pages. This will therefore be reserved for another volume, as announced in the Preface.

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OR,

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CALCULATED TO

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EACH QUARTER DEGREE OF BEARING.

Bearing.	1 Lat.	1 Dep.	2 Lat.	2 Dep.	3 Lat.	3 Dep.	4 Lat.	4 Dep.	5 Lat.	Bearing.
0°	1.000	0.000	2.000	0.000	3.000	0.000	4.000	0.000	5.000	90°
0¼	1.000	0.004	2.000	0.009	3.000	0.013	4.000	0.017	5.000	89¾
0½	1.000	0.009	2.000	0.017	3.000	0.026	4.000	0.035	5.000	89½
0¾	1.000	0.013	2.000	0.026	3.000	0.039	4.000	0.052	5.000	89¼
1°	1.000	0.017	2.000	0.035	3.000	0.052	3.999	0.070	4.999	89°
1¼	1.000	0.022	2.000	0.044	2.999	0.065	3.999	0.087	4.999	88¾
1½	1.000	0.026	1.999	0.052	2.999	0.079	3.999	0.105	4.998	88½
1¾	1.000	0.031	1.999	0.061	2.999	0.092	3.998	0.122	4.998	88¼
2°	0.999	0.035	1.999	0.070	2.998	0.105	3.998	0.140	4.997	88°
2¼	0.999	0.039	1.998	0.079	2.998	0.118	3.997	0.157	4.996	87¾
2½	0.999	0.044	1.998	0.087	2.997	0.131	3.996	0.174	4.995	87½
2¾	0.999	0.048	1.998	0.096	2.997	0.144	3.995	0.192	4.994	87¼
3°	0.999	0.052	1.997	0.105	2.996	0.157	3.995	0.209	4.993	87°
3¼	0.998	0.057	1.997	0.113	2.995	0.170	3.994	0.227	4.992	86¾
3½	0.998	0.061	1.996	0.122	2.994	0.183	3.993	0.244	4.991	86½
3¾	0.998	0.065	1.996	0.131	2.994	0.196	3.991	0.262	4.989	86¼
4°	0.998	0.070	1.995	0.140	2.993	0.209	3.990	0.279	4.988	86°
4¼	0.997	0.074	1.995	0.148	2.992	0.222	3.989	0.296	4.986	85¾
4½	0.997	0.078	1.994	0.157	2.991	0.235	3.988	0.314	4.985	85½
4¾	0.997	0.083	1.993	0.166	2.990	0.248	3.986	0.331	4.983	85¼
5°	0.996	0.087	1.992	0.174	2.989	0.261	3.985	0.349	4.981	85°
5¼	0.996	0.092	1.992	0.183	2.987	0.275	3.983	0.366	4.979	84¾
5½	0.995	0.096	1.991	0.192	2.986	0.288	3.982	0.383	4.977	84½
5¾	0.995	0.100	1.990	0.200	2.985	0.301	3.980	0.401	4.975	84¼
6°	0.995	0.105	1.989	0.209	2.984	0.314	3.978	0.418	4.973	84°
6¼	0.994	0.109	1.988	0.218	2.982	0.327	3.976	0.435	4.970	83¾
6½	0.994	0.113	1.987	0.226	2.981	0.340	3.974	0.453	4.968	83½
6¾	0.993	0.118	1.986	0.235	2.979	0.353	3.972	0.470	4.965	83¼
7°	0.993	0.122	1.985	0.244	2.978	0.366	3.970	0.487	4.963	83°
7¼	0.992	0.126	1.984	0.252	2.976	0.379	3.968	0.505	4.960	82¾
7½	0.991	0.131	1.983	0.261	2.974	0.392	3.966	0.522	4.957	82½
7¾	0.991	0.135	1.982	0.270	2.973	0.405	3.963	0.539	4.954	82¼
8°	0.990	0.139	1.981	0.278	2.971	0.418	3.961	0.557	4.951	82°
8¼	0.990	0.143	1.979	0.287	2.969	0.430	3.959	0.574	4.948	81¾
8½	0.989	0.148	1.978	0.296	2.967	0.443	3.956	0.591	4.945	81½
8¾	0.988	0.152	1.977	0.304	2.965	0.456	3.953	0.608	4.942	81¼
9°	0.988	0.156	1.975	0.313	2.963	0.469	3.951	0.626	4.938	81°
9¼	0.987	0.161	1.974	0.321	2.961	0.482	3.948	0.643	4.935	80¾
9½	0.986	0.165	1.973	0.330	2.959	0.495	3.945	0.660	4.931	80½
9¾	0.986	0.169	1.971	0.339	2.957	0.508	3.942	0.677	4.928	80¼
10°	0.985	0.174	1.970	0.347	2.954	0.521	3.939	0.695	4.924	80°
10¼	0.984	0.178	1.968	0.356	2.952	0.534	3.936	0.712	4.920	79¾
10½	0.983	0.182	1.967	0.364	2.950	0.547	3.933	0.729	4.916	79½
10¾	0.982	0.187	1.965	0.373	2.947	0.560	3.930	0.746	4.912	79¼
11°	0.982	0.191	1.963	0.382	2.945	0.572	3.927	0.763	4.908	79°
11¼	0.981	0.195	1.962	0.390	2.942	0.585	3.923	0.780	4.904	78¾
11½	0.980	0.199	1.960	0.399	2.940	0.598	3.920	0.797	4.900	78½
11¾	0.979	0.204	1.958	0.407	2.937	0.611	3.916	0.815	4.895	78¼
12°	0.978	0.208	1.956	0.416	2.934	0.624	3.913	0.832	4.891	78°
12¼	0.977	0.212	1.954	0.424	2.932	0.637	3.909	0.849	4.886	77¾
12½	0.976	0.216	1.953	0.433	2.929	0.649	3.905	0.866	4.881	77½
12¾	0.975	0.221	1.951	0.441	2.926	0.662	3.901	0.883	4.877	77¼
13°	0.974	0.225	1.949	0.450	2.923	0.675	3.897	0.900	4.872	77°
13¼	0.973	0.229	1.947	0.458	2.920	0.688	3.894	0.917	4.867	76¾
13½	0.972	0.233	1.945	0.467	2.917	0.700	3.889	0.934	4.862	76½
13¾	0.971	0.238	1.943	0.475	2.914	0.713	3.885	0.951	4.857	76¼
14°	0.970	0.242	1.941	0.484	2.911	0.726	3.881	0.968	4.851	76°
14¼	0.969	0.246	1.938	0.492	2.908	0.738	3.877	0.985	4.846	75¾
14½	0.968	0.250	1.936	0.501	2.904	0.751	3.873	1.002	4.841	75½
14¾	0.967	0.255	1.934	0.509	2.901	0.764	3.868	1.018	4.835	75¼
15°	0.966	0.259	1.932	0.518	2.898	0.776	3.864	1.035	4.830	75°
Bearing.	1 Dep.	1 Lat.	2 Dep.	2 Lat.	3 Dep.	3 Lat.	4 Dep.	4 Lat.	5 Dep.	Bearing.

Bearing.	5	6		7		8		9		Bearing.
	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	
0°	0.000	6.000	0.000	7.000	0.000	8.000	0.000	9.000	0.000	90°
0¼	0.022	6.000	0.026	7.000	0.031	8.000	0.035	9.000	0.039	89¾
0½	0.044	6.000	0.052	7.000	0.061	8.000	0.070	9.000	0.079	89½
0¾	0.065	5.999	0.079	6.999	0.092	7.999	0.105	8.999	0.118	89¼
1°	0.087	5.999	0.105	6.999	0.122	7.999	0.140	8.999	0.157	89°
1¼	0.109	5.999	0.131	6.998	0.153	7.998	0.175	8.998	0.196	88¾
1½	0.131	5.998	0.157	6.998	0.183	7.997	0.209	8.997	0.236	88½
1¾	0.153	5.997	0.183	6.997	0.214	7.996	0.244	8.996	0.275	88¼
2°	0.174	5.996	0.209	6.996	0.244	7.995	0.279	8.995	0.314	88°
2¼	0.196	5.995	0.236	6.995	0.275	7.994	0.314	8.993	0.353	87¾
2½	0.218	5.994	0.262	6.993	0.305	7.992	0.349	8.991	0.393	87½
2¾	0.240	5.993	0.288	6.992	0.336	7.991	0.384	8.990	0.432	87¼
3°	0.262	5.992	0.314	6.990	0.366	7.989	0.419	8.988	0.471	87°
3¼	0.283	5.990	0.340	6.989	0.397	7.987	0.454	8.986	0.510	86¾
3½	0.305	5.989	0.366	6.987	0.427	7.985	0.488	8.983	0.549	86½
3¾	0.327	5.987	0.392	6.985	0.458	7.983	0.523	8.981	0.589	86¼
4°	0.349	5.985	0.419	6.983	0.488	7.981	0.558	8.978	0.628	86°
4¼	0.371	5.984	0.445	6.981	0.519	7.978	0.593	8.975	0.667	85¾
4½	0.392	5.982	0.471	6.978	0.549	7.975	0.628	8.972	0.706	85½
4¾	0.414	5.979	0.497	6.976	0.580	7.973	0.662	8.969	0.745	85¼
5°	0.436	5.977	0.523	6.973	0.610	7.970	0.697	8.966	0.784	85°
5¼	0.458	5.975	0.549	6.971	0.641	7.966	0.732	8.962	0.824	84¾
5½	0.479	5.972	0.575	6.968	0.671	7.963	0.767	8.959	0.863	84½
5¾	0.501	5.970	0.601	6.965	0.701	7.960	0.802	8.955	0.902	84¼
6°	0.523	5.967	0.627	6.962	0.732	7.956	0.836	8.951	0.941	84°
6¼	0.544	5.964	0.653	6.958	0.762	7.952	0.871	8.947	0.980	83¾
6½	0.566	5.961	0.679	6.955	0.792	7.949	0.906	8.942	1.019	83½
6¾	0.588	5.958	0.705	6.951	0.823	7.945	0.940	8.938	1.058	83¼
7°	0.609	5.955	0.731	6.948	0.853	7.940	0.975	8.933	1.097	83°
7¼	0.631	5.952	0.757	6.944	0.883	7.936	1.010	8.928	1.136	82¾
7½	0.653	5.949	0.783	6.940	0.914	7.932	1.044	8.923	1.175	82½
7¾	0.674	5.945	0.809	6.936	0.944	7.927	1.079	8.918	1.214	82¼
8°	0.696	5.942	0.835	6.932	0.974	7.922	1.113	8.912	1.253	82°
8¼	0.717	5.938	0.861	6.928	1.004	7.917	1.148	8.907	1.291	81¾
8½	0.739	5.934	0.887	6.923	1.035	7.912	1.182	8.901	1.330	81½
8¾	0.761	5.930	0.913	6.919	1.065	7.907	1.217	8.895	1.369	81¼
9°	0.782	5.926	0.939	6.914	1.095	7.902	1.251	8.889	1.408	81°
9¼	0.804	5.922	0.964	6.909	1.125	7.896	1.286	8.883	1.447	80¾
9½	0.825	5.918	0.990	6.904	1.155	7.890	1.320	8.877	1.485	80½
9¾	0.847	5.913	1.016	6.899	1.185	7.884	1.355	8.870	1.524	80¼
10°	0.868	5.909	1.042	6.894	1.216	7.878	1.389	8.863	1.563	80°
10¼	0.890	5.904	1.068	6.888	1.246	7.872	1.424	8.856	1.601	79¾
10½	0.911	5.900	1.093	6.883	1.276	7.866	1.458	8.849	1.640	79½
10¾	0.933	5.895	1.119	6.877	1.306	7.860	1.492	8.842	1.679	79¼
11°	0.954	5.890	1.145	6.871	1.336	7.853	1.526	8.835	1.717	79°
11¼	0.975	5.885	1.171	6.866	1.366	7.846	1.561	8.827	1.756	78¾
11½	0.997	5.880	1.196	6.859	1.396	7.839	1.595	8.819	1.794	78½
11¾	1.018	5.874	1.222	6.853	1.425	7.832	1.629	8.811	1.833	78¼
12°	1.040	5.869	1.247	6.847	1.455	7.825	1.663	8.803	1.871	78°
12¼	1.061	5.863	1.273	6.841	1.485	7.818	1.697	8.795	1.910	77¾
12½	1.082	5.858	1.299	6.834	1.515	7.810	1.732	8.787	1.948	77½
12¾	1.103	5.852	1.324	6.827	1.545	7.803	1.766	8.778	1.986	77¼
13°	1.125	5.846	1.350	6.821	1.575	7.795	1.800	8.769	2.025	77°
13¼	1.146	5.840	1.375	6.814	1.604	7.787	1.834	8.760	2.063	76¾
13½	1.167	5.834	1.401	6.807	1.634	7.779	1.868	8.751	2.101	76½
13¾	1.188	5.828	1.426	6.799	1.664	7.771	1.902	8.742	2.139	76¼
14°	1.210	5.822	1.452	6.792	1.693	7.762	1.935	8.733	2.177	76°
14¼	1.231	5.815	1.477	6.785	1.723	7.754	1.969	8.723	2.215	75¾
14½	1.252	5.809	1.502	6.777	1.753	7.745	2.003	8.713	2.253	75½
14¾	1.273	5.802	1.528	6.769	1.782	7.736	2.037	8.703	2.291	75¼
15°	1.294	5.796	1.553	6.761	1.812	7.727	2.071	8.693	2.329	75°
	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	
Bearing.	5	6		7		8		9		Bearing.

Bearing.	1		2		3		4		5	Bearing.
	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	
15°	0·966	0·259	1·932	0·518	2·898	0·776	3·864	1·035	4·830	**75°**
15¼	0·965	0·263	1·930	0·526	2·894	0·789	3·859	1·052	4·824	74¾
15½	0·964	0·267	1·927	0·534	2·891	0·802	3·855	1·069	4·818	74½
15¾	0·962	0·271	1·925	0·543	2·887	0·814	3·850	1·086	4·812	74¼
16°	0·961	0·276	1·923	0·551	2·884	0·827	3·845	1·103	4·806	**74°**
16¼	0·960	0·280	1·920	0·560	2·880	0·839	3·840	1·119	4·800	73¾
16½	0·959	0·284	1·918	0·568	2·876	0·852	3·835	1·136	4·794	73½
16¾	0·958	0·288	1·915	0·576	2·873	0·865	3·830	1·153	4·788	73¼
17°	0·956	0·292	1·913	0·585	2·869	0·877	3·825	1·169	4·782	**73°**
17¼	0·955	0·297	1·910	0·593	2·865	0·890	3·820	1·186	4·775	72¾
17½	0·954	0·301	1·907	0·601	2·861	0·902	3·815	1·203	4·769	72½
17¾	0·952	0·305	1·905	0·610	2·857	0·915	3·810	1·220	4·762	72¼
18°	0·951	0·309	1·902	0·618	2·853	0·927	3·804	1·236	4·755	**72°**
18¼	0·950	0·313	1·899	0·626	2·849	0·939	3·799	1·253	4·748	71¾
18½	0·948	0·317	1·897	0·635	2·845	0·952	3·793	1·269	4·742	71½
18¾	0·947	0·321	1·894	0·643	2·841	0·964	3·788	1·286	4·735	71¼
19°	0·946	0·326	1·891	0·651	2·837	0·977	3·782	1·302	4·728	**71°**
19¼	0·944	0·330	1·888	0·659	2·832	0·989	3·776	1·319	4·720	70¾
19½	0·943	0·334	1·885	0·668	2·828	1·001	3·771	1·335	4·713	70½
19¾	0·941	0·338	1·882	0·676	2·824	1·014	3·765	1·352	4·706	70¼
20°	0·940	0·342	1·879	0·684	2·819	1·026	3·759	1·368	4·698	**70°**
20¼	0·938	0·346	1·876	0·692	2·815	1·038	3·753	1·384	4·691	69¾
20½	0·937	0·350	1·873	0·700	2·810	1·051	3·747	1·401	4·683	69½
20¾	0·935	0·354	1·870	0·709	2·805	1·063	3·741	1·417	4·676	69¼
21°	0·934	0·358	1·867	0·717	2·801	1·075	3·734	1·433	4·668	**69°**
21¼	0·932	0·362	1·864	0·725	2·796	1·087	3·728	1·450	4·660	68¾
21½	0·930	0·367	1·861	0·733	2·791	1·100	3·722	1·466	4·652	68½
21¾	0·929	0·371	1·858	0·741	2·786	1·112	3·715	1·482	4·644	68¼
22°	0·927	0·375	1·854	0·749	2·782	1·124	3·709	1·498	4·636	**68°**
22¼	0·926	0·379	1·851	0·757	2·777	1·136	3·702	1·515	4·628	67¾
22½	0·924	0·383	1·848	0·765	2·772	1·148	3·696	1·531	4·619	67½
22¾	0·922	0·387	1·844	0·773	2·767	1·160	3·689	1·547	4·611	67¼
23°	0·921	0·391	1·841	0·781	2·762	1·172	3·682	1·563	4·603	**67°**
23¼	0·919	0·395	1·838	0·789	2·756	1·184	3·675	1·579	4·594	66¾
23½	0·917	0·399	1·834	0·797	2·751	1·196	3·668	1·595	4·585	66½
23¾	0·915	0·403	1·831	0·805	2·746	1·208	3·661	1·611	4·577	66¼
24°	0·914	0·407	1·827	0·813	2·741	1·220	3·654	1·627	4·568	**66°**
24¼	0·912	0·411	1·824	0·821	2·735	1·232	3·647	1·643	4·559	65¾
24½	0·910	0·415	1·820	0·829	2·730	1·244	3·640	1·659	4·550	65½
24¾	0·908	0·419	1·816	0·837	2·724	1·256	3·633	1·675	4·541	65¼
25°	0·906	0·423	1·813	0·845	2·719	1·268	3·625	1·690	4·532	**65°**
25¼	0·904	0·427	1·809	0·853	2·713	1·280	3·618	1·706	4·522	64¾
25½	0·903	0·431	1·805	0·861	2·708	1·292	3·610	1·722	4·513	64½
25¾	0·901	0·434	1·801	0·869	2·702	1·303	3·603	1·738	4·503	64¼
26°	0·899	0·438	1·798	0·877	2·696	1·315	3·595	1·753	4·494	**64°**
26¼	0·897	0·442	1·794	0·885	2·691	1·327	3·587	1·769	4·484	63¾
26½	0·895	0·446	1·790	0·892	2·685	1·339	3·580	1·785	4·475	63½
26¾	0·893	0·450	1·786	0·900	2·679	1·350	3·572	1·800	4·465	63¼
27°	0·891	0·454	1·782	0·908	2·673	1·362	3·564	1·816	4·455	**63°**
27¼	0·889	0·458	1·778	0·916	2·667	1·374	3·556	1·831	4·445	62¾
27½	0·887	0·462	1·774	0·923	2·661	1·385	3·548	1·847	4·435	62½
27¾	0·885	0·466	1·770	0·931	2·655	1·397	3·540	1·862	4·425	62¼
28°	0·883	0·469	1·766	0·939	2·649	1·408	3·532	1·878	4·415	**62°**
28¼	0·881	0·473	1·762	0·947	2·643	1·420	3·524	1·893	4·404	61¾
28½	0·879	0·477	1·758	0·954	2·636	1·431	3·515	1·909	4·394	61½
28¾	0·877	0·481	1·753	0·962	2·630	1·443	3·507	1·924	4·384	61¼
29°	0·875	0·485	1·749	0·970	2·624	1·454	3·498	1·939	4·373	**61°**
29¼	0·872	0·489	1·745	0·977	2·617	1·466	3·490	1·954	4·362	60¾
29½	0·870	0·492	1·741	0·985	2·611	1·477	3·481	1·970	4·352	60½
29¾	0·868	0·496	1·736	0·992	2·605	1·489	3·473	1·985	4·341	60¼
30°	0·866	0·500	1·732	1·000	2·598	1·500	3·464	2·000	4·330	**60°**
	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	
Bearing.	1		2		3		4		5	Bearing.

Bearing.	5	6		7		8		9		Bearing.
	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	
15°	1·294	5·796	1·553	6·761	1·812	7·727	2·071	8·693	2·329	**75°**
15¼	1·315	5·789	1·578	6·754	1·841	7·718	2·104	8·683	2·367	74¾
15½	1·336	5·782	1·603	6·745	1·871	7·709	2·138	8·673	2·405	74½
15¾	1·357	5·775	1·629	6·737	1·900	7·700	2·172	8·662	2·443	74¼
16°	1·378	5·768	1·654	6·729	1·929	7·690	2·205	8·651	2·481	**74°**
16¼	1·399	5·760	1·679	6·720	1·959	7·680	2·239	8·640	2·518	73¾
16½	1·420	5·753	1·704	6·712	1·988	7·671	2·272	8·629	2·556	73½
16¾	1·441	5·745	1·729	6·703	2·017	7·661	2·306	8·618	2·594	73¼
17°	1·462	5·738	1·754	6·694	2·047	7·650	2·339	8·607	2·631	**73°**
17¼	1·483	5·730	1·779	6·685	2·076	7·640	2·372	8·595	2·669	72¾
17½	1·504	5·722	1·804	6·676	2·105	7·630	2·406	8·583	2·706	72½
17¾	1·524	5·714	1·829	6·667	2·134	7·619	2·439	8·572	2·744	72¼
18°	1·545	5·706	1·854	6·657	2·163	7·608	2·472	8·560	2·781	**72°**
18¼	1·566	5·698	1·879	6·648	2·192	7·598	2·505	8·547	2·818	71¾
18½	1·587	5·690	1·904	6·638	2·221	7·587	2·538	8·535	2·856	71½
18¾	1·607	5·682	1·929	6·629	2·250	7·575	2·572	8·522	2·893	71¼
19°	1·628	5·673	1·953	6·619	2·279	7·564	2·605	8·510	2·930	**71°**
19¼	1·648	5·665	1·978	6·609	2·308	7·553	2·638	8·497	2·967	70¾
19½	1·669	5·656	2·003	6·598	2·337	7·541	2·670	8·484	3·004	70½
19¾	1·690	5·647	2·028	6·588	2·365	7·529	2·703	8·471	3·041	70¼
20°	1·710	5·638	2·052	6·578	2·394	7·518	2·736	8·457	3·078	**70°**
20¼	1·731	5·629	2·077	6·567	2·423	7·506	2·769	8·444	3·115	69¾
20½	1·751	5·620	2·101	6·557	2·451	7·493	2·802	8·430	3·152	69½
20¾	1·771	5·611	2·126	6·546	2·480	7·481	2·834	8·416	3·189	69¼
21°	1·792	5·601	2·150	6·535	2·509	7·469	2·867	8·402	3·225	**69°**
21¼	1·812	5·592	2·175	6·524	2·537	7·456	2·900	8·388	3·262	68¾
21½	1·833	5·582	2·199	6·513	2·566	7·443	2·932	8·374	3·299	68½
21¾	1·853	5·573	2·223	6·502	2·594	7·430	2·964	8·359	3·335	68¼
22°	1·873	5·563	2·248	6·490	2·622	7·417	2·997	8·345	3·371	**68°**
22¼	1·893	5·553	2·272	6·479	2·651	7·404	3·029	8·330	3·408	67¾
22½	1·913	5·543	2·296	6·467	2·679	7·391	3·061	8·315	3·444	67½
22¾	1·934	5·533	2·320	6·455	2·707	7·378	3·094	8·300	3·480	67¼
23°	1·954	5·523	2·344	6·444	2·735	7·364	3·126	8·285	3·517	**67°**
23¼	1·974	5·513	2·368	6·432	2·763	7·350	3·158	8·269	3·553	66¾
23½	1·994	5·502	2·392	6·419	2·791	7·336	3·190	8·254	3·589	66½
23¾	2·014	5·492	2·416	6·407	2·819	7·322	3·222	8·238	3·625	66¼
24°	2·034	5·481	2·440	6·395	2·847	7·308	3·254	8·222	3·661	**66°**
24¼	2·054	5·471	2·464	6·382	2·875	7·294	3·286	8·206	3·696	65¾
24½	2·073	5·460	2·488	6·370	2·903	7·280	3·318	8·190	3·732	65½
24¾	2·093	5·449	2·512	6·357	2·931	7·265	3·349	8·173	3·768	65¼
25°	2·113	5·438	2·536	6·344	2·958	7·250	3·381	8·157	3·804	**65°**
25¼	2·133	5·427	2·559	6·331	2·986	7·236	3·413	8·140	3·839	64¾
25½	2·153	5·416	2·583	6·318	3·014	7·221	3·444	8·123	3·875	64½
25¾	2·172	5·404	2·607	6·305	3·041	7·206	3·476	8·106	3·910	64¼
26°	2·192	5·393	2·630	6·292	3·069	7·190	3·507	8·089	3·945	**64°**
26¼	2·211	5·381	2·654	6·278	3·096	7·175	3·538	8·072	3·981	63¾
26½	2·231	5·370	2·677	6·265	3·123	7·160	3·570	8·054	4·016	63½
26¾	2·250	5·358	2·701	6·251	3·151	7·144	3·601	8·037	4·051	63¼
27°	2·270	5·346	2·724	6·237	3·178	7·128	3·632	8·019	4·086	**63°**
27¼	2·289	5·334	2·747	6·223	3·205	7·112	3·663	8·001	4·121	62¾
27½	2·309	5·322	2·770	6·209	3·232	7·096	3·694	7·983	4·156	62½
27¾	2·328	5·310	2·794	6·195	3·259	7·080	3·725	7·965	4·190	62¼
28°	2·347	5·298	2·817	6·181	3·286	7·064	3·756	7·947	4·225	**62°**
28¼	2·367	5·285	2·840	6·166	3·313	7·047	3·787	7·928	4·260	61¾
28½	2·386	5·273	2·863	6·152	3·340	7·031	3·817	7·909	4·294	61½
28¾	2·405	5·260	2·886	6·137	3·367	7·014	3·848	7·891	4·329	61¼
29°	2·424	5·248	2·909	6·122	3·394	6·997	3·878	7·872	4·363	**61°**
29¼	2·443	5·235	2·932	6·107	3·420	6·980	3·909	7·852	4·398	60¾
29½	2·462	5·222	2·955	6·093	3·447	6·963	3·939	7·833	4·432	60½
29¾	2·481	5·209	2·977	6·077	3·474	6·946	3·970	7·814	4·466	60¼
30°	2·500	5·196	3·000	6·062	3·500	6·928	4·000	7·794	4·500	**60°**
	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	
Bearing.	5	6		7		8		9		Bearing.

Bearing.	1		2		3		4		5	
	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	
30°	0.866	0.500	1.732	1.000	2.598	1.500	3.464	2.000	4.330	60°
30¼	0.864	0.504	1.728	1.008	2.592	1.511	3.455	2.015	4.319	59¾
30½	0.862	0.508	1.723	1.015	2.585	1.523	3.447	2.030	4.308	59½
30¾	0.859	0.511	1.719	1.023	2.578	1.534	3.438	2.045	4.297	59¼
31°	0.857	0.515	1.714	1.030	2.572	1.545	3.429	2.060	4.286	**59°**
31¼	0.855	0.519	1.710	1.038	2.565	1.556	3.420	2.075	4.275	58¾
31½	0.853	0.522	1.705	1.045	2.558	1.567	3.411	2.090	4.263	58½
31¾	0.850	0.526	1.701	1.052	2.551	1.579	3.401	2.105	4.252	58¼
32°	0.848	0.530	1.696	1.060	2.544	1.590	3.392	2.120	4.240	**58°**
32¼	0.846	0.534	1.691	1.067	2.537	1.601	3.383	2.134	4.229	57¾
32½	0.843	0.537	1.687	1.075	2.530	1.612	3.374	2.149	4.217	57½
32¾	0.841	0.541	1.682	1.082	2.523	1.623	3.364	2.164	4.205	57¼
33°	0.839	0.545	1.677	1.089	2.516	1.634	3.355	2.179	4.193	**57°**
33¼	0.836	0.548	1.673	1.097	2.509	1.645	3.345	2.193	4.181	56¾
33½	0.834	0.552	1.668	1.104	2.502	1.656	3.336	2.208	4.169	56½
33¾	0.831	0.556	1.663	1.111	2.494	1.667	3.326	2.222	4.157	56¼
34°	0.829	0.559	1.658	1.118	2.487	1.678	3.316	2.237	4.145	**56°**
34¼	0.827	0.563	1.653	1.126	2.480	1.688	3.306	2.251	4.133	55¾
34½	0.824	0.566	1.648	1.133	2.472	1.699	3.297	2.266	4.121	55½
34¾	0.822	0.570	1.643	1.140	2.465	1.710	3.287	2.280	4.108	55¼
35°	0.819	0.574	1.638	1.147	2.457	1.721	3.277	2.294	4.096	**55°**
35¼	0.817	0.577	1.633	1.154	2.450	1.731	3.267	2.309	4.083	54¾
35½	0.814	0.581	1.628	1.161	2.442	1.742	3.257	2.323	4.071	54½
35¾	0.812	0.584	1.623	1.168	2.435	1.753	3.246	2.337	4.058	54¼
36°	0.809	0.588	1.618	1.176	2.427	1.763	3.236	2.351	4.045	**54°**
36¼	0.806	0.591	1.613	1.183	2.419	1.774	3.226	2.365	4.032	53¾
36½	0.804	0.595	1.608	1.190	2.412	1.784	3.215	2.379	4.019	53½
36¾	0.801	0.598	1.603	1.197	2.404	1.795	3.205	2.393	4.006	53¼
37°	0.799	0.602	1.597	1.204	2.396	1.805	3.195	2.407	3.993	**53°**
37¼	0.796	0.605	1.592	1.211	2.388	1.816	3.184	2.421	3.980	52¾
37½	0.793	0.609	1.587	1.218	2.380	1.826	3.173	2.435	3.967	52½
37¾	0.791	0.612	1.581	1.224	2.372	1.837	3.163	2.449	3.953	52¼
38°	0.788	0.616	1.576	1.231	2.364	1.847	3.152	2.463	3.940	**52°**
38¼	0.785	0.619	1.571	1.238	2.356	1.857	3.141	2.476	3.927	51¾
38½	0.783	0.623	1.565	1.245	2.348	1.868	3.130	2.490	3.913	51½
38¾	0.780	0.626	1.560	1.252	2.340	1.878	3.120	2.504	3.899	51¼
39°	0.777	0.629	1.554	1.259	2.331	1.888	3.109	2.517	3.886	**51°**
39¼	0.774	0.633	1.549	1.265	2.323	1.898	3.098	2.531	3.872	50¾
39½	0.772	0.636	1.543	1.272	2.315	1.908	3.086	2.544	3.858	50½
39¾	0.769	0.639	1.538	1.279	2.307	1.918	3.075	2.558	3.844	50¼
40°	0.766	0.643	1.532	1.286	2.298	1.928	3.064	2.571	3.830	**50°**
40¼	0.763	0.646	1.526	1.292	2.290	1.938	3.053	2.584	3.816	49¾
40½	0.760	0.649	1.521	1.299	2.281	1.948	3.042	2.598	3.802	49½
40¾	0.758	0.653	1.515	1.306	2.273	1.958	3.030	2.611	3.788	49¼
41°	0.755	0.656	1.509	1.312	2.264	1.968	3.019	2.624	3.774	**49°**
41¼	0.752	0.659	1.504	1.319	2.256	1.978	3.007	2.637	3.759	48¾
41½	0.749	0.663	1.498	1.325	2.247	1.988	2.996	2.650	3.745	48½
41¾	0.746	0.666	1.492	1.332	2.238	1.998	2.984	2.664	3.730	48¼
42°	0.743	0.669	1.486	1.338	2.229	2.007	2.973	2.677	3.716	**48°**
42¼	0.740	0.672	1.480	1.345	2.221	2.017	2.961	2.689	3.701	47¾
42½	0.737	0.676	1.475	1.351	2.212	2.027	2.949	2.702	3.686	47½
42¾	0.734	0.679	1.469	1.358	2.203	2.036	2.937	2.715	3.672	47¼
43°	0.731	0.682	1.463	1.364	2.194	2.046	2.925	2.728	3.657	**47°**
43¼	0.728	0.685	1.457	1.370	2.185	2.056	2.913	2.741	3.642	46¾
43½	0.725	0.688	1.451	1.377	2.176	2.065	2.901	2.753	3.627	46½
43¾	0.722	0.692	1.445	1.383	2.167	2.075	2.889	2.766	3.612	46¼
44°	0.719	0.695	1.439	1.389	2.158	2.084	2.877	2.779	3.597	**46°**
44¼	0.716	0.698	1.433	1.396	2.149	2.093	2.865	2.791	3.582	45¾
44½	0.713	0.701	1.427	1.402	2.140	2.103	2.853	2.804	3.566	45½
44¾	0.710	0.704	1.420	1.408	2.131	2.112	2.841	2.816	3.551	45¼
45°	0.707	0.707	1.414	1.414	2.121	2.121	2.828	2.828	3.536	**45°**
Bearing.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Bearing.
	1		2		3		4		5	

Bearing.	5	6		7		8		9		Bearing.
	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	
30°	2·500	5·196	3·000	6·062	3·500	6·928	4·000	7·794	4·500	60°
30¼	2·519	5·183	3·023	6·047	3·526	6·911	4·030	7·775	4·534	59¾
30½	2·538	5·170	3·045	6·031	3·553	6·893	4·060	7·755	4·568	59½
30¾	2·556	5·156	3·068	6·016	3·579	6·875	4·090	7·735	4·602	59¼
31°	2·575	5·143	3·090	6·000	3·605	6·857	4·120	7·715	4·635	59°
31¼	2·594	5·129	3·113	5·984	3·631	6·839	4·150	7·694	4·669	58¾
31½	2·612	5·116	3·135	5·968	3·657	6·821	4·180	7·674	4·702	58½
31¾	2·631	5·102	3·157	5·952	3·683	6·803	4·210	7·653	4·736	58¼
32°	2·650	5·088	3·180	5·936	3·709	6·784	4·239	7·632	4·769	58°
32¼	2·668	5·074	3·202	5·920	3·735	6·766	4·269	7·612	4·802	57¾
32½	2·686	5·060	3·224	5·904	3·761	6·747	4·298	7·591	4·836	57½
32¾	2·705	5·046	3·246	5·887	3·787	6·728	4·328	7·569	4·869	57¼
33°	2·723	5·032	3·268	5·871	3·812	6·709	4·357	7·548	4·902	57°
33¼	2·741	5·018	3·290	5·854	3·838	6·690	4·386	7·527	4·935	56¾
33½	2·760	5·003	3·312	5·837	3·864	6·671	4·416	7·505	4·967	56½
33¾	2·778	4·989	3·333	5·820	3·889	6·652	4·445	7·483	5·000	56¼
34°	2·796	4·974	3·355	5·803	3·914	6·632	4·474	7·461	5·033	56°
34¼	2·814	4·960	3·377	5·786	3·940	6·613	4·502	7·439	5·065	55¾
34½	2·832	4·945	3·398	5·769	3·965	6·593	4·531	7·417	5·098	55½
34¾	2·850	4·930	3·420	5·752	3·990	6·573	4·560	7·395	5·130	55¼
35°	2·868	4·915	3·441	5·734	4·015	6·553	4·589	7·372	5·162	55°
35¼	2·886	4·900	3·463	5·716	4·040	6·533	4·617	7·350	5·194	54¾
35½	2·904	4·885	3·484	5·699	4·065	6·513	4·646	7·327	5·226	54½
35¾	2·921	4·869	3·505	5·681	4·090	6·493	4·674	7·304	5·258	54¼
36°	2·939	4·854	3·527	5·663	4·115	6·472	4·702	7·281	5·290	54°
36¼	2·957	4·839	3·548	5·645	4·139	6·452	4·730	7·258	5·322	53¾
36½	2·974	4·823	3·569	5·627	4·164	6·431	4·759	7·235	5·353	53½
36¾	2·992	4·808	3·590	5·609	4·188	6·410	4·787	7·211	5·385	53¼
37°	3·009	4·792	3·611	5·590	4·213	6·389	4·815	7·188	5·416	53°
37¼	3·026	4·776	3·632	5·572	4·237	6·368	4·842	7·164	5·448	52¾
37½	3·044	4·760	3·653	5·554	4·261	6·347	4·870	7·140	5·479	52½
37¾	3·061	4·744	3·673	5·535	4·286	6·326	4·898	7·116	5·510	52¼
38°	3·078	4·728	3·694	5·516	4·310	6·304	4·925	7·092	5·541	52°
38¼	3·095	4·712	3·715	5·497	4·334	6·283	4·953	7·068	5·572	51¾
38½	3·113	4·696	3·735	5·478	4·358	6·261	4·980	7·043	5·603	51½
38¾	3·130	4·679	3·756	5·459	4·381	6·239	5·007	7·019	5·633	51¼
39°	3·147	4·663	3·776	5·440	4·405	6·217	5·035	6·994	5·664	51°
39¼	3·164	4·646	3·796	5·421	4·429	6·195	5·062	6·970	5·694	50¾
39½	3·180	4·630	3·816	5·401	4·453	6·173	5·089	6·945	5·725	50½
39¾	3·197	4·613	3·837	5·382	4·476	6·151	5·116	6·920	5·755	50¼
40°	3·214	4·596	3·857	5·362	4·500	6·128	5·142	6·894	5·785	50°
40¼	3·231	4·579	3·877	5·343	4·523	6·106	5·169	6·869	5·815	49¾
40½	3·247	4·562	3·897	5·323	4·546	6·083	5·196	6·844	5·845	49½
40¾	3·264	4·545	3·917	5·303	4·569	6·061	5·222	6·818	5·875	49¼
41°	3·280	4·528	3·936	5·283	4·592	6·038	5·248	6·792	5·905	49°
41¼	3·297	4·511	3·956	5·263	4·615	6·015	5·275	6·767	5·934	48¾
41½	3·313	4·494	3·976	5·243	4·638	5·992	5·301	6·741	5·964	48½
41¾	3·329	4·476	3·995	5·222	4·661	5·968	5·327	6·715	5·993	48¼
42°	3·346	4·459	4·015	5·202	4·684	5·945	5·353	6·688	6·022	48°
42¼	3·362	4·441	4·034	5·182	4·707	5·922	5·379	6·662	6·051	47¾
42½	3·378	4·424	4·054	5·161	4·729	5·898	5·405	6·635	6·080	47½
42¾	3·394	4·406	4·073	5·140	4·752	5·875	5·430	6·609	6·109	47¼
43°	3·410	4·388	4·092	5·119	4·774	5·851	5·456	6·582	6·138	47°
43¼	3·426	4·370	4·111	5·099	4·796	5·827	5·481	6·555	6·167	46¾
43½	3·442	4·352	4·130	5·078	4·818	5·803	5·507	6·528	6·195	46½
43¾	3·458	4·334	4·149	5·057	4·841	5·779	5·532	6·501	6·224	46¼
44°	3·473	4·316	4·168	5·035	4·863	5·755	5·557	6·474	6·252	46°
44¼	3·489	4·298	4·187	5·014	4·885	5·730	5·582	6·447	6·280	45¾
44½	3·505	4·280	4·206	4·993	4·906	5·706	5·607	6·419	6·308	45½
44¾	3·520	4·261	4·224	4·971	4·928	5·681	5·632	6·392	6·336	45¼
45°	3·536	4·243	4·243	4·950	4·950	5·657	5·657	6·364	6·364	45°
Bearing.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Bearing.
	5	6		7		8		9		

TABLE OF CHORDS: [Radius = 1.0000].

M.	0°	1°	2°	3°	4°	5°	6°	7°	8°	9°	10°	M.
0'	.0000	.0175	.0349	.0524	.0698	.0872	.1047	.1221	.1395	.1569	.1743	0'
1	.0003	.0177	.0352	.0526	.0701	.0875	.1050	.1224	.1398	.1572	.1746	1
2	.0006	.0180	.0355	.0529	.0704	.0878	.1053	.1227	.1401	.1575	.1749	2
3	.0009	.0183	.0358	.0532	.0707	.0881	.1055	.1230	.1404	.1578	.1752	3
4	.0012	.0186	.0361	.0535	.0710	.0884	.1058	.1233	.1407	.1581	.1755	4
5	.0015	.0189	.0364	.0538	.0713	.0887	.1061	.1235	.1410	.1584	.1758	5
6	.0017	.0192	.0366	.0541	.0715	.0890	.1064	.1238	.1413	.1587	.1761	6
7	.0020	.0195	.0369	.0544	.0718	.0893	.1067	.1241	.1415	.1589	.1763	7
8	.0023	.0198	.0372	.0547	.0721	.0896	.1070	.1244	.1418	.1592	.1766	8
9	.0026	.0201	.0375	.0550	.0724	.0899	.1073	.1247	.1421	.1595	.1769	9
10	.0029	.0204	.0378	.0553	.0727	.0901	.1076	.1250	.1424	.1598	.1772	10
11	.0032	.0207	.0381	.0556	.0730	.0904	.1079	.1253	.1427	.1601	.1775	11
12	.0035	.0209	.0384	.0558	.0733	.0907	.1082	.1256	.1430	.1604	.1778	12
13	.0038	.0212	.0387	.0561	.0736	.0910	.1084	.1259	.1433	.1607	.1781	13
14	.0041	.0215	.0390	.0564	.0739	.0913	.1087	.1262	.1436	.1610	.1784	14
15	.0044	.0218	.0393	.0567	.0742	.0916	.1090	.1265	.1439	.1613	.1787	15
16	.0047	.0221	.0396	.0570	.0745	.0919	.1093	.1267	.1442	.1616	.1789	16
17	.0049	.0224	.0398	.0573	.0747	.0922	.1096	.1270	.1444	.1618	.1792	17
18	.0052	.0227	.0401	.0576	.0750	.0925	.1099	.1273	.1447	.1621	.1795	18
19	.0055	.0230	.0404	.0579	.0753	.0928	.1102	.1276	.1450	.1624	.1798	19
20	.0058	.0233	.0407	.0582	.0756	.0931	.1105	.1279	.1453	.1627	.1801	20
21	.0061	.0236	.0410	.0585	.0759	.0933	.1108	.1282	.1456	.1630	.1804	21
22	.0064	.0239	.0413	.0588	.0762	.0936	.1111	.1285	.1459	.1633	.1807	22
23	.0067	.0241	.0416	.0590	.0765	.0939	.1114	.1288	.1462	.1636	.1810	23
24	.0070	.0244	.0419	.0593	.0768	.0942	.1116	.1291	.1465	.1639	.1813	24
25	.0073	.0247	.0422	.0596	.0771	.0945	.1119	.1294	.1468	.1642	.1816	25
26	.0076	.0250	.0425	.0599	.0774	.0948	.1122	.1296	.1471	.1645	.1818	26
27	.0079	.0253	.0428	.0602	.0776	.0951	.1125	.1299	.1473	.1647	.1821	27
28	.0081	.0256	.0430	.0605	.0779	.0954	.1128	.1302	.1476	.1650	.1824	28
29	.0084	.0259	.0433	.0608	.0782	.0957	.1131	.1305	.1479	.1653	.1827	29
30	.0087	.0262	.0436	.0611	.0785	.0960	.1134	.1308	.1482	.1656	.1830	30
31	.0090	.0265	.0439	.0614	.0788	.0962	.1137	.1311	.1485	.1659	.1833	31
32	.0093	.0268	.0442	.0617	.0791	.0965	.1140	.1314	.1488	.1662	.1836	32
33	.0096	.0271	.0445	.0619	.0794	.0968	.1143	.1317	.1491	.1665	.1839	33
34	.0099	.0273	.0448	.0622	.0797	.0971	.1145	.1320	.1494	.1668	.1842	34
35	.0102	.0276	.0451	.0625	.0800	.0974	.1148	.1323	.1497	.1671	.1845	35
36	.0105	.0279	.0454	.0628	.0803	.0977	.1151	.1325	.1500	.1674	.1847	36
37	.0108	.0282	.0457	.0631	.0806	.0980	.1154	.1328	.1502	.1676	.1850	37
38	.0111	.0285	.0460	.0634	.0808	.0983	.1157	.1331	.1505	.1679	.1853	38
39	.0113	.0288	.0462	.0637	.0811	.0986	.1160	.1334	.1508	.1682	.1856	39
40	.0116	.0291	.0465	.0640	.0814	.0989	.1163	.1337	.1511	.1685	.1859	40
41	.0119	.0294	.0468	.0643	.0817	.0992	.1166	.1340	.1514	.1688	.1862	41
42	.0122	.0297	.0471	.0646	.0820	.0994	.1169	.1343	.1517	.1691	.1865	42
43	.0125	.0300	.0474	.0649	.0823	.0997	.1172	.1346	.1520	.1694	.1868	43
44	.0128	.0303	.0477	.0651	.0826	.1000	.1175	.1349	.1523	.1697	.1871	44
45	.0131	.0305	.0480	.0654	.0829	.1003	.1177	.1352	.1526	.1700	.1873	45
46	.0134	.0308	.0483	.0657	.0832	.1006	.1180	.1355	.1529	.1703	.1876	46
47	.0137	.0311	.0486	.0660	.0835	.1009	.1183	.1357	.1531	.1705	.1879	47
48	.0140	.0314	.0489	.0663	.0838	.1012	.1186	.1360	.1534	.1708	.1882	48
49	.0143	.0317	.0492	.0666	.0840	.1015	.1189	.1363	.1537	.1711	.1885	49
50	.0145	.0320	.0494	.0669	.0843	.1018	.1192	.1366	.1540	.1714	.1888	50
51	.0148	.0323	.0497	.0672	.0846	.1021	.1195	.1369	.1543	.1717	.1891	51
52	.0151	.0326	.0500	.0675	.0849	.1023	.1198	.1372	.1546	.1720	.1894	52
53	.0154	.0329	.0503	.0678	.0852	.1026	.1201	.1375	.1549	.1723	.1897	53
54	.0157	.0332	.0506	.0681	.0855	.1029	.1204	.1378	.1552	.1726	.1900	54
55	.0160	.0335	.0509	.0683	.0858	.1032	.1206	.1381	.1555	.1729	.1902	55
56	.0163	.0337	.0512	.0686	.0861	.1035	.1209	.1384	.1558	.1732	.1905	56
57	.0166	.0340	.0515	.0689	.0864	.1038	.1212	.1386	.1560	.1734	.1908	57
58	.0169	.0343	.0518	.0692	.0867	.1041	.1215	.1389	.1563	.1737	.1911	58
59	.0172	0346	.0521	.0695	.0869	.1044	.1218	.1392	.1566	.1740	.1914	59
60	.0175	.0349	.0524	.0698	.0872	.1047	.1221	.1395	.1569	.1743	.1917	60

TABLE OF CHORDS: [Radius = 1.0000].

M.	11°	12°	13°	14°	15°	16°	17°	18°	19°	20°	21°	M.
0'	·1917	·2091	·2264	·2437	·2611	·2783	·2956	·3129	·3301	·3473	·3645	0'
1	·1920	·2093	·2267	·2440	·2613	·2786	·2959	·3132	·3304	·3476	·3648	1
2	·1923	·2096	·2270	·2443	·2616	·2789	·2962	·3134	·3307	·3479	·3650	2
3	·1926	·2099	·2273	·2446	·2619	·2792	·2965	·3137	·3310	·3482	·3653	3
4	·1928	·2102	·2276	·2449	·2622	·2795	·2968	·3140	·3312	·3484	·3656	4
5	·1931	·2105	·2279	·2452	·2625	·2798	·2971	·3143	·3315	·3487	·3659	5
6	·1934	·2108	·2281	·2455	·2628	·2801	·2973	·3146	·3318	·3490	·3662	6
7	·1937	·2111	·2284	·2458	·2631	·2804	·2976	·3149	·3321	·3493	·3665	7
8	·1940	·2114	·2287	·2460	·2634	·2807	·2979	·3152	·3324	·3496	·3668	8
9	·1943	·2117	·2290	·2463	·2636	·2809	·2982	·3155	·3327	·3499	·3670	9
10	·1946	·2119	·2293	·2466	·2639	·2812	·2985	·3157	·3330	·3502	·3673	10
11	·1949	·2122	·2296	·2469	·2642	·2815	·2988	·3160	·3333	·3504	·3676	11
12	·1952	·2125	·2299	·2472	·2645	·2818	·2991	·3163	·3335	·3507	·3679	12
13	·1955	·2128	·2302	·2475	·2648	·2821	·2994	·3166	·3338	·3510	·3682	13
14	·1957	·2131	·2305	·2478	·2651	·2824	·2996	·3169	·3341	·3513	·3685	14
15	·1960	·2134	·2307	·2481	·2654	·2827	·2999	·3172	·3344	·3516	·3688	15
16	·1963	·2137	·2310	·2484	·2657	·2830	·3002	·3175	·3347	·3519	·3690	16
17	·1966	·2140	·2313	·2486	·2660	·2832	·3005	·3178	·3350	·3522	·3693	17
18	·1969	·2143	·2316	·2489	·2662	·2835	·3008	·3180	·3353	·3525	·3696	18
19	·1972	·2146	·2319	·2492	·2665	·2838	·3011	3183	·3355	·3527	·3699	19
20	·1975	·2148	·2322	·2495	·2668	·2841	·3014	·3186	·3358	·3530	·3702	20
21	·1978	·2151	·2325	·2498	·2671	·2844	·3017	·3189	·3361	·3533	·3705	21
22	·1981	·2154	·2328	·2501	·2674	·2847	·3019	·3192	·3364	·3536	·3708	22
23	·1983	·2157	·2331	·2504	·2677	·2850	·3022	·3195	·3367	·3539	·3710	23
24	·1986	·2160	·2333	·2507	·2680	·2853	·3025	·3198	·3370	·3542	·3713	24
25	·1989	·2163	·2336	·2510	·2683	·2855	·3028	·3200	·3373	·3545	·3716	25
26	·1992	·2166	·2339	·2512	·2685	·2858	·3031	·3203	·3376	·3547	·3719	26
27	·1995	·2169	·2342	·2515	·2688	·2861	·3034	·3206	·3378	·3550	·3722	27
28	·1998	·2172	·2345	·2518	·2691	·2864	·3037	·3209	·3381	·3553	·3725	28
29	·2001	·2174	·2348	·2521	·2694	·2867	·3040	·3212	·3384	·3556	·3728	29
30	·2004	·2177	·2351	·2524	·2697	·2870	·3042	·3215	·3387	·3559	·3730	30
31	·2007	·2180	·2354	·2527	·2700	·2873	·3045	·3218	·3390	·3562	·3733	31
32	·2010	·2183	·2357	·2530	·2703	·2876	·3048	·3221	·3393	·3565	·3736	32
33	·2012	·2186	·2359	·2533	·2706	·2878	·3051	·3223	·3396	·3567	·3739	33
34	·2015	·2189	·2362	·2536	·2709	·2881	·3054	·3226	·3398	·3570	·3742	34
35	·2018	·2192	·2365	·2538	·2711	·2884	·3057	·3229	·3401	·3573	·3745	35
36	·2021	·2195	·2368	·2541	·2714	·2887	·3060	·3232	·3404	·3576	·3748	36
37	·2024	·2198	·2371	·2544	·2717	·2890	·3063	·3235	·3407	·3579	·3750	37
38	·2027	·2200	·2374	·2547	·2720	·2893	·3065	·3238	·3410	·3582	·3753	38
39	·2030	·2203	·2377	·2550	·2723	·2896	·3068	·3241	·3413	·3585	·3756	39
40	·2033	·2206	·2380	·2553	·2726	·2899	·3071	·3244	·3416	·3587	·3759	40
41	·2036	·2209	·2383	·2556	·2729	·2902	·3074	·3246	·3419	·3590	·3762	41
42	·2038	·2212	·2385	·2559	·2732	·2904	·3077	·3249	·3421	·3593	·3765	42
43	·2041	·2215	·2388	·2561	·2734	·2907	·3080	·3252	·3424	·3596	·3768	43
44	·2044	·2218	·2391	·2564	·2737	·2910	·3083	·3255	·3427	·3599	·3770	44
45	·2047	·2221	·2394	·2567	·2740	·2913	·3086	·3258	·3430	·3602	·3773	45
46	·2050	·2224	·2397	·2570	·2743	·2916	·3088	·3261	·3433	·3605	·3776	46
47	·2053	·2226	·2400	·2573	·2746	·2919	·3091	·3264	·3436	·3608	·3779	47
48	·2056	·2229	·2403	·2576	·2749	·2922	·3094	·3267	·3439	·3610	·3782	48
49	·2059	·2232	·2406	·2579	·2752	·2925	·3097	·3269	·3441	·3613	·3785	49
50	·2062	·2235	·2409	·2582	·2755	·2927	·3100	·3272	·3444	·3616	·3788	50
51	·2065	·2238	·2411	·2585	·2758	·2930	·3103	·3275	·3447	·3619	·3790	51
52	·2067	·2241	·2414	·2587	2760	·2933	·3106	·3278	·3450	·3622	·3793	52
53	·2070	·2244	·2417	·2590	·2763	·2936	·3109	·3281	·3453	·3625	·3796	53
54	·2073	·2247	·2420	·2593	·2766	·2939	·3111	·3284	·3456	·3628	·3799	54
55	·2076	·2250	·2423	·2596	·2769	·2942	·3114	·3287	·3459	·3630	·3802	55
56	·2079	·2253	·2426	·2599	·2772	·2945	·3117	·3289	·3462	·3633	·3805	56
57	·2082	·2255	·2429	·2602	·2775	·2948	·3120	·3292	·3464	·3636	·3808	57
58	·2085	·2258	·2432	·2605	·2778	·2950	·3123	·3295	·3467	·3639	·3810	58
59	·2088	·2261	·2434	·2608	·2781	·2953	·3126	·3298	·3470	·3642	·3813	59
60	·2091	·2264	·2437	·2611	·2783	·2956	·3129	·3301	·3473	·3645	·3816	60

TABLE OF CHORDS: [Radius = 1.0000].

M.	22°	23°	24°	25°	26°	27°	28°	29°	30°	31°	32°	M.
0'	.3816	.3987	.4158	.4329	.4499	.4669	.4838	.5008	.5176	.5345	.5513	0'
1	.3819	.3990	.4161	.4332	.4502	.4672	.4841	.5010	.5179	.5348	.5516	1
2	.3822	.3993	.4164	.4334	.4505	.4675	.4844	.5013	.5182	.5350	.5518	2
3	.3825	.3996	.4167	.4337	.4508	.4677	.4847	.5016	.5185	.5353	.5521	3
4	.3828	.3999	.4170	.4340	.4510	.4680	.4850	.5019	.5188	.5356	.5524	4
5	.3830	.4002	.4172	.4343	.4513	.4683	.4853	.5022	.5190	.5359	.5527	5
6	.3833	.4004	.4175	.4346	.4516	.4686	.4855	.5024	.5193	.5362	.5530	6
7	.3836	.4007	.4178	.4349	.4519	.4689	.4858	.5027	.5196	.5364	.5532	7
8	.3839	.4010	.4181	.4352	.4522	.4692	.4861	.5030	.5199	.5367	.5535	8
9	.3842	.4013	.4184	.4354	.4525	.4694	.4864	.5033	.5202	.5370	.5538	9
10	.3845	.4016	.4187	.4357	.4527	.4697	.4867	.5036	.5204	.5373	.5541	10
11	.3848	.4019	.4190	.4360	.4530	.4700	.4869	.5039	.5207	.5376	.5543	11
12	.3850	.4022	.4192	.4363	.4533	.4703	.4872	.5041	.5210	.5378	.5546	12
13	.3853	.4024	.4195	.4366	.4536	.4706	.4875	.5044	.5213	.5381	.5549	13
14	.3856	.4027	.4198	.4369	.4539	.4708	.4878	.5047	.5216	.5384	.5552	14
15	.3859	.4030	.4201	.4371	.4542	.4711	.4881	.5050	.5219	.5387	.5555	15
16	.3862	.4033	.4204	.4374	.4544	.4714	.4884	.5053	.5221	.5390	.5557	16
17	.3865	.4036	.4207	.4377	.4547	.4717	.4886	.5055	.5224	.5392	.5560	17
18	.3868	.4039	.4209	.4380	.4550	.4720	.4889	.5058	.5227	.5395	.5563	18
19	.3870	.4042	.4212	.4383	.4553	.4723	.4892	.5061	.5230	.5398	.5566	19
20	.3873	.4044	.4215	.4386	.4556	.4725	.4895	.5064	.5233	.5401	.5569	20
21	.3876	.4047	.4218	.4388	.4559	.4728	.4898	.5067	.5235	.5404	.5571	21
22	.3879	.4050	.4221	.4391	.4561	.4731	.4901	.5070	.5238	.5406	.5574	22
23	.3882	.4053	.4224	.4394	.4564	.4734	.4903	.5072	.5241	.5409	.5577	23
24	.3885	.4056	.4226	.4397	.4567	.4737	.4906	.5075	.5244	.5412	.5580	24
25	.3888	.4059	.4229	.4400	.4570	.4740	.4909	.5078	.5247	.5415	.5583	25
26	.3890	.4061	.4232	.4403	.4573	.4742	.4912	.5081	.5249	.5418	.5585	26
27	.3893	.4064	.4235	.4405	.4576	.4745	.4915	.5084	.5252	.5420	.5588	27
28	.3896	.4067	.4238	.4408	.4578	.4748	.4917	.5086	.5255	.5423	.5591	28
29	.3899	.4070	.4241	.4411	.4581	.4751	.4920	.5089	.5258	.5426	.5594	29
30	.3902	.4073	.4244	.4414	.4584	.4754	.4923	.5092	.5261	.5429	.5597	30
31	.3905	.4076	.4246	.4417	.4587	.4757	.4926	.5095	.5263	.5432	.5599	31
32	.3908	.4079	.4249	.4420	.4590	.4759	.4929	.5098	.5266	.5434	.5602	32
33	.3910	.4081	.4252	.4422	.4593	.4762	.4932	.5100	.5269	.5437	.5605	33
34	.3913	.4084	.4255	.4425	.4595	.4765	.4934	.5103	.5272	.5440	.5608	34
35	.3916	.4087	.4258	.4428	.4598	.4768	.4937	.5106	.5275	.5443	.5611	35
36	.3919	.4090	.4261	.4431	.4601	.4771	.4940	.5109	.5277	.5446	.5613	36
37	.3922	.4093	.4263	.4434	.4604	.4773	.4943	.5112	.5280	.5448	.5616	37
38	.3925	.4096	.4266	.4437	.4607	.4776	.4946	.5115	.5283	.5451	.5619	38
39	.3927	.4098	.4269	.4439	.4609	.4779	.4948	.5117	.5286	.5454	.5622	39
40	.3930	.4101	.4272	.4442	.4612	.4782	.4951	.5120	.5289	.5457	.5625	40
41	.3933	.4104	.4275	.4445	.4615	.4785	.4954	.5123	.5291	.5460	.5627	41
42	.3936	.4107	.4278	.4448	.4618	.4788	.4957	.5126	.5294	.5462	.5630	42
43	.3939	.4110	.4280	.4451	.4621	.4790	.4960	.5129	.5297	.5465	.5633	43
44	.3942	.4113	.4283	.4454	.4624	.4793	.4963	.5131	.5300	.5468	.5636	44
45	.3945	.4116	.4286	.4456	.4626	.4796	.4965	.5134	.5303	.5471	.5638	45
46	.3947	.4118	.4289	.4459	.4629	.4799	.4968	.5137	.5306	.5474	.5641	46
47	.3950	.4121	.4292	.4462	.4632	.4802	.4971	.5140	.5308	.5476	.5644	47
48	.3953	.4124	.4295	.4465	.4635	.4805	.4974	.5143	.5311	.5479	.5647	48
49	.3956	.4127	.4298	.4468	.4638	.4807	.4977	.5145	.5314	.5482	.5650	49
50	.3959	.4130	.4300	.4471	.4641	.4810	.4979	.5148	.5317	.5485	.5652	50
51	.3962	.4133	.4303	.4474	.4643	.4813	.4982	.5151	.5320	.5488	.5655	51
52	.3965	.4135	.4306	.4476	.4646	.4816	.4985	.5154	.5322	.5490	.5658	52
53	.3967	.4138	.4309	.4479	.4649	.4819	.4988	.5157	.5325	.5493	.5661	53
54	.3970	.4141	.4312	.4482	.4652	.4822	.4991	.5160	.5328	.5496	.5664	54
55	.3973	.4144	.4315	.4485	.4655	.4824	.4994	.5162	.5331	.5499	.5666	55
56	.3976	.4147	.4317	.4488	.4658	.4827	.4996	.5165	.5334	.5502	.5669	56
57	.3979	.4150	.4320	.4491	.4660	.4830	.4999	.5168	.5336	.5504	.5672	57
58	.3982	.4153	.4323	.4493	.4663	.4833	.5002	.5171	.5339	.5507	.5675	58
59	.3985	.4155	.4326	.4496	.4666	.4836	.5005	.5174	.5342	.5510	.5678	59
60	.3987	.4158	.4329	.4499	.4669	.4838	.5008	.5176	.5345	.5513	.5680	60

TABLE OF CHORDS: [Radius = 1.0000].

M.	33°	34°	35°	36°	37°	38°	39°	40°	41°	42°	43°	M.
0′	.5680	.5847	.6014	.6180	.6346	.6511	.6676	.6840	.7004	.7167	.7330	0′
1	.5683	.5850	.6017	.6183	.6349	.6514	.6679	.6843	.7007	.7170	.7333	1
2	.5686	.5853	.6020	.6186	.6352	.6517	.6682	.6846	.7010	.7173	.7335	2
3	.5689	.5856	.6022	.6189	.6354	.6520	.6684	.6849	.7012	.7176	.7338	3
4	.5691	.5859	.6025	.6191	.6357	.6522	.6687	.6851	.7015	.7178	.7341	4
5	.5694	.5861	.6028	.6194	.6360	.6525	.6690	.6854	.7018	.7181	.7344	5
6	.5697	.5864	.6031	.6197	.6363	.6528	.6693	.6857	.7020	.7184	.7346	6
7	.5700	.5867	.6034	.6200	.6365	.6531	.6695	.6860	.7023	.7186	.7349	7
8	.5703	.5870	.6036	.6202	.6368	.6533	.6698	.6862	.7026	.7189	.7352	8
9	.5705	.5872	.6039	.6205	.6371	.6536	.6701	.6865	.7029	.7192	.7354	9
10	.5708	.5875	.6042	.6208	.6374	.6539	.6704	.6868	.7031	.7195	.7357	10
11	.5711	.5878	.6045	.6211	.6376	.6542	.6706	.6870	.7034	.7197	.7360	11
12	.5714	.5881	.6047	.6214	.6379	.6544	.6709	.6873	.7037	.7200	.7362	12
13	.5717	.5884	.6050	.6216	.6382	.6547	.6712	.6876	.7040	.7203	.7365	13
14	.5719	.5886	.6053	.6219	.6385	.6550	.6715	.6879	.7042	.7205	.7368	14
15	.5722	.5889	.6056	.6222	.6387	.6553	.6717	.6881	.7045	.7208	.7371	15
16	.5725	.5892	.6058	.6225	.6390	.6555	.6720	.6884	.7048	.7211	.7373	16
17	.5728	.5895	.6061	.6227	.6393	.6558	.6723	.6887	.7050	.7214	.7376	17
18	.5730	.5897	.6064	.6230	.6396	.6561	.6725	.6890	.7053	.7216	.7379	18
19	.5733	.5900	.6067	.6233	.6398	.6564	.6728	.6892	.7056	.7219	.7381	19
20	.5736	.5903	.6070	.6236	.6401	.6566	.6731	.6895	.7059	.7222	.7384	20
21	.5739	.5906	.6072	.6238	.6404	.6569	.6734	.6898	.7061	.7224	.7387	21
22	.5742	.5909	.6075	.6241	.6407	.6572	.6736	.6901	.7064	.7227	.7390	22
23	.5744	.5911	.6078	.6244	.6410	.6575	.6739	.6903	.7067	.7230	.7392	23
24	.5747	.5914	.6081	.6247	.6412	.6577	.6742	.6906	.7069	.7232	.7395	24
25	.5750	.5917	.6083	.6249	.6415	.6580	.6745	.6909	.7072	.7235	.7398	25
26	.5753	.5920	.6086	.6252	.6418	.6583	.6747	.6911	.7075	.7238	.7400	26
27	.5756	.5922	.6089	.6255	.6421	.6586	.6750	.6914	.7078	.7241	.7403	27
28	.5758	.5925	.6092	.6258	.6423	.6588	.6753	.6917	.7080	.7243	.7406	28
29	.5761	.5928	.6095	.6260	.6426	.6591	.6756	.6920	.7083	.7246	.7408	29
30	.5764	.5931	.6097	.6263	.6429	.6594	.6758	.6922	.7086	.7249	.7411	30
31	.5767	.5934	.6100	.6266	.6432	.6597	.6761	.6925	.7089	.7251	.7414	31
32	.5769	.5936	.6103	.6269	.6434	.6599	.6764	.6928	.7091	.7254	.7417	32
33	.5772	.5939	.6106	.6272	.6437	.6602	.6767	.6931	.7094	.7257	.7419	33
34	.5775	.5942	.6108	.6274	.6440	.6605	.6769	.6933	.7097	.7260	.7422	34
35	.5778	.5945	.6111	.6277	.6443	.6608	.6772	.6936	.7099	.7262	.7425	35
36	.5781	.5947	.6114	.6280	.6445	.6610	.6775	.6939	.7102	.7265	.7427	36
37	.5783	.5950	.6117	.6283	.6448	.6613	.6777	.6941	.7105	.7268	.7430	37
38	.5786	.5953	.6119	.6285	.6451	.6616	.6780	.6944	.7108	.7270	.7433	38
39	.5789	.5956	.6122	.6288	.6454	.6619	.6783	.6947	.7110	.7273	.7435	39
40	.5792	.5959	.6125	.6291	.6456	.6621	.6786	.6950	.7113	.7276	.7438	40
41	.5795	.5961	.6128	.6294	.6459	.6624	.6788	.6952	.7116	.7279	.7441	41
42	.5797	.5964	.6130	.6296	.6462	.6627	.6791	.6955	.7118	.7281	.7443	42
43	.5800	.5967	.6133	.6299	.6465	.6630	.6794	.6958	.7121	.7284	.7446	43
44	.5803	.5970	.6136	.6302	.6467	.6632	.6797	.6961	.7124	.7287	.7449	44
45	.5806	.5972	.6139	.6305	.6470	.6635	.6799	.6963	.7127	.7289	.7452	45
46	.5808	.5975	.6142	.6307	.6473	.6638	.6802	.6966	.7129	.7292	.7454	46
47	.5811	.5978	.6144	.6310	.6476	.6640	.6805	.6969	.7132	.7295	.7457	47
48	.5814	.5981	.6147	.6313	.6478	.6643	.6808	.6971	.7135	.7298	.7460	48
49	.5817	.5984	.6150	.6316	.6481	.6646	.6810	.6974	.7137	.7300	.7462	49
50	.5820	.5986	.6153	.6318	.6484	.6649	.6813	.6977	.7140	.7303	.7465	50
51	.5822	.5989	.6155	.6321	.6487	.6651	.6816	.6980	.7143	.7306	.7468	51
52	.5825	.5992	.6158	.6324	.6489	.6654	.6819	.6982	.7146	.7308	.7471	52
53	.5828	.5995	.6161	.6327	.6492	.6657	.6821	.6985	.7148	.7311	.7473	53
54	.5831	.5997	.6164	.6330	.6495	.6660	.6824	.6988	.7151	.7314	.7476	54
55	.5834	.6000	.6166	.6332	.6498	.6662	.6827	.6991	.7154	.7316	.7479	55
56	.5836	.6003	.6169	.6335	.6500	.6665	.6829	.6993	.7156	.7319	.7481	56
57	.5839	.6006	.6172	.6338	.6503	.6668	.6832	.6996	.7159	.7322	.7484	57
58	.5842	.6009	.6175	.6341	.6506	.6671	.6835	.6999	.7162	.7325	.7487	58
59	.5845	.6011	.6178	.6343	.6509	.6673	.6838	.7001	.7165	.7327	.7489	59
60	.5847	.6014	.6180	.6346	.6511	.6676	.6840	.7004	.7167	.7330	.7492	60

TABLE OF CHORDS: [Radius = 1.0000].

M.	44°	45°	46°	47°	48°	49°	50°	51°	52°	53°	54°	M.
0'	·7492	·7654	·7815	·7975	·8135	·8294	·8452	·8610	·8767	·8924	·9080	0'
1	·7495	·7656	·7817	·7978	·8137	·8297	·8455	·8613	·8770	·8927	·9082	1
2	·7498	·7659	·7820	·7980	·8140	·8299	·8458	·8615	·8773	·8929	·9085	2
3	·7500	·7662	·7823	·7983	·8143	·8302	·8460	·8618	·8775	·8932	·9088	3
4	·7503	·7664	·7825	·7986	·8145	·8304	·8463	·8621	·8778	·8934	·9090	4
5	·7506	·7667	·7828	·7988	·8148	·8307	·8466	·8623	·8780	·8937	·9093	5
6	·7508	·7670	·7831	·7991	·8151	·8310	·8468	·8626	·8783	·8940	·9095	6
7	·7511	·7672	·7833	·7994	·8153	·8312	·8471	·8629	·8786	·8942	·9098	7
8	·7514	·7675	·7836	·7996	·8156	·8315	·8473	·8631	·8788	·8945	·9101	8
9	·7516	·7678	·7839	·7999	·8159	·8318	·8476	·8634	·8791	·8947	·9103	9
10	·7519	·7681	·7841	·8002	·8161	·8320	·8479	·8636	·8794	·8950	·9106	10
11	·7522	·7683	·7844	·8004	·8164	·8323	·8481	·8639	·8796	·8953	·9108	11
12	·7524	·7686	·7847	·8007	·8167	·8326	·8484	·8642	·8799	·8955	·9111	12
13	·7527	·7689	·7849	·8010	·8169	·8328	·8487	·8644	·8801	·8958	·9113	13
14	·7530	·7691	·7852	·8012	·8172	·8331	·8489	·8647	·8804	·8960	·9116	14
15	·7533	·7694	·7855	·8015	·8175	·8334	·8492	·8650	·8807	·8963	·9119	15
16	·7535	·7697	·7857	·8018	·8177	·8336	·8495	·8652	·8809	·8966	·9121	16
17	·7538	·7699	·7860	·8020	·8180	·8339	·8497	·8655	·8812	·8968	·9124	17
18	·7541	·7702	·7863	·8023	·8183	·8341	·8500	·8657	·8814	·8971	·9126	18
19	·7543	·7705	·7865	·8026	·8185	·8344	·8502	·8660	·8817	·8973	·9129	19
20	·7546	·7707	·7868	·8028	·8188	·8347	·8505	·8663	·8820	·8976	·9132	20
21	·7549	·7710	·7871	·8031	·8190	·8349	·8508	·8665	·8822	·8979	·9134	21
22	·7551	·7713	·7873	·8034	·8193	·8352	·8510	·8668	·8825	·8981	·9137	22
23	·7554	·7715	·7876	·8036	·8196	·8355	·8513	·8671	·8828	·8984	·9139	23
24	·7557	·7718	·7879	·8039	·8198	·8357	·8516	·8673	·8830	·8986	·9142	24
25	·7560	·7721	·7882	·8042	·8201	·8360	·8518	·8676	·8833	·8989	·9145	25
26	·7562	·7723	·7884	·8044	·8204	·8363	·8521	·8678	·8835	·8992	·9147	26
27	·7565	·7726	·7887	·8047	·8206	·8365	·8523	·8681	·8838	·8994	·9150	27
28	·7568	·7729	·7890	·8050	·8209	·8368	·8526	·8684	·8841	·8997	·9152	28
29	·7570	·7731	·7892	·8052	·8212	·8371	·8529	·8686	·8843	·8999	·9155	29
30	·7573	·7734	·7895	·8055	·8214	·8373	·8531	·8689	·8846	·9002	·9157	30
31	·7576	·7737	·7898	·8058	·8217	·8376	·8534	·8692	·8848	·9005	·9160	31
32	·7578	·7740	·7900	·8060	·8220	·8378	·8537	·8694	·8851	·9007	·9163	32
33	·7581	·7742	·7903	·8063	·8222	·8381	·8539	·8697	·8854	·9010	·9165	33
34	·7584	·7745	·7906	·8066	·8225	·8384	·8542	·8699	·8856	·9012	·9168	34
35	·7586	·7748	·7908	·8068	·8228	·8386	·8545	·8702	·8859	·9015	·9170	35
36	·7589	·7750	·7911	·8071	·8230	·8389	·8547	·8705	·8861	·9018	·9173	36
37	·7592	·7753	·7914	·8074	·8233	·8392	·8550	·8707	·8864	·9020	·9176	37
38	·7595	·7756	·7916	·8076	·8236	·8394	·8552	·8710	·8867	·9023	·9178	38
39	·7597	·7758	·7919	·8079	·8238	·8397	·8555	·8712	·8869	·9025	·9181	39
40	·7600	·7761	·7922	·8082	·8241	·8400	·8558	·8715	·8872	·9028	·9183	40
41	·7603	·7764	·7924	·8084	·8244	·8402	·8560	·8718	·8874	·9031	·9186	41
42	·7605	·7766	·7927	·8087	·8246	·8405	·8563	·8720	·8877	·9033	·9188	42
43	·7608	·7769	·7930	·8090	·8249	·8408	·8566	·8723	·8880	·9036	·9191	43
44	·7611	·7772	·7932	·8092	·8251	·8410	·8568	·8726	·8882	·9038	·9194	44
45	·7613	·7774	·7935	·8095	·8254	·8413	·8571	·8728	·8885	·9041	·9196	45
46	·7616	·7777	·7938	·8098	·8257	·8415	·8573	·8731	·8887	·9044	·9199	46
47	·7619	·7780	·7940	·8100	·8259	·8418	·8576	·8734	·8890	·9046	·9201	47
48	·7621	·7782	·7943	·8103	·8262	·8421	·8579	·8736	·8893	·9049	·9204	48
49	·7624	·7785	·7946	·8105	·8265	·8423	·8581	·8739	·8895	·9051	·9207	49
50	·7627	·7788	·7948	·8108	·8267	·8426	·8584	·8741	·8898	·9054	·9209	50
51	·7629	·7791	·7951	·8111	·8270	·8429	·8587	·8744	·8900	·9056	·9212	51
52	·7632	·7793	·7954	·8113	·8273	·8431	·8589	·8747	·8903	·9059	·9214	52
53	·7635	·7796	·7956	·8116	·8275	·8434	·8592	·8749	·8906	·9062	·9217	53
54	·7638	·7799	·7959	·8119	·8278	·8437	·8594	·8752	·8908	·9064	·9219	54
55	·7640	·7801	·7962	·8121	·8281	·8439	·8597	·8754	·8911	·9067	·9222	55
56	·7643	·7804	·7964	·8124	·8283	·8442	·8600	·8757	·8914	·9069	·9225	56
57	·7646	·7807	·7967	·8127	·8286	·8444	·8602	·8760	·8916	·9072	·9227	57
58	·7648	·7809	·7970	·8129	·8289	·8447	·8605	·8762	·8919	·9075	·9230	58
59	·7651	·7812	·7972	·8132	·8291	·8450	·8608	·8765	·8921	·9077	·9232	59
60	·7654	·7815	·7975	·8135	·8294	·8452	·8610	·8767	·8924	·9080	·9235	60

TABLE OF CHORDS: [Radius = 1.0000].

M.	55°	56°	57°	58°	59°	60°	61°	62°	63°	64°	M.
0′	.9235	.9389	.9543	.9696	.9848	1.0000	1.0151	1.0301	1.0450	1.0598	0′
1	.9238	.9392	.9546	.9699	.9851	1.0003	1.0153	1.0303	1.0452	1.0601	1
2	.9240	.9395	.9548	.9701	.9854	1.0005	1.0156	1.0306	1.0455	1.0603	2
3	.9243	.9397	.9551	.9704	.9856	1.0008	1.0158	1.0308	1.0457	1.0606	3
4	.9245	.9400	.9553	.9706	.9859	1.0010	1.0161	1.0311	1.0460	1.0608	4
5	.9248	.9402	.9556	.9709	.9861	1.0013	1.0163	1.0313	1.0462	1.0611	5
6	.9250	.9405	.9559	.9711	.9864	1.0015	1.0166	1.0316	1.0465	1.0613	6
7	.9253	.9407	.9561	.9714	.9866	1.0018	1.0168	1.0318	1.0467	1.0616	7
8	.9256	.9410	.9564	.9717	.9869	1.0020	1.0171	1.0321	1.0470	1.0618	8
9	.9258	.9413	.9566	.9719	.9871	1.0023	1.0173	1.0323	1.0472	1.0621	9
10	.9261	.9415	.9569	.9722	.9874	1.0025	1.0176	1.0326	1.0475	1.0623	10
11	.9263	.9418	.9571	.9724	.9876	1.0028	1.0178	1.0328	1.0477	1.0626	11
12	.9266	.9420	.9574	.9727	.9879	1.0030	1.0181	1.0331	1.0480	1.0628	12
13	.9268	.9423	.9576	.9729	.9881	1.0033	1.0183	1.0333	1.0482	1.0630	13
14	.9271	.9425	.9579	.9732	.9884	1.0035	1.0186	1.0336	1.0485	1.0633	14
15	.9274	.9428	.9581	.9734	.9886	1.0038	1.0188	1.0338	1.0487	1.0635	15
16	.9276	.9430	.9584	.9737	.9889	1.0040	1.0191	1.0341	1.0490	1.0638	16
17	.9279	.9433	.9587	.9739	.9891	1.0043	1.0193	1.0343	1.0492	1.0640	17
18	.9281	.9436	.9589	.9742	.9894	1.0045	1.0196	1.0346	1.0495	1.0643	18
19	.9284	.9438	.9592	.9744	.9897	1.0048	1.0198	1.0348	1.0497	1.0645	19
20	.9287	.9441	.9594	.9747	.9899	1.0050	1.0201	1.0351	1.0500	1.0648	20
21	.9289	.9443	.9597	.9750	.9902	1.0053	1.0203	1.0353	1.0502	1.0650	21
22	.9292	.9446	.9599	.9752	.9904	1.0055	1.0206	1.0356	1.0504	1.0653	22
23	.9294	.9448	.9602	.9755	.9907	1.0058	1.0208	1.0358	1.0507	1.0655	23
24	.9297	.9451	.9604	.9757	.9909	1.0060	1.0211	1.0361	1.0509	1.0658	24
25	.9299	.9454	.9607	.9760	.9912	1.0063	1.0213	1.0363	1.0512	1.0660	25
26	.9302	.9456	.9610	.9762	.9914	1.0065	1.0216	1.0366	1.0514	1.0662	26
27	.9305	.9459	.9612	.9765	.9917	1.0068	1.0218	1.0368	1.0517	1.0665	27
28	.9307	.9461	.9615	.9767	.9919	1.0070	1.0221	1.0370	1.0519	1.0667	28
29	.9310	.9464	.9617	.9770	.9922	1.0073	1.0223	1.0373	1.0522	1.0670	29
30	.9312	.9466	.9620	.9772	.9924	1.0075	1.0226	1.0375	1.0524	1.0672	30
31	.9315	.9469	.9622	.9775	.9927	1.0078	1.0228	1.0378	1.0527	1.0675	31
32	.9317	.9472	.9625	.9778	.9929	1.0080	1.0231	1.0380	1.0529	1.0677	32
33	.9320	.9474	.9627	.9780	.9932	1.0083	1.0233	1.0383	1.0532	1.0680	33
34	.9323	.9477	.9630	.9783	.9934	1.0086	1.0236	1.0385	1.0534	1.0682	34
35	.9325	.9479	.9633	.9785	.9937	1.0088	1.0238	1.0388	1.0537	1.0685	35
36	.9328	.9482	.9635	.9788	.9939	1.0091	1.0241	1.0390	1.0539	1.0687	36
37	.9330	.9484	.9638	.9790	.9942	1.0093	1.0243	1.0393	1.0542	1.0690	37
38	.9333	.9487	.9640	.9793	.9945	1.0096	1.0246	1.0395	1.0544	1.0692	38
39	.9335	.9489	.9643	.9795	.9947	1.0098	1.0248	1.0398	1.0547	1.0694	39
40	.9338	.9492	.9645	.9798	.9950	1.0101	1.0251	1.0400	1.0549	1.0697	40
41	.9341	.9495	.9648	.9800	.9952	1.0103	1.0253	1.0403	1.0551	1.0699	41
42	.9343	.9497	.9650	.9803	.9955	1.0106	1.0256	1.0405	1.0554	1.0702	42
43	.9346	.9500	.9653	.9805	.9957	1.0108	1.0258	1.0408	1.0556	1.0704	43
44	.9348	.9502	.9655	.9808	.9960	1.0111	1.0261	1.0410	1.0559	1.0707	44
45	.9351	.9505	.9658	.0810	.9962	1.0113	1.0263	1.0413	1.0561	1.0709	45
46	.9353	.9507	.9661	.9813	.9965	1.0116	1.0266	1.0415	1.0564	1.0712	46
47	.9356	.9510	.9663	.9816	.9967	1.0118	1.0268	1.0418	1.0566	1.0714	47
48	.9359	.9512	.9666	.9818	.9970	1.0121	1.0271	1.0420	1.0569	1.0717	48
49	.9361	.9515	.9668	.9821	.9972	1.0123	1.0273	1.0423	1.0571	1.0719	49
50	.9364	.9518	.9671	.9823	.9975	1.0126	1.0276	1.0425	1.0574	1.0721	50
51	.9366	.9520	.9673	.9826	.9977	1.0128	1.0278	1.0428	1.0576	1.0724	51
52	.9369	.9523	.9676	.9828	.9980	1.0131	1.0281	1.0430	1.0579	1.0726	52
53	.9371	.9525	.9678	.9831	.9982	1.0133	1.0283	1.0433	1.0581	1.0729	53
54	.9374	.9528	.9681	.9833	.9985	1.0136	1.0286	1.0435	1.0584	1.0731	54
55	.9377	.9530	.9683	.9836	.9987	1.0138	1.0288	1.0438	1.0586	1.0734	55
56	.9379	.9533	.9686	.9838	.9990	1.0141	1.0291	1.0440	1.0589	1.0736	56
57	.9382	.9536	.9689	.9841	.9992	1.0143	1.0293	1.0443	1.0591	1.0739	57
58	.9384	.9538	.9691	.9843	.9995	1.0146	1.0296	1.0445	1.0593	1.0741	58
59	.9387	.9541	.9694	.9846	.9998	1.0148	1.0298	1.0447	1.0596	1.0744	59
60	.9389	.9543	.9696	.9848	10000	1.0151	1.0301	1.0450	1.0598	1.0746	60

TABLE OF CHORDS: [Radius = 1.0000].

M.	65°	66°	67°	68°	69°	70°	71°	72°	73°	M.
0′	1·0746	1·0893	1·1039	1·1184	1·1328	1·1472	1·1614	1·1756	1·1896	0′
1	1·0748	1·0895	1·1041	1·1186	1·1331	1·1474	1·1616	1·1758	1·1899	1
2	1·0751	1·0898	1·1044	1·1189	1·1333	1·1476	1·1619	1·1760	1·1901	2
3	1·0753	1·0900	1·1046	1·1191	1·1335	1·1479	1·1621	1·1763	1·1903	3
4	1·0756	1·0903	1·1048	1·1194	1·1338	1·1481	1·1624	1·1765	1·1906	4
5	1·0758	1·0905	1·1051	1·1196	1·1340	1·1483	1·1626	1·1767	1·1908	5
6	1·0761	1·0907	1·1053	1·1198	1·1342	1·1486	1·1628	1·1770	1·1910	6
7	1·0763	1·0910	1·1056	1·1201	1·1345	1·1488	1·1631	1·1772	1·1913	7
8	1·0766	1·0912	1·1058	1·1203	1·1347	1·1491	1·1633	1·1775	1·1915	8
9	1·0768	1·0915	1·1061	1·1206	1·1350	1·1493	1·1635	1·1777	1·1917	9
10	1·0771	1·0917	1·1063	1·1208	1·1352	1·1495	1·1638	1·1779	1·1920	10
11	1·0773	1·0920	1·1065	1·1210	1·1354	1·1498	1·1640	1·1782	1·1922	11
12	1·0775	1·0922	1·1068	1·1213	1·1357	1·1500	1·1642	1·1784	1·1924	12
13	1·0778	1·0924	1·1070	1·1215	1·1359	1·1502	1·1645	1·1786	1·1927	13
14	1·0780	1·0927	1·1073	1·1218	1·1362	1·1505	1·1647	1·1789	1·1929	14
15	1·0783	1·0929	1·1075	1·1220	1·1364	1·1507	1·1650	1·1791	1·1931	15
16	1·0785	1·0932	1·1078	1·1222	1·1366	1·1510	1·1652	1·1793	1·1934	16
17	1·0788	1·0934	1·1080	1·1225	1·1369	1·1512	1·1654	1·1796	1·1936	17
18	1·0790	1·0937	1·1082	1·1227	1·1371	1·1514	1·1657	1·1798	1·1938	18
19	1·0793	1·0939	1·1085	1·1230	1·1374	1·1517	1·1659	1·1800	1·1941	19
20	1·0795	1·0942	1·1087	1·1232	1·1376	1·1519	1·1661	1·1803	1·1943	20
21	1·0797	1·0944	1·1090	1·1234	1·1378	1·1522	1·1664	1·1805	1·1946	21
22	1·0800	1·0946	1·1092	1·1237	1·1381	1·1524	1·1666	1·1807	1·1948	22
23	1·0802	1·0949	1·1094	1·1239	1·1383	1·1526	1·1668	1·1810	1·1950	23
24	1·0805	1·0951	1·1097	1·1242	1·1386	1·1529	1·1671	1·1812	1·1952	24
25	1·0807	1·0954	1·1099	1·1244	1·1388	1·1531	1·1673	1·1814	1·1955	25
26	1·0810	1·0956	1·1102	1·1246	1·1390	1·1533	1·1676	1·1817	1·1957	26
27	1·0812	1·0959	1·1104	1·1249	1·1393	1·1536	1·1678	1·1819	1·1959	27
28	1·0815	1·0961	1·1107	1·1251	1·1395	1·1538	1·1680	1·1821	1·1962	28
29	1·0817	1·0963	1·1109	1·1254	1·1398	1·1541	1·1683	1·1824	1·1964	29
30	1·0820	1·0966	1·1111	1·1256	1·1400	1·1543	1·1685	1·1826	1·1966	30
31	1·0822	1·0968	1·1114	1·1258	1·1402	1·1545	1·1687	1·1829	1·1969	31
32	1·0824	1·0971	1·1116	1·1261	1·1405	1·1548	1·1690	1·1831	1·1971	32
33	1·0827	1·0973	1·1119	1·1263	1·1407	1·1550	1·1692	1·1833	1·1973	33
34	1·0829	1·0976	1·1121	1·1266	1·1409	1·1552	1·1694	1·1836	1·1976	34
35	1·0832	1·0978	1·1123	1·1268	1·1412	1·1555	1·1697	1·1838	1·1978	35
36	1·0834	1·0980	1·1126	1·1271	1·1414	1·1557	1·1699	1·1840	1·1980	36
37	1·0837	1·0983	1·1128	1·1273	1·1417	1·1560	1·1702	1·1843	1·1983	37
38	1·0839	1·0985	1·1131	1·1275	1·1419	1·1562	1·1704	1·1845	1·1985	38
39	1·0841	1·0988	1·1133	1·1278	1·1421	1·1564	1·1706	1·1847	1·1987	39
40	1·0844	1·0990	1·1136	1·1280	1·1424	1·1567	1·1709	1·1850	1·1990	40
41	1·0846	1·0993	1·1138	1·1283	1·1426	1·1569	1·1711	1·1852	1·1992	41
42	1·0849	1·0995	1·1140	1·1285	1·1429	1·1571	1·1713	1·1854	1·1994	42
43	1·0851	1·0997	1·1143	1·1287	1·1431	1·1574	1·1716	1·1857	1·1997	43
44	1·0854	1·1000	1·1145	1·1290	1·1433	1·1576	1·1718	1·1859	1·1999	44
45	1·0856	1·1002	1·1148	1·1292	1·1436	1·1579	1·1720	1·1861	1·2001	45
46	1·0859	1·1005	1·1150	1·1295	1·1438	1·1581	1·1723	1·1864	1·2004	46
47	1·0861	1·1007	1·1152	1·1297	1·1441	1·1583	1·1725	1·1866	1·2006	47
48	1·0863	1·1010	1·1155	1·1299	1·1443	1·1586	1·1727	1·1868	1·2008	48
49	1·0866	1·1012	1·1157	1·1302	1·1445	1·1588	1·1730	1·1871	1·2011	49
50	1·0868	1·1014	1·1160	1·1304	1·1448	1·1590	1·1732	1·1873	1·2013	50
51	1·0871	1·1017	1·1162	1·1307	1·1450	1·1593	1·1735	1·1875	1·2015	51
52	1·0873	1·1019	1·1165	1·1309	1·1452	1·1595	1·1737	1·1878	1·2018	52
53	1·0876	1·1022	1·1167	1·1311	1·1455	1·1598	1·1739	1·1880	1·2020	53
54	1·0878	1·1024	1·1169	1·1314	1·1457	1·1600	1·1742	1·1882	1·2022	54
55	1·0881	1·1027	1·1172	1·1316	1·1460	1·1602	1·1744	1·1885	1·2025	55
56	1·0883	1·1029	1·1174	1·1319	1·1462	1·1605	1·1746	1·1887	1·2027	56
57	1·0885	1·1031	1·1177	1·1321	1·1464	1·1607	1·1749	1·1889	1·2029	57
58	1·0888	1·1034	1·1179	1·1323	1·1467	1·1609	1·1751	1·1892	1·2032	58
59	1·0890	1·1036	1·1181	1·1326	1·1469	1·1612	1·1753	1·1894	1·2034	59
60	1·0893	1·1039	1·1184	1·1328	1·1472	1·1614	1·1756	1·1896	1·2036	60

TABLE OF CHORDS: [Radius = 1.0000].

M.	74°	75°	76°	77°	78°	79°	80°	81°	82°	M.
0′	1·2036	1·2175	1·2313	1·2450	1·2586	1·2722	1·2856	1·2989	1·3121	0′
1	1·2039	1·2178	1·2316	1·2453	1·2589	1·2724	1·2858	1·2991	1·3123	1
2	1·2041	1·2180	1·2318	1·2455	1·2591	1·2726	1·2860	1·2993	1·3126	2
3	1·2043	1·2182	1·2320	1·2457	1·2593	1·2728	1·2862	1·2996	1·3128	3
4	1·2046	1·2184	1·2322	1·2459	1·2595	1·2731	1·2865	1·2998	1·3130	4
5	1·2048	1·2187	1·2325	1·2462	1·2598	1·2733	1·2867	1·3000	1·3132	5
6	1·2050	1·2189	1·2327	1·2464	1·2600	1·2735	1·2869	1·3002	1·3134	6
7	1·2053	1·2191	1·2329	1·2466	1·2602	1·2737	1·2871	1·3004	1·3137	7
8	1·2055	1·2194	1·2332	1·2468	1·2604	1·2740	1·2874	1·3007	1·3139	8
9	1·2057	1·2196	1·2334	1·2471	1·2607	1·2742	1·2876	1·3009	1·3141	9
10	1·2060	1·2198	1·2336	1·2473	1·2609	1·2744	1·2878	1·3011	1·3143	10
11	1·2062	1·2201	1·2338	1·2475	1·2611	1·2746	1·2880	1·3013	1·3145	11
12	1·2064	1·2203	1·2341	1·2478	1·2614	1·2748	1·2882	1·3015	1·3147	12
13	1·2066	1·2205	1·2343	1·2480	1·2616	1·2751	1·2885	1·3018	1·3150	13
14	1·2069	1·2208	1·2345	1·2482	1·2618	1·2753	1·2887	1·3020	1·3152	14
15	1·2071	1·2210	1·2348	1·2484	1·2620	1·2755	1·2889	1·3022	1·3154	15
16	1·2073	1·2212	1·2350	1·2487	1·2623	1·2757	1·2891	1·3024	1·3156	16
17	1·2076	1·2214	1·2352	1·2489	1·2625	1·2760	1·2894	1·3027	1·3158	17
18	1·2078	1·2217	1·2354	1·2491	1·2627	1·2762	1·2896	1·3029	1·3161	18
19	1·2080	1·2219	1·2357	1·2493	1·2629	1·2764	1·2898	1·3031	1·3163	19
20	1·2083	1·2221	1·2359	1·2496	1·2632	1·2766	1·2900	1·3033	1·3165	20
21	1·2085	1·2224	1·2361	1·2498	1·2634	1·2769	1·2903	1·3035	1·3167	21
22	1·2087	1·2226	1·2364	1·2500	1·2636	1·2771	1·2905	1·3038	1·3169	22
23	1·2090	1·2228	1·2366	1·2503	1·2638	1·2773	1·2907	1·3040	1·3172	23
24	1·2092	1·2231	1·2368	1·2505	1·2641	1·2775	1·2909	1·3042	1·3174	24
25	1·2094	1·2233	1·2370	1·2507	1·2643	1·2778	1·2911	1·3044	1·3176	25
26	1·2097	1·2235	1·2373	1·2509	1·2645	1·2780	1·2914	1·3046	1·3178	26
27	1·2099	1·2237	1·2375	1·2512	1·2648	1·2782	1·2916	1·3049	1·3180	27
28	1·2101	1·2240	1·2377	1·2514	1·2650	1·2784	1·2918	1·3051	1·3183	28
29	1·2104	1·2242	1·2380	1·2516	1·2652	1·2787	1·2920	1·3053	1·3185	29
30	1·2106	1·2244	1·2382	1·2518	1·2654	1·2789	1·2922	1·3055	1·3187	30
31	1·2108	1·2247	1·2384	1·2521	1·2656	1·2791	1·2925	1·3057	1·3189	31
32	1·2111	1·2249	1·2386	1·2523	1·2659	1·2793	1·2927	1·3060	1·3191	32
33	1·2113	1·2251	1·2389	1·2525	1·2661	1·2795	1·2929	1·3062	1·3193	33
34	1·2115	1·2254	1·2391	1·2528	1·2663	1·2798	1·2931	1·3064	1·3196	34
35	1·2117	1·2256	1·2393	1·2530	1·2665	1·2800	1·2934	1·3066	1·3198	35
36	1·2120	1·2258	1·2396	1·2532	1·2668	1·2802	1·2936	1·3068	1·3200	36
37	1·2122	1·2260	1·2398	1·2534	1·2670	1·2804	1·2938	1·3071	1·3202	37
38	1·2124	1·2263	1·2400	1·2537	1·2672	1·2807	1·2940	1·3073	1·3204	38
39	1·2127	1·2265	1·2402	1·2539	1·2674	1·2809	1·2942	1·3075	1·3207	39
40	1·2129	1·2267	1·2405	1·2541	1·2677	1·2811	1·2945	1·3077	1·3209	40
41	1·2131	1·2270	1·2407	1·2543	1·2679	1·2813	1·2947	1·3079	1·3211	41
42	1·2134	1·2272	1·2409	1·2546	1·2681	1·2816	1·2949	1·3082	1·3213	42
43	1·2136	1·2274	1·2412	1·2548	1·2683	1·2818	1·2951	1·3084	1·3215	43
44	1·2138	1·2277	1·2414	1·2550	1·2686	1·2820	1·2954	1·3086	1·3218	44
45	1·2141	1·2279	1·2416	1·2552	1·2688	1·2822	1·2956	1·3088	1·3220	45
46	1·2143	1·2281	1·2418	1·2555	1·2690	1·2825	1·2958	1·3090	1·3222	46
47	1·2145	1·2283	1·2421	1·2557	1·2692	1·2827	1·2960	1·3093	1·3224	47
48	1·2148	1·2286	1·2423	1·2559	1·2695	1·2829	1·2962	1·3095	1·3226	48
49	1·2150	1·2288	1·2425	1·2562	1·2697	1·2831	1·2965	1·3097	1·3228	49
50	1·2152	1·2290	1·2428	1·2564	1·2699	1·2833	1·2967	1·3099	1·3231	50
51	1·2154	1·2293	1·2430	1·2566	1·2701	1·2836	1·2969	1·3101	1·3233	51
52	1·2157	1·2295	1·2432	1·2568	1·2704	1·2838	1·2971	1·3104	1·3235	52
53	1·2159	1·2297	1·2434	1·2571	1·2706	1·2840	1·2973	1·3106	1·3237	53
54	1·2161	1·2299	1·2437	1·2573	1·2708	1·2842	1·2976	1·3108	1·3239	54
55	1·2164	1·2302	1·2439	1·2575	1·2710	1·2845	1·2978	1·3110	1·3242	55
56	1·2166	1·2304	1·2441	1·2577	1·2713	1·2847	1·2980	1·3112	1·3244	56
57	1·2168	1·2306	1·2443	1·2580	1·2715	1·2849	1·2982	1·3115	1·3246	57
58	1·2171	1·2309	1·2446	1·2582	1·2717	1·2851	1·2985	1·3117	1·3248	58
59	1·2173	1·2311	1·2448	1·2584	1·2719	1·2854	1·2987	1·3119	1·3250	59
60	1·2175	1·2313	1·2450	1·2586	1·2722	1·2856	1·2989	1·3121	1·3252	60

TABLE OF CHORDS: [Radius = 1.0000].

M.	83°	84°	85°	86°	87°	88°	89°	M.
0′	1·3252	1·3383	1·3512	1·3640	1·3767	1·3893	1·4018	0′
1	1·3255	1·3385	1·3514	1·3642	1·3769	1·3895	1·4020	1
2	1·3257	1·3387	1·3516	1·3644	1·3771	1·3897	1·4022	2
3	1·3259	1·3389	1·3518	1·3646	1·3773	1·3899	1·4024	3
4	1·3261	1·3391	1·3520	1·3648	1·3776	1·3902	1·4026	4
5	1·3263	1·3393	1·3523	1·3651	1·3778	1·3904	1·4029	5
6	1·3265	1·3396	1·3525	1·3653	1·3780	1·3906	1·4031	6
7	1·3268	1·3398	1·3527	1·3655	1·3782	1·3908	1·4033	7
8	1·3270	1·3400	1·3529	1·3657	1·3784	1·3910	1·4035	8
9	1·3272	1·3402	1·3531	1·3659	1·3786	1·3912	1·4037	9
10	1·3274	1·3404	1·3533	1·3661	1·3788	1·3914	1·4039	10
11	1·3276	1·3406	1·3535	1·3663	1·3790	1·3916	1·4041	11
12	1·3279	1·3409	1·3538	1·3665	1·3792	1·3918	1·4043	12
13	1·3281	1·3411	1·3540	1·3668	1·3794	1·3920	1·4045	13
14	1·3283	1·3413	1·3542	1·3670	1·3797	1·3922	1·4047	14
15	1·3285	1·3415	1·3544	1·3672	1·3799	1·3925	1·4049	15
16	1·3287	1·3417	1·3546	1·3674	1·3801	1·3927	1·4051	16
17	1·3289	1·3419	1·3548	1·3676	1·3803	1·3929	1·4053	17
18	1·3292	1·3421	1·3550	1·3678	1·3805	1·3931	1·4055	18
19	1·3294	1·3424	1·3552	1·3680	1·3807	1·3933	1·4058	19
20	1·3296	1·3426	1·3555	1·3682	1·3809	1·3935	1·4060	20
21	1·3298	1·3428	1·3557	1·3685	1·3811	1·3937	1·4062	21
22	1·3300	1·3430	1·3559	1·3687	1·3813	1·3939	1·4064	22
23	1·3302	1·3432	1·3561	1·3689	1·3816	1·3941	1·4066	23
24	1·3305	1·3434	1·3563	1·3691	1·3818	1·3943	1·4068	24
25	1·3307	1·3437	1·3565	1·3693	1·3820	1·3945	1·4070	25
26	1·3309	1·3439	1·3567	1·3695	1·3822	1·3947	1·4072	26
27	1·3311	1·3441	1·3570	1·3697	1·3824	1·3950	1·4074	27
28	1·3313	1·3443	1·3572	1·3699	1·3826	1·3952	1·4076	28
29	1·3315	1·3445	1·3574	1·3702	1·3828	1·3954	1·4078	29
30	1·3318	1·3447	1·3576	1·3704	1·3830	1·3956	1·4080	30
31	1·3320	1·3449	1·3578	1·3706	1·3832	1·3958	1·4082	31
32	1·3322	1·3452	1·3580	1·3708	1·3834	1·3960	1·4084	32
33	1·3324	1·3454	1·3582	1·3710	1·3837	1·3962	1·4086	33
34	1·3326	1·3456	1·3585	1·3712	1·3839	1·3964	1·4089	34
35	1·3328	1·3458	1·3587	1·3714	1·3841	1·3966	1·4091	35
36	1·3331	1·3460	1·3589	1·3716	1·3843	1·3968	1·4093	36
37	1·3333	1·3462	1·3591	1·3718	1·3845	1·3970	1·4095	37
38	1·3335	1·3465	1·3593	1·3721	1·3847	1·3972	1·4097	38
39	1·3337	1·3467	1·3595	1·3723	1·3849	1·3975	1·4099	39
40	1·3339	1·3469	1·3597	1·3725	1·3851	1·3977	1·4101	40
41	1·3341	1·3471	1·3599	1·3727	1·3853	1·3979	1·4103	41
42	1·3344	1·3473	1·3602	1·3729	1·3855	1·3981	1·4105	42
43	1·3346	1·3475	1·3604	1·3731	1·3858	1·3983	1·4107	43
44	1·3348	1·3477	1·3606	1·3733	1·3860	1·3985	1·4109	44
45	1·3350	1·3480	1·3608	1·3735	1·3862	1·3987	1·4111	45
46	1·3352	1·3482	1·3610	1·3738	1·3864	1·3989	1·4113	46
47	1·3354	1·3484	1·3612	1·3740	1·3866	1·3991	1·4115	47
48	1·3357	1·3486	1·3614	1·3742	1·3868	1·3993	1·4117	48
49	1·3359	1·3488	1·3617	1·3744	1·3870	1·3995	1·4119	49
50	1·3361	1·3490	1·3619	1·3746	1·3872	1·3997	1·4122	50
51	1·3363	1·3492	1·3621	1·3748	1·3874	1·3999	1·4124	51
52	1·3365	1·3495	1·3623	1·3750	1·3876	1·4002	1·4126	52
53	1·3367	1·3497	1·3625	1·3752	1·3879	1·4004	1·4128	53
54	1·3370	1·3499	1·3627	1·3754	1·3881	1·4006	1·4130	54
55	1·3372	1·3501	1·3629	1·3757	1·3883	1·4008	1·4132	55
56	1·3374	1·3503	1·3631	1·3759	1·3885	1·4010	1·4134	56
57	1·3376	1·3505	1·3634	1·3761	1·3887	1·4012	1·4136	57
58	1·3378	1·3508	1·3636	1·3763	1·3889	1·4014	1·4138	58
59	1·3380	1·3510	1·3638	1·3765	1·3891	1·4016	1·4140	59
60	1·3383	1·3512	1·3640	1·3767	1·3893	1·4018	1·4142	60

TABLE

OF

NATURAL SINES AND TANGENTS;

TO

EVERY DEGREE AND MINUTE OF THE QUADRANT.

IF the given angle is less than 45°, look for the degrees and the title of the column, at the *top* of the page; and for the minutes on the *left*. But if the angle is between 45° and 90°, look for the degrees and the title of the column, at the *bottom;* and for the minutes on the *right*.

The *Secants and Cosecants*, which are not inserted in this table, may be easily supplied. If 1 be divided by the cosine of an arc, the quotient will be the secant of that arc. And if 1 be divided by the sine, the quotient will be the cosecant.

The values of the Sines and Cosines are less than a unit, and are given in decimals, although the decimal point is not printed. So also, the tangents of arcs less than 45°, and cotangents of arcs greater than 45°, are less than a unit and are expressed in decimals with the decimal point omitted.

NATURAL SINES AND COSINES.

′	0° Sine.	0° Cosine.	1° Sine.	1° Cosine.	2° Sine.	2° Cosine.	3° Sine.	3° Cosine.	4° Sine.	4° Cosine.	′
0	00000	Unit.	01745	99985	03490	99939	05234	99863	06976	99756	60
1	00029	Unit.	01774	99984	03519	99938	05263	99861	07005	99754	59
2	00058	Unit.	01803	99984	03548	99937	05292	99860	07034	99752	58
3	00087	Unit.	01832	99983	03577	99936	05321	99858	07063	99750	57
4	00116	Unit.	01862	99983	03606	99935	05350	99857	07092	99748	56
5	00145	Unit.	01891	99982	03635	99934	05379	99855	07121	99746	55
6	00175	Unit.	01920	99982	03664	99933	05408	99854	07150	99744	54
7	00204	Unit.	01949	99981	03693	99932	05437	99852	07179	99742	53
8	00233	Unit.	01978	99980	03723	99931	05466	99851	07208	99740	52
9	00262	Unit.	02007	99980	03752	99930	05495	99849	07237	99738	51
10	00291	Unit.	02036	99979	03781	99929	05524	99847	07266	99736	50
11	00320	99999	02065	99979	03810	99927	05553	99846	07295	99734	49
12	00349	99999	02094	99978	03839	99926	05582	99844	07324	99731	48
13	00378	99999	02123	99977	03868	99925	05611	99842	07353	99729	47
14	00407	99999	02152	99977	03897	99924	05640	99841	07382	99727	46
15	00436	99999	02181	99976	03926	99923	05669	99839	07411	99725	45
16	00465	99999	02211	99976	03955	99922	05698	99838	07440	99723	44
17	00495	99999	02240	99975	03984	99921	05727	99836	07469	99721	43
18	00524	99999	02269	99974	04013	99919	05756	99834	07498	99719	42
19	00553	99998	02298	99974	04042	99918	05785	99833	07527	99716	41
20	00582	99998	02327	99973	04071	99917	05814	99831	07556	99714	40
21	00611	99998	02356	99972	04100	99916	05844	99829	07585	99712	39
22	00640	99998	02385	99972	04129	99915	05873	99827	07614	99710	38
23	00669	99998	02414	99971	04159	99913	05902	99826	07643	99708	37
24	00698	99998	02443	99970	04188	99912	05931	99824	07672	99705	36
25	00727	99997	02472	99969	04217	99911	05960	99822	07701	99703	35
26	00756	99997	02501	99969	04246	99910	05989	99821	07730	99701	34
27	00785	99997	02530	99968	04275	99909	06018	99819	07759	99699	33
28	00814	99997	02560	99967	04304	99907	06047	99817	07788	99696	32
29	00844	99996	02589	99966	04333	99906	06076	99815	07817	99694	31
30	00873	99996	02618	99966	04362	99905	06105	99813	07846	99692	30
31	00902	99996	02647	99965	04391	99904	06134	99812	07875	99689	29
32	00931	99996	02676	99964	04420	99902	06163	99810	07904	99687	28
33	00960	99995	02705	99963	04449	99901	06192	99808	07933	99685	27
34	00989	99995	02734	99963	04478	99900	06221	99806	07962	99683	26
35	01018	99995	02763	99962	04507	99898	06250	99804	07991	99680	25
36	01047	99995	02792	99961	04536	99897	06279	99803	08020	99678	24
37	01076	99994	02821	99960	04565	99896	06308	99801	08049	99676	23
38	01105	99994	02850	99959	04594	99894	06337	99799	08078	99673	22
39	01134	99994	02879	99959	04623	99893	06366	99797	08107	99671	21
40	01164	99993	02908	99958	04653	99892	06395	99795	08136	99668	20
41	01193	99993	02938	99957	04682	99890	06424	99793	08165	99666	19
42	01222	99993	02967	99956	04711	99889	06453	99792	08194	99664	18
43	01251	99992	02996	99955	04740	99888	06482	99790	08223	99661	17
44	01280	99992	03025	99954	04769	99886	06511	99788	08252	99659	16
45	01309	99991	03054	99953	04798	99885	06540	99786	08281	99657	15
46	01338	99991	03083	99952	04827	99883	06569	99784	08310	99654	14
47	01367	99991	03112	99952	04856	99882	06598	99782	08339	99652	13
48	01396	99990	03141	99951	04885	99881	06627	99780	08368	99649	12
49	01425	99990	03170	99950	04914	99879	06656	99778	08397	99647	11
50	01454	99989	03199	99949	04943	99878	06685	99776	08426	99644	10
51	01483	99989	03228	99948	04972	99876	06714	99774	08455	99642	9
52	01513	99989	03257	99947	05001	99875	06743	99772	08484	99639	8
53	01542	99988	03286	99946	05030	99873	06773	99770	08513	99637	7
54	01571	99988	03316	99945	05059	99872	06802	99768	08542	99635	6
55	01600	99987	03345	99944	05088	99870	06831	99766	08571	99632	5
56	01629	99987	03374	99943	05117	99869	06860	99764	08600	99630	4
57	01658	99986	03403	99942	05146	99867	06889	99762	08629	99627	3
58	01687	99986	03432	99941	05175	99866	06918	99760	08658	99625	2
59	01716	99985	03461	99940	05205	99864	06947	99758	08687	99622	1
60	01745	99985	03490	99939	05234	99863	06976	99756	08716	99619	0
′	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	′
	89°		88°		87°		86°		85°		

NATURAL SINES AND COSINES.

′	5° Sine	5° Cosine	6° Sine	6° Cosine	7° Sine	7° Cosine	8° Sine	8° Cosine	9° Sine	9° Cosine	′
0	08716	99619	10453	99452	12187	99255	13917	99027	15643	98769	60
1	08745	99617	10482	99449	12216	99251	13946	99023	15672	98764	59
2	08774	99614	10511	99446	12245	99248	13975	99019	15701	98760	58
3	08803	99612	10540	99443	12274	99244	14004	99015	15730	98755	57
4	08831	99609	10569	99440	12302	99240	14033	99011	15758	98751	56
5	08860	99607	10597	99437	12331	99237	14061	99006	15787	98746	55
6	08889	99604	10626	99434	12360	99233	14090	99002	15816	98741	54
7	08918	99602	10655	99431	12389	99230	14119	98998	15845	98737	53
8	08947	99599	10684	99428	12418	99226	14148	98994	15873	98732	52
9	08976	99596	10713	99424	12447	99222	14177	98990	15902	98728	51
10	09005	99594	10742	99421	12476	99219	14205	98986	15931	98723	50
11	09034	99591	10771	99418	12504	99215	14234	98982	15959	98718	49
12	09063	99588	10800	99415	12533	99211	14263	98978	15988	98714	48
13	09092	99586	10829	99412	12562	99208	14292	98973	16017	98709	47
14	09121	99583	10858	99409	12591	99204	14320	98969	16046	98704	46
15	09150	99580	10887	99406	12620	99200	14349	98965	16074	98700	45
16	09179	99578	10916	99402	12649	99197	14378	98961	16103	98695	44
17	09208	99575	10945	99399	12678	99193	14407	98957	16132	98690	43
18	09237	99572	10973	99396	12706	99189	14436	98953	16160	98686	42
19	09266	99570	11002	99393	12735	99186	14464	98948	16189	98681	41
20	09295	99567	11031	99390	12764	99182	14493	98944	16218	98676	40
21	09324	99564	11060	99386	12793	99178	14522	98940	16246	98671	39
22	09353	99562	11089	99383	12822	99175	14551	98936	16275	98667	38
23	09382	99559	11118	99380	12851	99171	14580	98931	16304	98662	37
24	09411	99556	11147	99377	12880	99167	14608	98927	16333	98657	36
25	09440	99553	11176	99374	12908	99163	14637	98923	16361	98652	35
26	09469	99551	11205	99370	12937	99160	14666	98919	16390	98648	34
27	09498	99548	11234	99367	12966	99156	14695	98914	16419	98643	33
28	09527	99545	11263	99364	12995	99152	14723	98910	16447	98638	32
29	09556	99542	11291	99360	13024	99148	14752	98906	16476	98633	31
30	09585	99540	11320	99357	13053	99144	14781	98902	16505	98629	30
31	09614	99537	11349	99354	13081	99141	14810	98897	16533	98624	29
32	09642	99534	11378	99351	13110	99137	14838	98893	16562	98619	28
33	09671	99531	11407	99347	13139	99133	14867	98889	16591	98614	27
34	09700	99528	11436	99344	13168	99129	14896	98884	16620	98609	26
35	09729	99526	11465	99341	13197	99125	14925	98880	16648	98604	25
36	09758	99523	11494	99337	13226	99122	14954	98876	16677	98600	24
37	09787	99520	11523	99334	13254	99118	14982	98871	16706	98595	23
38	09816	99517	11552	99331	13283	99114	15011	98867	16734	98590	22
39	09845	99514	11580	99327	13312	99110	15040	98863	16763	98585	21
40	09874	99511	11609	69324	13341	99106	15069	98858	16792	98580	20
41	09903	99508	11638	99320	13370	99102	15097	98854	16820	98575	19
42	09932	99506	11667	99317	13399	99098	15126	98849	16849	98570	18
43	09961	99503	11696	99314	13427	99094	15155	98845	16878	98565	17
44	09990	99500	11725	99310	13456	99091	15184	98841	16906	98561	16
45	10019	99497	11754	99307	13485	99087	15212	98836	16935	98556	15
46	10048	99494	11783	99303	13514	99083	15241	98832	16964	98551	14
47	10077	99491	11812	99300	13543	99079	15270	98827	16992	98546	13
48	10106	99488	11840	99297	13572	99075	15299	98823	17021	98541	12
49	10135	99485	11869	99293	13600	99071	15327	98818	17050	98536	11
50	10164	99482	11898	99290	13629	99067	15356	98814	17078	98531	10
51	10192	99479	11927	99286	13658	99063	15385	98809	17107	98526	9
52	10221	99476	11956	99283	13687	99059	15414	98805	17136	98521	8
53	10250	99473	11985	99279	13716	99055	15442	98800	17164	98516	7
54	10279	99470	12014	99276	13744	99051	15471	98796	17193	98511	6
55	10308	99467	12043	99272	13773	99047	15500	98791	17222	98506	5
56	10337	99464	12071	99269	13802	99043	15529	98787	17250	98501	4
57	10366	99461	12100	99265	13831	99039	15557	98782	17279	98496	3
58	10395	99458	12129	99262	13860	99035	15586	98778	17308	98491	2
59	10424	99455	12158	99258	13889	99031	15615	98773	17336	98486	1
60	10453	99452	12187	99255	13917	99027	15643	98769	17365	98481	0
′	84° Cosine	84° Sine	83° Cosine	83° Sine	82° Cosine	82° Sine	81° Cosine	81° Sine	80° Cosine	80° Sine	′

NATURAL SINES AND COSINES.

′	10° Sine.	10° Cosine.	11° Sine.	11° Cosine.	12° Sine.	12° Cosine.	13° Sine.	13° Cosine.	14° Sine.	14° Cosine.	′
0	17365	98481	19081	98163	20791	97815	22495	97437	24192	97030	60
1	17393	98476	19109	98157	20820	97809	22523	97430	24220	97023	59
2	17422	98471	19138	98152	20848	97803	22552	97424	24249	97015	58
3	17451	98466	19167	98146	20877	97797	22580	97417	24277	97008	57
4	17479	98461	19195	98140	20905	97791	22608	97411	24305	97001	56
5	17508	98455	19224	98135	20933	97784	22637	97404	24333	96994	55
6	17537	98450	19252	98129	20962	97778	22665	97398	24362	96987	54
7	17565	98445	19281	98124	20990	97772	22693	97391	24390	96980	53
8	17594	98440	19309	98118	21019	97766	22722	97384	24418	96973	52
9	17623	98435	19338	98112	21047	97760	22750	97378	24446	96966	51
10	17651	98430	19366	98107	21076	97754	22778	97371	24474	96959	50
11	17680	98425	19395	98101	21104	97748	22807	97365	24503	96952	49
12	17708	98420	19423	98096	21132	97742	22835	97358	24531	96945	48
13	17737	98414	19452	98090	21161	97735	22863	97351	24559	96937	47
14	17766	98409	19481	98084	21189	97729	22892	97345	24587	96930	46
15	17794	98404	19509	98079	21218	97723	22920	97338	24615	96923	45
16	17823	98399	19538	98073	21246	97717	22948	97331	24644	96916	44
17	17852	98394	19566	98067	21275	97711	22977	97325	24672	96909	43
18	17880	98389	19595	98061	21303	97705	23005	97318	24700	96902	42
19	17909	98383	19623	98056	21331	97698	23033	97311	24728	96894	41
20	17937	98378	19652	98050	21360	97692	23062	97304	24756	96887	40
21	17966	98373	19680	98044	21388	97686	23090	97298	24784	96880	39
22	17995	98368	19709	98039	21417	97680	23118	97291	24813	96873	38
23	18023	98362	19737	98033	21445	97673	23146	97284	24841	96866	37
24	18052	98357	19766	98027	21474	97667	23175	97278	24869	96858	36
25	18081	98352	19794	98021	21502	97661	23203	97271	24897	96851	35
26	18109	98347	19823	98016	21530	97655	23231	97264	24925	96844	34
27	18138	98341	19851	98010	21559	97648	23260	97257	24954	96837	33
28	18166	98336	19880	98004	21587	97642	23288	97251	24982	96829	32
29	18195	98331	19908	97998	21616	97636	23316	97244	25010	96822	31
30	18224	98325	19937	97992	21644	97630	23345	97237	25038	96815	30
31	18252	98320	19965	97987	21672	97623	23373	97230	25066	96807	29
32	18281	98315	19994	97981	21701	97617	23401	97223	25094	96800	28
33	18309	98310	20022	97975	21729	97611	23429	97217	25122	96793	27
34	18338	98304	20051	97969	21758	97604	23458	97210	25151	96786	26
35	18367	98299	20079	97963	21786	97598	23486	97203	25179	96778	25
36	18395	98294	20108	97958	21814	97592	23514	97196	25207	96771	24
37	18424	98288	20136	97952	21843	97585	23542	97189	25235	96764	23
38	18452	98283	20165	97946	21871	97579	23571	97182	25263	96756	22
39	18481	98277	20193	97940	21899	97573	23599	97176	25291	96749	21
40	18509	98272	20222	97934	21928	97566	23627	97169	25320	96742	20
41	18538	98267	20250	97928	21956	97560	23656	97162	25348	96734	19
42	18567	98261	20279	97922	21985	97553	23684	97155	25376	96727	18
43	18595	98256	20307	97916	22013	97547	23712	97148	25404	96719	17
44	18624	98250	20336	97910	22041	97541	23740	97141	25432	96712	16
45	18652	98245	20364	97905	22070	97534	23769	97134	25460	96705	15
46	18681	98240	20393	97899	22098	97528	23797	97127	25488	96697	14
47	18710	98234	20421	97893	22126	97521	23825	97120	25516	96690	13
48	18738	98229	20450	97887	22155	97515	23853	97113	25545	96682	12
49	18767	98223	20478	97881	22183	97508	23882	97106	25573	96675	11
50	18795	98218	20507	97875	22212	97502	23910	97100	25601	96667	10
51	18824	98212	20535	97869	22240	97496	23938	97093	25629	96660	9
52	18852	98207	20563	97863	22268	97489	23966	97086	25657	96653	8
53	18881	98201	20592	97857	22297	97483	23995	97079	25685	96645	7
54	18910	98196	20620	97851	22325	97476	24023	97072	25713	96638	6
55	18938	98190	20649	97845	22353	97470	24051	97065	25741	96630	5
56	18967	98185	20677	97839	22382	97463	24079	97058	25769	96623	4
57	18995	98179	20706	97833	22410	97457	24108	97051	25798	96615	3
58	19024	98174	20734	97827	22438	97450	24136	97044	25826	96608	2
59	19052	98168	20763	97821	22467	97444	24164	97037	25854	96600	1
60	19081	98163	20791	97815	22495	97437	24192	97030	25882	96593	0
′	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	′
	79°		78°		77°		76°		75°		

′	15°		16°		17°		18°		19°		′
	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	
0	25882	96593	27564	96126	29237	95630	30902	95106	32557	94552	60
1	25910	96585	27592	96118	29265	95622	30929	95097	32584	94542	59
2	25938	96578	27620	96110	29293	95613	30957	95088	32612	94533	58
3	25966	96570	27648	96102	29321	95605	30985	95079	32639	94523	57
4	25994	96562	27676	96094	29348	95596	31012	95070	32667	94514	56
5	26022	96555	27704	96086	29376	95588	31040	95061	32694	94504	55
6	26050	96547	27731	96078	29404	95579	31068	95052	32722	94495	54
7	26079	96540	27759	96070	29432	95571	31095	95043	32749	94485	53
8	26107	96532	27787	96062	29460	95562	31123	95033	32777	94476	52
9	26135	96524	27815	96054	29487	95554	31151	95024	32804	94466	51
10	26163	96517	27843	96046	29515	95545	31178	95015	32832	94457	50
11	26191	96509	27871	96037	29543	95536	31206	95006	32859	94447	49
12	26219	96502	27899	96029	29571	95528	31233	94997	32887	94438	48
13	26247	96494	27927	96021	29599	95519	31261	94988	32914	94428	47
14	26275	96486	27955	96013	29626	95511	31289	94979	32942	94418	46
15	26303	96479	27983	96005	29654	95502	31316	94970	32969	94409	45
16	26331	96471	28011	95997	29682	95493	31344	94961	32997	94399	44
17	26359	96463	28039	95989	29710	95485	31372	94952	33024	94390	43
18	26387	96456	28067	95981	29737	95476	31399	94943	33051	94380	42
19	26415	96448	28095	95972	29765	95467	31427	94933	33079	94370	41
20	26443	96440	28123	95964	29793	95459	31454	94924	33106	94361	40
21	26471	96433	28150	95956	29821	95450	31482	94915	33134	94351	39
22	26500	96425	28178	95948	29849	95441	31510	94906	33161	94342	38
23	26528	96417	28206	95940	29876	95433	31537	94897	33189	94332	37
24	26556	96410	28234	95931	29904	95424	31565	94888	33216	94322	36
25	26584	96402	28262	95923	29932	95415	31593	94878	33244	94313	35
26	26612	96394	28290	95915	29960	95407	31620	94869	33271	94303	34
27	26640	96386	28318	95907	29987	95398	31648	94860	33298	94293	33
28	26668	96379	28346	95898	30015	95389	31675	94851	33326	94284	32
29	26696	96371	28374	95890	30043	95380	31703	94842	33353	94274	31
30	26724	96363	28402	95882	30071	95372	31730	94832	33381	94264	30
31	26752	96355	28429	95874	30098	95363	31758	94823	33408	94254	29
32	26780	96347	28457	95865	30126	95354	31786	94814	33436	94245	28
33	26808	96340	28485	95857	30154	95345	31813	94805	33463	94235	27
34	26836	96332	28513	95849	30182	95337	31841	94795	33490	94225	26
35	26864	96324	28541	95841	30209	95328	31868	94786	33518	94215	25
36	26892	96316	28569	95832	30237	95319	31896	94777	33545	94206	24
37	26920	96308	28597	95824	30265	95310	31923	94768	33573	94196	23
38	26948	96301	28625	95816	30292	95301	31951	94758	33600	94186	22
39	26976	96293	28652	95807	30320	95293	31979	94749	33627	94176	21
40	27004	96285	28680	95799	30348	95284	32006	94740	33655	94167	20
41	27032	96277	28708	95791	30376	95275	32034	94730	33682	94157	19
42	27060	96269	28736	95782	30403	95266	32061	94721	33710	94147	18
43	27088	96261	28764	95774	30431	95257	32089	94712	33737	94137	17
44	27116	96253	28792	95766	30459	95248	32116	94702	33764	94127	16
45	27144	96246	28820	95757	30486	95240	32144	94693	33792	94118	15
46	27172	96238	28847	95749	30514	95231	32171	94684	33819	94108	14
47	27200	96230	28875	95740	30542	95222	32199	94674	33846	94098	13
48	27228	96222	28903	95732	30570	95213	32227	94665	33874	94088	12
49	27256	96214	28931	95724	30597	95204	32254	94656	33901	94078	11
50	27284	96206	28959	95715	30625	95195	32282	94646	33929	94068	10
51	27312	96198	28987	95707	30653	95186	32309	94637	33956	94058	9
52	27340	96190	29015	95698	30680	95177	32337	94627	33983	94049	8
53	27368	96182	29042	95690	30708	95168	32364	94618	34011	94039	7
54	27396	96174	29070	95681	30736	95159	32392	94609	34038	94029	6
55	27424	96166	29098	95673	30763	95150	32419	94599	34065	94019	5
56	27452	96158	29126	95664	30791	95142	32447	94590	34093	94009	4
57	27480	96150	29154	95656	30819	95133	32474	94580	34120	93999	3
58	27508	96142	29182	95647	30846	95124	32502	94571	34147	93989	2
59	27536	96134	29209	95639	30874	95115	32529	94561	34175	93979	1
60	27564	96126	29237	95630	30902	95106	32557	94552	34202	93969	0
	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	
′	74°		73°		72°		71°		70°		′

′	20°		21°		22°		23°		24°		′
	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	
0	34202	93969	35837	93358	37461	92718	39073	92050	40674	91355	60
1	34229	93959	35864	93348	37488	92707	39100	92039	40700	91343	59
2	34257	93949	35891	93337	37515	92697	39127	92028	40727	91331	58
3	34284	93939	35918	93327	37542	92686	39153	92016	40753	91319	57
4	34311	93929	35945	93316	37569	92675	39180	92005	40780	91307	56
5	34339	93919	35973	93306	37595	92664	39207	91994	40806	91295	55
6	34366	93909	36000	93295	37622	92653	39234	91982	40833	91283	54
7	34393	93899	36027	93285	37649	92642	39260	91971	40860	91272	53
8	34421	93889	36054	93274	37676	92631	39287	91959	40886	91260	52
9	34448	93879	36081	93264	37703	92620	39314	91948	40913	91248	51
10	34475	93869	36108	93253	37730	92609	39341	91936	40939	91236	50
11	34503	93859	36135	93243	37757	92598	39367	91925	40966	91224	49
12	34530	93849	36162	93232	37784	92587	39394	91914	40992	91212	48
13	34557	93839	36190	93222	37811	92576	39421	91902	41019	91200	47
14	34584	93829	36217	93211	37838	92565	39448	91891	41045	91188	46
15	34612	93819	36244	93201	37865	92554	39474	91879	41072	91176	45
16	34639	93809	36271	93190	37892	92543	39501	91868	41098	91164	44
17	34666	93799	36298	93180	37919	92532	39528	91856	41125	91152	43
18	34694	93789	36325	93169	37946	92521	39555	91845	41151	91140	42
19	34721	93779	36352	93159	37973	92510	39581	91833	41178	91128	41
20	34748	93769	36379	93148	37999	92499	39608	91822	41204	91116	40
21	34775	93759	36406	93137	38026	92488	39636	91810	41231	91104	39
22	34803	93748	36434	93127	38053	92477	39661	91799	41257	91092	38
23	34830	93738	36461	93116	38080	92466	39688	91787	41284	91080	37
24	34857	93728	36488	93106	38107	92455	39715	91775	41310	91068	36
25	34884	93718	36515	93095	38134	92444	39741	91764	41337	91056	35
26	34912	93708	36542	93084	38161	92432	39768	91752	41363	91044	34
27	34939	93698	36569	93074	38188	92421	39795	91741	41390	91032	33
28	34966	93688	36596	93063	38215	92410	39822	91729	41416	91020	32
29	34993	93677	36623	93052	38241	92399	39848	91718	41443	91008	31
30	35021	93667	36650	93042	38268	92388	39875	91706	41469	90996	30
31	35048	93657	36677	93031	38295	92377	39902	91694	41496	90984	29
32	35075	93647	36704	93020	38322	92366	39928	91683	41522	90972	28
33	35102	93637	36731	93010	38349	92355	39955	91671	41549	90960	27
34	35130	93626	36758	92999	38376	92343	39982	91660	41575	90948	26
35	35157	93616	36785	92988	38403	92332	40008	91648	41602	90936	25
36	35184	93606	36812	92978	38430	92321	40035	91636	41628	90924	24
37	35211	93596	36839	92967	38456	92310	40062	91625	41655	90911	23
38	35239	93585	36867	92956	38483	92299	40088	91613	41681	90899	22
39	35266	93575	36894	92945	38510	92287	40115	91601	41707	90887	21
40	35293	93565	36921	92935	38537	92276	40141	91590	41734	90875	20
41	35320	93555	36948	92924	38564	92265	40168	91578	41760	90863	19
42	35347	93544	36975	92913	38591	92254	40195	91566	41787	90851	18
43	35375	93534	37002	92902	38617	92243	40221	91555	41813	90839	17
44	35402	93524	37029	92892	38644	92231	40248	91543	41840	90826	16
45	35429	93514	37056	92881	38671	92220	40275	91531	41866	90814	15
46	35456	93503	37083	92870	38698	92209	40301	91519	41892	90802	14
47	35484	93493	37110	92859	38725	92198	40328	91508	41919	90790	13
48	35511	93483	37137	92849	38752	92186	40355	91496	41945	90778	12
49	35538	93472	37164	92838	38778	92175	40381	91484	41972	90766	11
50	35565	93462	37191	92827	38805	92164	40408	91472	41998	90753	10
51	35592	93452	37218	92816	38832	92152	40434	91461	42024	90741	9
52	35619	93441	37245	92805	38859	92141	40461	91449	42051	90729	8
53	35647	93431	37272	92794	38886	92130	40488	91437	42077	90717	7
54	35674	93420	37299	92784	38912	92119	40514	91425	42104	90704	6
55	35701	93410	37326	92773	38939	92107	40541	91414	42130	90692	5
56	35728	93400	37353	92762	38966	92096	40567	91402	42156	90680	4
57	35755	93389	37380	92751	38993	92085	40594	91390	42183	90668	3
58	35782	93379	37407	92740	39020	92073	40621	91378	42209	90655	2
59	35810	93368	37434	92729	39046	92062	40647	91366	42235	90643	1
60	35837	93358	37461	92718	39073	92050	40674	91355	42262	90631	0
	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	
′	69°		68°		67°		66°		65°		′

NATURAL SINES AND COSINES.

′	25° Sine.	25° Cosine.	26° Sine.	26° Cosine.	27° Sine.	27° Cosine.	28° Sine.	28° Cosine.	29° Sine.	29° Cosine.	′
0	42262	90631	43837	89879	45399	89101	46947	88295	48481	87462	60
1	42288	90618	43863	89867	45425	89087	46973	88281	48506	87448	59
2	42315	90606	43889	89854	45451	89074	46999	88267	48532	87434	58
3	42341	90594	43916	89841	45477	89061	47024	88254	48557	87420	57
4	42367	90582	43942	89828	45503	89048	47050	88240	48583	87406	56
5	42394	90569	43968	89816	45529	89035	47076	88226	48608	87391	55
6	42420	90557	43994	89803	45554	89021	47101	88213	48634	87377	54
7	42446	90545	44020	89790	45580	89008	47127	88199	48659	87363	53
8	42473	90532	44046	89777	45606	88995	47153	88185	48684	87349	52
9	42499	90520	44072	89764	45632	88981	47178	88172	48710	87335	51
10	42525	90507	44098	89752	45658	88968	47204	88158	48735	87321	50
11	42552	90495	44124	89739	45684	88955	47229	88144	48761	87306	49
12	42578	90483	44151	89726	45710	88942	47255	88130	48786	87292	48
13	42604	90470	44177	89713	45736	88928	47281	88117	48811	87278	47
14	42631	90458	44203	89700	45762	88915	47306	88103	48837	87264	46
15	42657	90446	44229	89687	45787	88902	47332	88089	48862	87250	45
16	42683	90433	44255	89674	45813	88888	47358	88075	48888	87235	44
17	42709	90421	44281	89662	45839	88875	47383	88062	48913	87221	43
18	42736	90408	44307	89649	45865	88862	47409	88048	48938	87207	42
19	42762	90396	44333	89636	45891	88848	47434	88034	48964	87193	41
20	42788	90383	44359	89623	45917	88835	47460	88020	48989	87178	40
21	42815	90371	44385	89610	45942	88822	47486	88006	49014	87164	39
22	42841	90358	44411	89597	45968	88808	47511	87993	49040	87150	38
23	42867	90346	44437	89584	45994	88795	47537	87979	49065	87136	37
24	42894	90334	44464	89571	46020	88782	47562	87965	49090	87121	36
25	42920	90321	44490	89558	46046	88768	47588	87951	49116	87107	35
26	42946	90309	44516	89545	46072	88755	47614	87937	49141	87093	34
27	42972	90296	44542	89532	46097	88741	47639	87923	49166	87079	33
28	42999	90284	44568	89519	46123	88728	47665	87909	49192	87064	32
29	43025	90271	44594	89506	46149	88715	47690	87896	49217	87050	31
30	43051	90259	44620	89493	46175	88701	47716	87882	49242	87036	30
31	43077	90246	44646	89480	46201	88688	47741	87868	49268	87021	29
32	43104	90233	44672	89467	46226	88674	47767	87854	49293	87007	28
33	43130	90221	44698	89454	46252	88661	47793	87840	49318	86993	27
34	43156	90208	44724	89441	46278	88647	47818	87826	49344	86978	26
35	43182	90196	44750	89428	46304	88634	47844	87812	49369	86964	25
36	43209	90183	44776	89415	46330	88620	47869	87798	49394	86949	24
37	43235	90171	44802	89402	46355	88607	47895	87784	49419	86935	23
38	43261	90158	44828	89389	46381	88593	47920	87770	49445	86921	22
39	43287	90146	44854	89376	46407	88580	47946	87756	49470	86906	21
40	43313	90133	44880	89363	46433	88566	47971	87743	49495	86892	20
41	43340	90120	44906	89350	46458	88553	47997	87729	49521	86878	19
42	43366	90108	44932	89337	46484	88539	48022	87715	49546	86863	18
43	43392	90095	44958	89324	46510	88526	48048	87701	49571	86849	17
44	43418	90082	44984	89311	46536	88512	48073	87687	49596	86834	16
45	43445	90070	45010	89298	46561	88499	48099	87673	49622	86820	15
46	43471	90057	45036	89285	46587	88485	48124	87659	49647	86805	14
47	43497	90045	45062	89272	46613	88472	48150	87645	49672	86791	13
48	43523	90032	45088	89259	46639	88458	48175	87631	49697	86777	12
49	43549	90019	45114	89245	46664	88445	48201	87617	49723	86762	11
50	43575	90007	45140	89232	46690	88431	48226	87603	49748	86748	10
51	43602	89994	45166	89219	46716	88417	48252	87589	49773	86733	9
52	43628	89981	45192	89206	46742	88404	48277	87575	49798	86719	8
53	43654	89968	45218	89193	46767	88390	48303	87561	49824	86704	7
54	43680	89956	45243	89180	46793	88377	48328	87546	49849	86690	6
55	43706	89943	45269	89167	46819	88363	48354	87532	49874	86675	5
56	43733	89930	45295	89153	46844	88349	48379	87518	49899	86661	4
57	43759	89918	45321	89140	46870	88336	48405	87504	49924	86646	3
58	43785	89905	45347	89127	46896	88322	48430	87490	49950	86632	2
59	43811	89892	45373	89114	46921	88308	48456	87476	49975	86617	1
60	43837	89879	45399	89101	46947	88295	48481	87462	50000	86603	0
′	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	′
	64°		63°		62°		61°		60°		

NATURAL SINES AND COSINES.

′	30°		31°		32°		33°		34°		′
	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	
0	50000	86603	51504	85717	52992	84805	54464	83867	55919	82904	60
1	50025	86588	51529	85702	53017	84789	54488	83851	55943	82887	59
2	50050	86573	51554	85687	53041	84774	54513	83835	55968	82871	58
3	50076	86559	51579	85672	53066	84759	54537	83819	55992	82855	57
4	50101	86544	51604	85657	53091	84743	54561	83804	56016	82839	56
5	50126	86530	51628	85642	53115	84728	54586	83788	56040	82822	55
6	50151	86515	51653	85627	53140	84712	54610	83772	56064	82806	54
7	50176	86501	51678	85612	53164	84697	54635	83756	56088	82790	53
8	50201	86486	51703	85597	53189	84681	54659	83740	56112	82773	52
9	50227	86471	51728	85582	53214	84666	54683	83724	56136	82757	51
10	50252	86457	51753	85567	53238	84650	54708	83708	56160	82741	50
11	50277	86442	51778	85551	53263	84635	54732	83692	56184	82724	49
12	50302	86427	51803	85536	53288	84619	54756	83676	56208	82708	48
13	50327	86413	51828	85521	53312	84604	54781	83660	56232	82692	47
14	50352	86398	51852	85506	53337	84588	54805	83645	56256	82675	46
15	50377	86384	51877	85491	53361	84573	54829	83629	56280	82659	45
16	50403	86369	51902	85476	53386	84557	54854	83613	56305	82643	44
17	50428	86354	51927	85461	53411	84542	54878	83597	56329	82626	43
18	50453	86340	51952	85446	53435	84526	54902	83581	56353	82610	42
19	50478	86325	51977	85431	53460	84511	54927	83565	56377	82593	41
20	50503	86310	52002	85416	53484	84495	54951	83549	56401	82577	40
21	50528	86295	52026	85401	53509	84480	54975	83533	56425	82561	39
22	50553	86281	52051	85385	53534	84464	54999	83517	56449	82544	38
23	50578	86266	52076	85370	53558	84448	55024	83501	56473	82528	37
24	50603	86251	52101	85355	53583	84433	55048	83485	56497	82511	36
25	50628	86237	52126	85340	53607	84417	55072	83469	56521	82495	35
26	50654	86222	52151	85325	53632	84402	55097	83453	56545	82478	34
27	50679	86207	52175	85310	53656	84386	55121	83437	56569	82462	33
28	50704	86192	52200	85294	53681	84370	55145	83421	56593	82446	32
29	50729	86178	52225	85279	53705	84355	55169	83405	56617	82429	31
30	50754	86163	52250	85264	53730	84339	55194	83389	56641	82413	30
31	50779	86148	52275	85249	53754	84324	55218	83373	56665	82396	29
32	50804	86133	52299	85234	53779	84308	55242	83356	56689	82380	28
33	50829	86119	52324	85218	53804	84292	55266	83340	56713	82363	27
34	50854	86104	52349	85203	53828	84277	55291	83324	56736	82347	26
35	50879	86089	52374	85188	53853	84261	55315	83308	56760	82330	25
36	50904	86074	52399	85173	53877	84245	55339	83292	56784	82314	24
37	50929	86059	52423	85157	53902	84230	55363	83276	56808	82297	23
38	50954	86045	52448	85142	53926	84214	55388	83260	56832	82281	22
39	50979	86030	52473	85127	53951	84198	55412	83244	56856	82264	21
40	51004	86015	52498	85112	53975	84182	55436	83228	56880	82248	20
41	51029	86000	52522	85096	54000	84167	55460	83212	56904	82231	19
42	51054	85985	52547	85081	54024	84151	55484	83195	56928	82214	18
43	51079	85970	52572	85066	54049	84135	55509	83179	56952	82198	17
44	51104	85956	52597	85051	54073	84120	55533	83163	56976	82181	16
45	51129	85941	52621	85035	54097	84104	55557	83147	57000	82165	15
46	51154	85926	52646	85020	54122	84088	55581	83131	57024	82148	14
47	51179	85911	52671	85005	54146	84072	55605	83115	57047	82132	13
48	51204	85896	52696	84989	54171	84057	55630	83098	57071	82115	12
49	51229	85881	52720	84974	54195	84041	55654	83082	57095	82098	11
50	51254	85866	52745	84959	54220	84025	55678	83066	57119	82082	10
51	51279	85851	52770	84943	54244	84009	55702	83050	57143	82065	9
52	51304	85836	52794	84928	54269	83994	55726	83034	57167	82048	8
53	51329	85821	52819	84913	54293	83978	55750	83017	57191	82032	7
54	51354	85806	52844	84897	54317	83962	55775	83001	57215	82015	6
55	51379	85792	52869	84882	54342	83946	55799	82985	57238	81999	5
56	51404	85777	52893	84866	54366	83930	55823	82969	57262	81982	4
57	51429	85762	52918	84851	54391	83915	55847	82953	57286	81965	3
58	51454	85747	52943	84836	54415	83899	55871	82936	57310	81949	2
59	51479	85732	52967	84820	54440	83883	55895	82920	57334	81932	1
60	51504	85717	52992	84805	54464	83867	55919	82904	57358	81915	0
′	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	′
	59°		58°		57°		56°		55°		

′	35° Sine.	35° Cosine.	36° Sine.	36° Cosine.	37° Sine.	37° Cosine.	38° Sine.	38° Cosine.	39° Sine.	39° Cosine.	′
0	57358	81915	58779	80902	60182	79864	61566	78801	62932	77715	60
1	57381	81899	58802	80885	60205	79846	61589	78783	62955	77696	59
2	57405	81882	58826	80867	60228	79829	61612	78765	62977	77678	58
3	57429	81865	58849	80850	60251	79811	61635	78747	63000	77660	57
4	57453	81848	58873	80833	60274	79793	61658	78729	63022	77641	56
5	57477	81832	58896	80816	60298	79776	61681	78711	63045	77623	55
6	57501	81815	58920	80799	60321	79758	61704	78694	63068	77605	54
7	57524	81798	58943	80782	60344	79741	61726	78676	63090	77586	53
8	57548	81782	58967	80765	60367	79723	61749	78658	63113	77568	52
9	57572	81765	58990	80748	60390	79706	61772	78640	63135	77550	51
10	57596	81748	59014	80730	60414	79688	61795	78622	63158	77531	50
11	57619	81731	59037	80713	60437	79671	61818	78604	63180	77513	49
12	57643	81714	59061	80696	60460	79653	61841	78586	63203	77494	48
13	57667	81698	59084	80679	60483	79635	61864	78568	63225	77476	47
14	57691	81681	59108	80662	60506	79618	61887	78550	63248	77458	46
15	57715	81664	59131	80644	60529	79600	61909	78532	63271	77439	45
16	57738	81647	59154	80627	60553	79583	61932	78514	63293	77421	44
17	57762	81631	59178	80610	60576	79565	61955	78496	63316	77402	43
18	57786	81614	59201	80593	60599	79547	61978	78478	63338	77384	42
19	57810	81597	59225	80576	60622	79530	62001	78460	63361	77366	41
20	57833	81580	59248	80558	60645	79512	62024	78442	63383	77347	40
21	57857	81563	59272	80541	60668	79494	62046	78424	63406	77329	39
22	57881	81546	59295	80524	60691	79477	62069	78405	63428	77310	38
23	57904	81530	59318	80507	60714	79459	62092	78387	63451	77292	37
24	57928	81513	59342	80489	60738	79441	62115	78369	63473	77273	36
25	57952	81496	59365	80472	60761	79424	62138	78351	63496	77255	35
26	57976	81479	59389	80455	60784	79406	62160	78333	63518	77236	34
27	57999	81462	59412	80438	60807	79388	62183	78315	63540	77218	33
28	58023	81445	59436	80420	60830	79371	62206	78297	63563	77199	32
29	58047	81428	59459	80403	60853	79353	62229	78279	63585	77181	31
30	58070	81412	59482	80386	60876	79335	62251	78261	63608	77162	30
31	58094	81395	59506	80368	60899	79318	62274	78243	63630	77144	29
32	58118	81378	59529	80351	60922	79300	62297	78225	63653	77125	28
33	58141	81361	59552	80334	60945	79282	62320	78206	63675	77107	27
34	58165	81344	59576	80316	60968	79264	62342	78188	63698	77088	26
35	58189	81327	59599	80299	60991	79247	62365	78170	63720	77070	25
36	58212	81310	59622	80282	61015	79229	62388	78152	63742	77051	24
37	58236	81293	59646	80264	61038	79211	62411	78134	63765	77033	23
38	58260	81276	59669	80247	61061	79193	62433	78116	63787	77014	22
39	58283	81259	59693	80230	61084	79176	62456	78098	63810	76996	21
40	58307	81242	59716	80212	61107	79158	62479	78079	63832	76977	20
41	58330	81225	59739	80195	61130	79140	62502	78061	63854	76959	19
42	58354	81208	59763	80178	61153	79122	62524	78043	63877	76940	18
43	58378	81191	59786	80160	61176	79105	62547	78025	63899	76921	17
44	58401	81174	59809	80143	61199	79087	62570	78007	63922	76903	16
45	58425	81157	59832	80125	61222	79069	62592	77988	63944	76884	15
46	58449	81140	59856	80108	61245	79051	62615	77970	63966	76866	14
47	58472	81123	59879	80091	61268	79033	62638	77952	63989	76847	13
48	58496	81106	59902	80073	61291	79016	62660	77934	64011	76828	12
49	58519	81089	59926	80056	61314	78998	62683	77916	64033	76810	11
50	58543	81072	59949	80038	61337	78980	62706	77897	64056	76791	10
51	58567	81055	59972	80021	61360	78962	62728	77879	64078	76772	9
52	58590	81038	59995	80003	61383	78944	62751	77861	64100	76754	8
53	58614	81021	60019	79986	61406	78926	62774	77843	64123	76735	7
54	58637	81004	60042	79968	61429	78908	62796	77824	64145	76717	6
55	58661	80987	60065	79951	61451	78891	62819	77806	64167	76698	5
56	58684	80970	60089	79934	61474	78873	62842	77788	64190	76679	4
57	58708	80953	60112	79916	61497	78855	62864	77769	64212	76661	3
58	58731	80936	60135	79899	61520	78837	62887	77751	64234	76642	2
59	58755	80919	60158	79881	61543	78819	62909	77733	64256	76623	1
60	58779	80902	60182	79864	61566	78801	62932	77715	64279	76604	0
′	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	′
	54°		53°		52°		51°		50°		

NATURAL SINES AND COSINES.

′	40°		41°		42°		43°		44°		′
	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	
0	64279	76604	65606	75471	66913	74314	68200	73135	69466	71934	60
1	64301	76586	65628	75452	66935	74295	68221	73116	69487	71914	59
2	64323	76567	65650	75433	66956	74276	68242	73096	69508	71894	58
3	64346	76548	65672	75414	66978	74256	68264	73076	69529	71873	57
4	64368	76530	65694	75395	66999	74237	68285	73056	69549	71853	56
5	64390	76511	65716	75375	67021	74217	68306	73036	69570	71833	55
6	64412	76492	65738	75356	67043	74198	68327	73016	69591	71813	54
7	64435	76473	65759	75337	67064	74178	68349	72996	69612	71792	53
8	64457	76455	65781	75318	67086	74159	68370	72976	69633	71772	52
9	64479	76436	65803	75299	67107	74139	68391	72957	69654	71752	51
10	64501	76417	65825	75280	67129	74120	68412	72937	69675	71732	50
11	64524	76398	65847	75261	67151	74100	68434	72917	69696	71711	49
12	64546	76380	65869	75241	67172	74080	68455	72897	69717	71691	48
13	64568	76361	65891	75222	67194	74061	68476	72877	69737	71671	47
14	64590	76342	65913	75203	67215	74041	68497	72857	69758	71650	46
15	64612	76323	65935	75184	67237	74022	68518	72837	69779	71630	45
16	64635	76304	65956	75165	67258	74002	68539	72817	69800	71610	44
17	64657	76286	65978	75146	67280	73983	68561	72797	69821	71590	43
18	64679	76267	66000	75126	67301	73963	68582	72777	69842	71569	42
19	64701	76248	66022	75107	67323	73944	68603	72757	69862	71549	41
20	64723	76229	66044	75088	67344	73924	68624	72737	69883	71529	40
21	64746	76210	66066	75069	67366	73904	68645	72717	69904	71508	39
22	64768	76192	66088	75050	67387	73885	68666	72697	69925	71488	38
23	64790	76173	66109	75030	67409	73865	68688	72677	69946	71468	37
24	64812	76154	66131	75011	67430	73846	68709	72657	69966	71447	36
25	64834	76135	66153	74992	67452	73826	68730	72637	69987	71427	35
26	64856	76116	66175	74973	67473	73806	68751	72617	70008	71407	34
27	64878	76097	66197	74953	67495	73787	68772	72597	70029	71386	33
28	64901	76078	66218	74934	67516	73767	68793	72577	70049	71366	32
29	64923	76059	66240	74915	67538	73747	68814	72557	70070	71345	31
30	64945	76041	66262	74896	67559	73728	68835	72537	70091	71325	30
31	64967	76022	66284	74876	67580	73708	68857	72517	70112	71305	29
32	64989	76003	66306	74857	67602	73688	68878	72497	70132	71284	28
33	65011	75984	66327	74838	67623	73669	68899	72477	70153	71264	27
34	65033	75965	66349	74818	67645	73649	68920	72457	70174	71243	26
35	65055	75946	66371	74799	67666	73629	68941	72437	70195	71223	25
36	65077	75927	66393	74780	67688	73610	68962	72417	70215	71203	24
37	65100	75908	66414	74760	67709	73590	68983	72397	70236	71182	23
38	65122	75889	66436	74741	67730	73570	69004	72377	70257	71162	22
39	65144	75870	66458	74722	67752	73551	69025	72357	70277	71141	21
40	65166	75851	66480	74703	67773	73531	69046	72337	70298	71121	20
41	65188	75832	66501	74683	67795	73511	69067	72317	70319	71100	19
42	65210	75813	66523	74664	67816	73491	69088	72297	70339	71080	18
43	65232	75794	66545	74644	67837	73472	69109	72277	70360	71059	17
44	65254	75775	66566	74625	67859	73452	69130	72257	70381	71039	16
45	65276	75756	66588	74606	67880	73432	69151	72236	70401	71019	15
46	65298	75738	66610	74586	67901	73413	69172	72216	70422	70998	14
47	65320	75719	66632	74567	67923	73393	69193	72196	70443	70978	13
48	65342	75700	66653	74548	67944	73373	69214	72176	70463	70957	12
49	65364	75680	66675	74528	67965	73353	69235	72156	70484	70937	11
50	65386	75661	66697	74509	67987	73333	69256	72136	70505	70916	10
51	65408	75642	66718	74489	68008	73314	69277	72116	70525	70896	9
52	65430	75623	66740	74470	68029	73294	69298	72095	70546	70875	8
53	65452	75604	66762	74451	68051	73274	69319	72075	70567	70855	7
54	65474	75585	66783	74431	68072	73254	69340	72055	70587	70834	6
55	65496	75566	66805	74412	68093	73234	69361	72035	70608	70813	5
56	65518	75547	66827	74392	68115	73215	69382	72015	70628	70793	4
57	65540	75528	66848	74373	68136	73195	69403	71995	70649	70772	3
58	65562	75509	66870	74353	68157	73175	69424	71974	70670	70752	2
59	65584	75490	66891	74334	68179	73155	69445	71954	70690	70731	1
60	65606	75471	66913	74314	68200	73135	69466	71934	70711	70711	0
	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	Cosine.	Sine.	
′	49°		48°		47°		46°		45°		′

NATURAL TANGENTS AND COTANGENTS.

′	0°		1°		2°		3°		′
	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	
0	00000	Infinite.	01746	57·2900	03492	28·6363	05241	19·0811	60
1	00029	3437·75	01775	56·3506	03521	28·3994	05270	18·9755	59
2	00058	1718·87	01804	55·4415	03550	28·1664	05299	18·8711	58
3	00087	1145·92	01833	54·5613	03579	27·9372	05328	18·7678	57
4	00116	859·436	01862	53·7086	03609	27·7117	05357	18·6656	56
5	00145	687·549	01891	52·8821	03638	27·4899	05387	18·5645	55
6	00175	572·957	01920	52·0807	03667	27·2715	05416	18·4645	54
7	00204	491·106	01949	51·3032	03696	27·0566	05445	18·3655	53
8	00233	429·718	01978	50·5485	03725	26·8450	05474	18·2677	52
9	00262	381·971	02007	49·8157	03754	26·6367	05503	18·1708	51
10	00291	343·774	02036	49·1039	03783	26·4316	05533	18·0750	50
11	00320	312·521	02066	48·4121	03812	26·2296	05562	17·9802	49
12	00349	286·478	02095	47·7395	03842	26·0307	05591	17·8863	48
13	00378	264·441	02124	47·0853	03871	25·8348	05620	17·7934	47
14	00407	245·552	02153	46·4489	03900	25·6418	05649	17·7015	46
15	00436	229·182	02182	45·8294	03929	25·4517	05678	17·6106	45
16	00465	214·858	02211	45·2261	03958	25·2644	05708	17·5205	44
17	00495	202·219	02240	44·6386	03987	25·0798	05737	17·4314	43
18	00524	190·984	02269	44·0661	04016	24·8978	05766	17·3432	42
19	00553	180·932	02298	43·5081	04046	24·7185	05795	17·2558	41
20	00582	171·885	02328	42·9641	04075	24·5418	05824	17·1693	40
21	00611	163·700	02357	42·4335	04104	24·3675	05854	17·0837	39
22	00640	156·259	02386	41·9158	04133	24·1957	05883	16·9990	38
23	00669	149·465	02415	41·4106	04162	24·0263	05912	16·9150	37
24	00698	143·237	02444	40·9174	04191	23·8593	05941	16·8319	36
25	00727	137·507	02473	40·4358	04220	23·6945	05970	16·7496	35
26	00756	132·219	02502	39·9655	04250	23·5321	05999	16·6681	34
27	00785	127·321	02531	39·5059	04279	23·3718	06029	16·5874	33
28	00814	122·774	02560	39·0568	04308	23·2137	06058	16·5075	32
29	00844	118·540	02589	38·6177	04337	23·0577	06087	16·4283	31
30	00873	114·589	02619	38·1885	04366	22·9038	06116	16·3499	30
31	00902	110·892	02648	37·7686	04395	22·7519	06145	16·2722	29
32	00931	107·426	02677	37·3579	04424	22·6020	06175	16·1952	28
33	00960	104·171	02706	36·9560	04454	22·4541	06204	16·1190	27
34	00989	101·107	02735	36·5627	04483	22·3081	06233	16·0435	26
35	01018	98·2179	02764	36·1776	04512	22·1640	06262	15·9687	25
36	01047	95·4895	02793	35·8006	04541	22·0217	06291	15·8945	24
37	01076	92·9085	02822	35·4313	04570	21·8813	06321	15·8211	23
38	01105	90·4633	02851	35·0695	04599	21·7426	06350	15·7483	22
39	01135	88·1436	02881	34·7151	04628	21·6056	06379	15·6762	21
40	01164	85·9398	02910	34·3678	04658	21·4704	06408	15·6048	20
41	01193	83·8435	02939	34·0273	04687	21·3369	06437	15·5340	19
42	01222	81·8470	02968	33·6935	04716	21·2049	06467	15·4638	18
43	01251	79·9434	02997	33·3662	04745	21·0747	06496	15·3943	17
44	01280	78·1263	03026	33·0452	04774	20·9460	06525	15·3254	16
45	01309	76·3900	03055	32·7303	04803	20·8188	06554	15·2571	15
46	01338	74·7292	03084	32·4213	04832	20·6932	06584	15·1893	14
47	01367	73·1390	03114	32·1181	04862	20·5691	06613	15·1222	13
48	01396	71·6151	03143	31·8205	04891	20·4465	06642	15·0557	12
49	01425	70·1533	03172	31·5284	04920	20·3253	06671	14·9898	11
50	01455	68·7501	03201	31·2416	04949	20·2056	06700	14·9244	10
51	01484	67·4019	03230	30·9599	04978	20·0872	06730	14·8596	9
52	01513	66·1055	03259	30·6833	05007	19·9702	06759	14·7954	8
53	01542	64·8580	03288	30·4116	05037	19·8546	06788	14·7317	7
54	01571	63·6567	03317	30·1446	05066	19·7403	06817	14·6685	6
55	01600	62·4992	03346	29·8823	05095	19·6273	06847	14·6059	5
56	01629	61·3829	03376	29·6245	05124	19·5156	06876	14·5438	4
57	01658	60·3058	03405	29·3711	05153	19·4051	06905	14·4823	3
58	01687	59·2659	03434	29·1220	05182	19·2959	06934	14·4212	2
59	01716	58·2612	03463	28·8771	05212	19·1879	06963	14·3607	1
60	01746	57·2900	03492	28·6363	05241	19·0811	06993	14·3007	0
	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	
′	89°		88°		87°		86°		′

′	4°		5°		6°		7°		′
	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	
0	06993	14·3007	08749	11·4301	10510	9·51436	12278,	8·14435	60
1	07022	14·2411	08778	11·3919	10540	9·48781	12308	8·12481	59
2	07051	14·1821	08807	11·3540	10569	9·46141	12338	8·10536	58
3	07080	14·1235	08837	11·3163	10599	9·43515	12367	8·08600	57
4	07110	14·0655	08866	11·2789	10628	9·40904	12397	8·06674	56
5	07139	14·0079	08895	11·2417	10657	9·38307	12426	8·04756	55
6	07168	13·9507	08925	11·2048	10687	9·35724	12456	8·02848	54
7	07197	13·8940	08954	11·1681	10716	9·33154	12485	8·00948	53
8	07227	13·8378	08983	11·1316	10746	9·30599	12515	7·99058	52
9	07256	13·7821	09013	11·0954	10775	9·28058	12544	7·97176	51
10	07285	13·7267	09042	11·0594	10805	9·25530	12574	7·95302	50
11	07314	13·6719	09071	11·0237	10834	9·23016	12603	7·93438	49
12	07344	13·6174	09101	10·9882	10863	9·20516	12633	7·91582	48
13	07373	13·5634	09130	10·9529	10893	9·18028	12662	7·89734	47
14	07402	13·5098	09159	10·9178	10922	9·15554	12692	7·87895	46
15	07431	13·4566	09189	10·8829	10952	9·13093	12722	7·86064	45
16	07461	13·4039	09218	10·8483	10981	9·10646	12751	7·84242	44
17	07490	13·3515	09247	10·8139	11011	9·08211	12781	7·82428	43
18	07519	13·2996	09277	10·7797	11040	9·05789	12810	7·80622	42
19	07548	13·2480	09306	10·7457	11070	9·03379	12840	7·78825	41
20	07578	13·1969	09335	10·7119	11099	9·00983	12869	7·77035	40
21	07607	13·1461	09365	10·6783	11128	8·98598	12899	7·75254	39
22	07636	13·0958	09394	10·6450	11158	8·96227	12929	7·73480	38
23	07665	13·0458	09423	10·6118	11187	8·93867	12958	7·71715	37
24	07695	12·9962	09453	10·5789	11217	8·91520	12988	7·69957	36
25	07724	12·9469	09482	10·5462	11246	8·89185	13017	7·68208	35
26	07753	12·8981	09511	10·5136	11276	8·86862	13047	7·66466	34
27	07782	12·8496	09541	10·4813	11305	8·84551	13076	7·64732	33
28	07812	12·8014	09570	10·4491	11335	8·82252	13106	7·63005	32
29	07841	12·7536	09600	10·4172	11364	8·79964	13136	7·61287	31
30	07870	12·7062	09629	10·3854	11394	8·77689	13165	7·59575	30
31	07899	12·6591	09658	10·3538	11423	8·75425	13195	7·57872	29
32	07929	12·6124	09688	10·3224	11452	8·73172	13224	7·56176	28
33	07958	12·5660	09717	10·2913	11482	8·70931	13254	7·54487	27
34	07987	12·5199	09746	10·2602	11511	8·68701	13284	7·52806	26
35	08017	12·4742	09776	10·2294	11541	8·66482	13313	7·51132	25
36	08046	12·4288	09805	10·1988	11570	8·64275	13343	7·49465	24
37	08075	12·3838	09834	10·1683	11600	8·62078	13372	7·47806	23
38	08104	12·3390	09864	10·1381	11629	8·59893	13402	7·46154	22
39	08134	12·2946	09893	10·1080	11659	8·57718	13432	7·44509	21
40	08163	12·2505	09923	10·0780	11688	8·55555	13461	7·42871	20
41	08192	12·2067	09952	10·0483	11718	8·53402	13491	7·41240	19
42	08221	12·1632	09981	10·0187	11747	8·51259	13521	7·39616	18
43	08251	12·1201	10011	9·98930	11777	8·49128	13550	7·37999	17
44	08280	12·0772	10040	9·96007	11806	8·47007	13580	7·36389	16
45	08309	12·0346	10069	9·93101	11836	8·44896	13609	7·34786	15
46	08339	11·9923	10099	9·90211	11865	8·42795	13639	7·33190	14
47	08368	11·9504	10128	9·87338	11895	8·40705	13669	7·31600	13
48	08397	11·9087	10158	9·84482	11924	8·38625	13698	7·30018	12
49	08427	11·8673	10187	9·81641	11954	8·36555	13728	7·28442	11
50	08456	11·8262	10216	9·78817	11983	8·34496	13758	7·26873	10
51	08485	11·7853	10246	9·76009	12013	8·32446	13787	7·25310	9
52	08514	11·7448	10275	9·73217	12042	8·30406	13817	7·23754	8
53	08544	11·7045	10305	9·70441	12072	8·28376	13846	7·22204	7
54	08573	11·6645	10334	9·67680	12101	8·26355	13876	7·20661	6
55	08602	11·6248	10363	9·64935	12131	8·24345	13906	7·19125	5
56	08632	11·5853	10393	9·62205	12160	8·22344	13935	7·17594	4
57	08661	11·5461	10422	9·59490	12190	8·20352	13965	7·16071	3
58	08690	11·5072	10452	9·56791	12219	8·18370	13995	7·14553	2
59	08720	11·4685	10481	9·54106	12249	8·16398	14024	7·13042	1
60	08749	11·4301	10510	9·51436	12278	8·14435	14054	7·11537	0
′	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	′
	85°		84°		83°		82°		

NATURAL TANGENTS AND COTANGENTS.

′	8° Tangent.	8° Cotang.	9° Tangent.	9° Cotang.	10° Tangent.	10° Cotang.	11° Tangent.	11° Cotang.	′
0	14054	7·11537	15838	6·31375	17633	5·67128	19438	5·14455	60
1	14084	7·10038	15868	6·30189	17663	5·66165	19468	5·13658	59
2	14113	7·08546	15898	6·29007	17693	5·65205	19498	5·12862	58
3	14143	7·07059	15928	6·27829	17723	5·64248	19529	5·12069	57
4	14173	7·05579	15958	6·26655	17753	5·63295	19559	5·11279	56
5	14202	7·04105	15988	6·25486	17783	5·62344	19589	5·10490	55
6	14232	7·02637	16017	6·24321	17813	5·61397	19619	5·09704	54
7	14262	7·01174	16047	6·23160	17843	5·60452	19649	5·08921	53
8	14291	6·99718	16077	6·22003	17873	5·59511	19680	5·08139	52
9	14321	6·98268	16107	6·20851	17903	5·58573	19710	5·07360	51
10	14351	6·96823	16137	6·19703	17933	5·57638	19740	5·06584	50
11	14381	6·95385	16167	6·18559	17963	5·56706	19770	5·05809	49
12	14410	6·93952	16196	6·17419	17993	5·55777	19801	5·05037	48
13	14440	6·92525	16226	6·16283	18023	5·54851	19831	5·04267	47
14	14470	6·91104	16256	6·15151	18053	5·53927	19861	5·03499	46
15	14499	6·89688	16286	6·14023	18083	5·53007	19891	5·02734	45
16	14529	6·88278	16316	6·12899	18113	5·52090	19921	5·01971	44
17	14559	6·86874	16346	6·11779	18143	5·51176	19952	5·01210	43
18	14588	6·85475	16376	6·10664	18173	5·50264	19982	5·00451	42
19	14618	6·84082	16405	6·09552	18203	5·49356	20012	4·99695	41
20	14648	6·82694	16435	6·08444	18233	5·48451	20042	4·98940	40
21	14678	6·81312	16465	6·07340	18263	5·47548	20073	4·98188	39
22	14707	6·79936	16495	6·06240	18293	5·46648	20103	4·97438	38
23	14737	6·78564	16525	6·05143	18323	5·45751	20133	4·96690	37
24	14767	6·77199	16555	6·04051	18353	5·44857	20164	4·95945	36
25	14796	6·75838	16585	6·02962	18383	5·43966	20194	4·95201	35
26	14826	6·74483	16615	6·01878	18414	5·43077	20224	4·94460	34
27	14856	6·73133	16645	6·00797	18444	5·42192	20254	4·93721	33
28	14886	6·71789	16674	5·99720	18474	5·41309	20285	4·92984	32
29	14915	6·70450	16704	5·98646	18504	5·40429	20315	4·92249	31
30	14945	6·69116	16734	5·97576	18534	5·39552	20345	4·91516	30
31	14975	6·67787	16764	5·96510	18564	5·38677	20376	4·90785	29
32	15005	6·66463	16794	5·95448	18594	5·37805	20406	4·90056	28
33	15034	6·65144	16824	5·94390	18624	5·36936	20436	4·89330	27
34	15064	6·63831	16854	5·93335	18654	5·36070	20466	4·88605	26
35	15094	6·62523	16884	5·92283	18684	5·35206	20497	4·87882	25
36	15124	6·61219	16914	5·91235	18714	5·34345	20527	4·87162	24
37	15153	6·59921	16944	5·90191	18745	5·33487	20557	4·86444	23
38	15183	6·58627	16974	5·89151	18775	5·32631	20588	4·85727	22
39	15213	6·57339	17004	5·88114	18805	5·31778	20618	4·85013	21
40	15243	6·56055	17033	5·87080	18835	5·30928	20648	4·84300	20
41	15272	6·54777	17063	5·86051	18865	5·30080	20679	4·83590	19
42	15302	6·53503	17093	5·85024	18895	5·29235	20709	4·82882	18
43	15332	6·52234	17123	5·84001	18925	5·28393	20739	4·82175	17
44	15362	6·50970	17153	5·82982	18955	5·27553	20770	4·81471	16
45	15391	6·49710	17183	5·81966	18986	5·26715	20800	4·80769	15
46	15421	6·48456	17213	5·80953	19016	5·25880	20830	4·80068	14
47	15451	6·47206	17243	5·79944	19046	5·25048	20861	4·79370	13
48	15481	6·45961	17273	5·78938	19076	5·24218	20891	4·78673	12
49	15511	6·44720	17303	5·77936	19106	5·23391	20921	4·77978	11
50	15540	6·43484	17333	5·76937	19136	5·22566	20952	4·77286	10
51	15570	6·42253	17363	5·75941	19166	5·21744	20982	4·76595	9
52	15600	6·41026	17393	5·74949	19197	5·20925	21013	4·75906	8
53	15630	6·39804	17423	5·73960	19227	5·20107	21043	4·75219	7
54	15660	6·38587	17453	5·72974	19257	5·19293	21073	4·74534	6
55	15689	6·37374	17483	5·71992	19287	5·18480	21104	4·73851	5
56	15719	6·36165	17513	5·71013	19317	5·17671	21134	4·73170	4
57	15749	6·34961	17543	5·70037	19347	5·16863	21164	4·72490	3
58	15779	6·33761	17573	5·69064	19378	5·16058	21195	4·71813	2
59	15809	6·32566	17603	5·68094	19408	5·15256	21225	4·71137	1
60	15838	6·31375	17633	5·67128	19438	5·14455	21256	4·70463	0
′	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	′
	81°		80°		79°		78°		

NATURAL TANGENTS AND COTANGENTS.

′	12°		13°		14°		15°		′
	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	
0	21256	4·70463	23087	4·33148	24933	4·01078	26795	3·73205	60
1	21286	4·69791	23117	4·32573	24964	4·00582	26826	3·72771	59
2	21316	4·69121	23148	4·32001	24995	4·00086	26857	3·72338	58
3	21347	4·68452	23179	4·31430	25026	3·99592	26888	3·71907	57
4	21377	4·67786	23209	4·30860	25056	3·99099	26920	3·71476	56
5	21408	4·67121	23240	4·30291	25087	3·98607	26951	3·71046	55
6	21438	4·66458	23271	4·29724	25118	3·98117	26982	3·70616	54
7	21469	4·65797	23301	4·29159	25149	3·97627	27013	3·70188	53
8	21499	4·65138	23332	4·28595	25180	3·97139	27044	3·69761	52
9	21529	4·64480	23363	4·28032	25211	3·96651	27076	3·69335	51
10	21560	4·63825	23393	4·27471	25242	3·96165	27107	3·68909	50
11	21590	4·63171	23424	4·26911	25273	3·95680	27138	3·68485	49
12	21621	4·62518	23455	4·26352	25304	3·95196	27169	3·68061	48
13	21651	4·61868	23485	4·25795	25335	3·94713	27201	3·67638	47
14	21682	4·61219	23516	4·25239	25366	3·94232	27232	3·67217	46
15	21712	4·60572	23547	4·24685	25397	3·93751	27263	3·66796	45
16	21743	4·59927	23578	4·24132	25428	3·93271	27294	3·66376	44
17	21773	4·59283	23608	4·23580	25459	3·92793	27326	3·65957	43
18	21804	4·58641	23639	4·23030	25490	3·92316	27357	3·65538	42
19	21834	4·58001	23670	4·22481	25521	3·91839	27388	3·65121	41
20	21864	4·57363	•3700	4·21933	25552	3·91364	27419	3·64705	40
21	21895	4·56726	23731	4·21387	25583	3·90890	27451	3·64289	39
22	21925	4·56091	23762	4·20842	25614	3·90417	27482	3·63874	38
23	21956	4·55458	23793	4·20298	25645	3·89945	27513	3·63461	37
24	21986	4·54826	23823	4·19756	25676	3·89474	27545	3·63048	36
25	22017	4·54196	23854	4·19215	25707	3·89004	27576	3·62636	35
26	22047	4·53568	23885	4·18675	25738	3·88536	27607	3·62224	34
27	22078	4·52941	23916	4·18137	25769	3·88068	27638	3·61814	33
28	22108	4·52316	23946	4·17600	25800	3·87601	27670	3·61405	32
29	22139	4·51693	23977	4·17064	25831	3·87136	27701	3·60996	31
30	22169	4·51071	24008	4·16530	25862	3·86671	27732	3·60588	30
31	22200	4·50451	24039	4·15997	25893	3·86208	27764	3·60181	29
32	22231	4·49832	24069	4·15465	25924	3·85745	27795	3·59775	28
33	22261	4·49215	24100	4·14934	25955	3·85284	27826	3·59370	27
34	22292	4·48600	24131	4·14405	25986	3·84824	27858	3·58966	26
35	22322	4·47986	24162	4·13877	26017	3·84364	27889	3·58562	25
36	22353	4·47374	24193	4·13350	26048	3·83906	27920	3·58160	24
37	22383	4·46764	24223	4·12825	26079	3·83449	27952	3·57758	23
38	22414	4·46155	24254	4·12301	26110	3·82992	27983	3·57357	22
39	22444	4·45548	24285	4·11778	26141	3·82537	28015	3·56957	21
40	22475	4·44942	24316	4·11256	26172	3·82083	28046	3·56557	20
41	22505	4·44338	24347	4·10736	26203	3·81630	28077	3·56159	19
42	22536	4·43735	24377	4·10216	26235	3·81177	28109	3·55761	18
43	22567	4·43134	24408	4·09699	26266	3·80726	28140	3·55364	17
44	22597	4·42534	24439	4·09182	26297	3·80276	28172	3·54968	16
45	22628	4·41936	24470	4·08666	26328	3·79827	28203	3·54573	15
46	22658	4·41340	24501	4·08152	26359	3·79378	28234	3·54179	14
47	22689	4·40745	24532	4·07639	26390	3·78931	28266	3·53785	13
48	22719	4·40152	24562	4·07127	26421	3·78485	28297	3·53393	12
49	22750	4·39560	24593	4·06616	26452	3·78040	28329	3·53001	11
50	22781	4·38969	24624	4·06107	26483	3·77595	28360	3·52609	10
51	22811	4·38381	24655	4·05599	26515	3·77152	28391	3·52219	9
52	22842	4·37793	24686	4·05092	26546	3·76709	28423	3·51829	8
53	22872	4·37207	24717	4·04586	26577	3·76268	28454	3·51441	7
54	22903	4·36623	24747	4·04081	26608	3·75828	28486	3·51053	6
55	22934	4·36040	24778	4·03578	26639	3·75388	28517	3·50666	5
56	22964	4·35459	24809	4·03075	26670	3·74950	28549	3·50279	4
57	22995	4·34879	24840	4·02574	26701	3·74512	28580	3·49894	3
58	23026	4·34300	24871	4·02074	26733	3·74075	28612	3·49509	2
59	23056	4·33723	24902	4·01576	26764	3·73640	28643	3·49125	1
60	23087	4·33148	24933	4·01078	26795	3·73205	28675	3·48741	0
′	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	′
	77°		76°		75°		74°		

NATURAL TANGENTS AND COTANGENTS.

′	16°		17°		18°		19°		′
	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	
0	28675	3·48741	30573	3·27085	32492	3·07768	34433	2·90421	60
1	28706	3·48359	30605	3·26745	32524	3·07464	34465	2·90147	59
2	28738	3·47977	30637	3·26406	32556	3·07160	34498	2·89873	58
3	28769	3·47596	30669	3·26067	32588	3·06857	34530	2·89600	57
4	28800	3·47216	30700	3·25729	32621	3·06554	34563	2·89327	56
5	28832	3·46837	30732	3·25392	32653	3·06252	34596	2·89055	55
6	28864	3·46458	30764	3·25055	32685	3·05950	34628	2·88783	54
7	28895	3·46080	30796	3·24719	32717	3·05649	34661	2·88511	53
8	28927	3·45703	30828	3·24383	32749	3·05349	34693	2·88240	52
9	28958	3·45327	30860	3·24049	32782	3·05049	34726	2·87970	51
10	28990	3·44951	30891	3·23714	32814	3·04749	34758	2·87700	50
11	29021	3·44576	30923	3·23381	32846	3·04450	34791	2·87430	49
12	29053	3·44202	30955	3·23048	32878	3·04152	34824	2·87161	48
13	29084	3·43829	30987	3·22715	32911	3·03854	34856	2·86892	47
14	29116	3·43456	31019	3·22384	32943	3·03556	34889	2·86624	46
15	29147	3·43084	31051	3·22053	32975	3·03260	34922	2·86356	45
16	29179	3·42713	31083	3·21722	33007	3·02963	34954	2·86089	44
17	29210	3·42343	31115	3·21392	33040	3·02667	34987	2·85822	43
18	29242	3·41973	31147	3·21063	33072	3·02372	35019	2·85555	42
19	29274	3·41604	31178	3·20734	33104	3·02077	35052	2·85289	41
20	29305	3·41236	31210	3·20406	33136	3·01783	35085	2·85023	40
21	29337	3·40869	31242	3·20079	33169	3·01489	35117	2·84758	39
22	29368	3·40502	31274	3·19752	33201	3·01196	35150	2·84494	38
23	29400	3·40136	31306	3·19426	33233	3·00903	35183	2·84229	37
24	29432	3·39771	31338	3·19100	33266	3·00611	35216	2·83965	36
25	29463	3·39406	31370	3·18775	33298	3·00319	35248	2·83702	35
26	29495	3·39042	31402	3·18451	33330	3·00028	35281	2·83439	34
27	29526	3·38679	31434	3·18127	33363	2·99738	35314	2·83176	33
28	29558	3·38317	31466	3·17804	33395	2·99447	35346	2·82914	32
29	29590	3·37955	31498	3·17481	33427	2·99158	35379	2·82653	31
30	29621	3·37594	31530	3·17159	33460	2·98868	35412	2·82391	30
31	29653	3·37234	31562	3·16838	33492	2·98580	35445	2·82130	29
32	29685	3·36875	31594	3·16517	33524	2·98292	35477	2·81870	28
33	29716	3·36516	31626	3·16197	33557	2·98004	35510	2·81610	27
34	29748	3·36158	31658	3·15877	33589	2·97717	35543	2·81350	26
35	29780	3·35800	31690	3·15558	33621	2·97430	35576	2·81091	25
36	29811	3·35443	31722	3·15240	33654	2·97144	35608	2·80833	24
37	29843	3·35087	31754	3·14922	33686	2·96858	35641	2·80574	23
38	29875	3·34732	31786	3·14605	33718	2·96573	35674	2·80316	22
39	29906	3·34377	31818	3·14288	33751	2·96288	35707	2·80059	21
40	29938	3·34023	31850	3·13972	33783	2·96004	35740	2·79802	20
41	29970	3·33670	31882	3·13656	33816	2·95721	35772	2·79545	19
42	30001	3·33317	31914	3·13341	33848	2·95437	35805	2·79289	18
43	30033	3·32965	31946	3·13027	33881	2·95155	35838	2·79033	17
44	30065	3·32614	31978	3·12713	33913	2·94872	35871	2·78778	16
45	30097	3·32264	32010	3·12400	33945	2·94590	35904	2·78523	15
46	30128	3·31914	32042	3·12087	33978	2·94309	35937	2·78269	14
47	30160	3·31565	32074	3·11775	34010	2·94028	35969	2·78014	13
48	30192	3·31216	32106	3·11464	34043	2·93748	36002	2·77761	12
49	30224	3·30868	32139	3·11153	34075	2·93468	36035	2·77507	11
50	30255	3·30521	32171	3·10842	34108	2·93189	36068	2·77254	10
51	30287	3·30174	32203	3·10532	34140	2·92910	36101	2·77002	9
52	30319	3·29829	32235	3·10223	34173	2·92632	36134	2·76750	8
53	30351	3·29483	32267	3·09914	34205	2·92354	36167	2·76498	7
54	30382	3·29139	32299	3·09606	34238	2·92076	36199	2·76247	6
55	30414	3·28795	32331	3·09298	34270	2·91799	36232	2·75996	5
56	30446	3·28452	32363	3·08991	34303	2·91523	36265	2·75746	4
57	30478	3·28109	32396	3·08685	34335	2·91246	36298	2·75496	3
58	30509	3·27767	32428	3·08379	34368	2·90971	36331	2·75246	2
59	30541	3·27426	32460	3·08073	34400	2·90696	36364	2·74997	1
60	30573	3·27085	32492	3·07768	34433	2·90421	36397	2·74748	0
′	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	′
	73°		72°		71°		70°		

′	20°		21°		22°		23°		′
	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	
0	36397	2·74748	38386	2·60509	40403	2·47509	42447	2·35585	60
1	36430	2·74499	38420	2·60283	40436	2·47302	42482	2·35395	59
2	36463	2·74251	38453	2·60057	40470	2·47095	42516	2·35205	58
3	36496	2·74004	38487	2·59831	40504	2·46888	42551	2·35015	57
4	36529	2·73756	38520	2·59606	40538	2·46682	42585	2·34825	56
5	36562	2·73509	38553	2·59381	40572	2·46476	42619	2·34636	55
6	36595	2·73263	38587	2·59156	40606	2·46270	42654	2·34447	54
7	36628	2·73017	38620	2·58932	40640	2·46065	42688	2·34258	53
8	36661	2·72771	38654	2·58708	40674	2·45860	42722	2·34069	52
9	36694	2·72526	38687	2·58484	40707	2·45655	42757	2·33881	51
10	36727	2·72281	38721	2·58261	40741	2·45451	42791	2·33693	50
11	36760	2·72036	38754	2·58038	40775	2·45246	42826	2·33505	49
12	36793	2·71792	38787	2·57815	40809	2·45043	42860	2·33317	48
13	36826	2·71548	38821	2·57593	40843	2·44839	42894	2·33130	47
14	36859	2·71305	38854	2·57371	40877	2·44636	42929	2·32943	46
15	36892	2·71062	38888	2·57150	40911	2·44433	42963	2·32756	45
16	36925	2·70819	38921	2·56928	40945	2·44230	42998	2·32570	44
17	36958	2·70577	38955	2·56707	40979	2·44027	43032	2·32383	43
18	36991	2·70335	38988	2·56487	41013	2·43825	43067	2·32197	42
19	37024	2·70094	39022	2·56266	41047	2·43623	43101	2·32012	41
20	37057	2·69853	39055	2·56046	41081	2·43422	43136	2·31826	40
21	37090	2·69612	39089	2·55827	41115	2·43220	43170	2·31641	39
22	37124	2·69371	39122	2·55608	41149	2·43019	43205	2·31456	38
23	37157	2·69131	39156	2·55389	41183	2·42819	43239	2·31271	37
24	37190	2·68892	39190	2·55170	41217	2·42618	43274	2·31086	36
25	37223	2·68653	39223	2·54952	41251	2·42418	43308	2·30902	35
26	37256	2·68414	39257	2·54734	41285	2·42218	43343	2·30718	34
27	37289	2·68175	39290	2·54516	41319	2·42019	43378	2·30534	33
28	37322	2·67937	39324	2·54299	41353	2·41819	43412	2·30351	32
29	37355	2·67700	39357	2·54082	41387	2·41620	43447	2·30167	31
30	37388	2·67462	39391	2·53865	41421	2·41421	43481	2·29984	30
31	37422	2·67225	39425	2·53648	41455	2·41223	43516	2·29801	29
32	37455	2·66989	39458	2·53432	41490	2·41025	43550	2·29619	28
33	37488	2·66752	39492	2·53217	41524	2·40827	43585	2·29437	27
34	37521	2·66516	39526	2·53001	41558	2·40629	43620	2·29254	26
35	37554	2·66281	39559	2·52786	41592	2·40432	43654	2·29073	25
36	37588	2·66046	39593	2·52571	41626	2·40235	43689	2·28891	24
37	37621	2·65811	39626	2·52357	41660	2·40038	43724	2·28710	23
38	37654	2·65576	39660	2·52142	41694	2·39841	43758	2·28528	22
39	37687	2·65342	39694	2·51929	41728	2·39645	43793	2·28348	21
40	37720	2·65109	39727	2·51715	41763	2·39449	43828	2·28167	20
41	37754	2·64875	39761	2·51502	41797	2·39253	43862	2·27987	19
42	37787	2·64642	39795	2·51289	41831	2·39058	43897	2·27806	18
43	37820	2·64410	39829	2·51076	41865	2·38862	43932	2·27626	17
44	37853	2·64177	39862	2·50864	41899	2·38668	43966	2·27447	16
45	37887	2·63945	39896	2·50652	41933	2·38473	44001	2·27267	15
46	37920	2·63714	39930	2·50440	41968	2·38279	44036	2·27088	14
47	37953	2·63483	39963	2·50229	42002	2·38084	44071	2·26909	13
48	37986	2·63252	39997	2·50018	42036	2·37891	44105	2·26730	12
49	38020	2·63021	40031	2·49807	42070	2·37697	44140	2·26552	11
50	38053	2·62791	40065	2·49597	42105	2·37504	44175	2·26374	10
51	38086	2·62561	40098	2·49386	42139	2·37311	44210	2·26196	9
52	38120	2·62332	40132	2·49177	42173	2·37118	44244	2·26018	8
53	38153	2·62103	40166	2·48967	42207	2·36925	44279	2·25840	7
54	38186	2·61874	40200	2·48758	42242	2·36733	44314	2·25663	6
55	38220	2·61646	40234	2·48549	42276	2·36541	44349	2·25486	5
56	38253	2·61418	40267	2·48340	42310	2·36349	44384	2·25309	4
57	38286	2·61190	40301	2·48132	42345	2·36158	44418	2·25132	3
58	38320	2·60963	40335	2·47924	42379	2·35967	44453	2·24956	2
59	38353	2·60736	40369	2·47716	42413	2·35776	44488	2·24780	1
60	38386	2·60509	40403	2·47509	42447	2·35585	44523	2·24604	0
′	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	′
	69°		68°		67°		66°		

NATURAL TANGENTS AND COTANGENTS.

′	24°		25°		26°		27°		′
	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	
0	44523	2·24604	46631	2·14451	48773	2·05030	50953	1·96261	60
1	44558	2·24428	46666	2·14288	48809	2·04879	50989	1·96120	59
2	44593	2·24252	46702	2·14125	48845	2·04728	51026	1·95979	58
3	44627	2·24077	46737	2·13963	48881	2·04577	51063	1·95838	57
4	44662	2·23902	46772	2·13801	48917	2·04426	51099	1·95698	56
5	44697	2·23727	46808	2·13639	48953	2·04276	51136	1·95557	55
6	44732	2·23553	46843	2·13477	48989	2·04125	51173	1·95417	54
7	44767	2·23378	46879	2·13316	49026	2·03975	51209	1·95277	53
8	44802	2·23204	46914	2·13154	49062	2·03825	51246	1·95137	52
9	44837	2·23030	46950	2·12993	49098	2·03675	51283	1·94997	51
10	44872	2·22857	46985	2·12832	49134	2·03526	51319	1·94858	50
11	44907	2·22683	47021	2·12671	49170	2·03376	51356	1·94718	49
12	44942	2·22510	47056	2·12511	49206	2·03227	51393	1·94579	48
13	44977	2·22337	47092	2·12350	49242	2·03078	51430	1·94440	47
14	45012	2·22164	47128	2·12190	49278	2·02929	51467	1·94301	46
15	45047	2·21992	47163	2·12030	49315	2·02780	51503	1·94162	45
16	45082	2·21819	47199	2·11871	49351	2·02631	51540	1·94023	44
17	45117	2·21647	47234	2·11711	49387	2·02483	51577	1·93885	43
18	45152	2·21475	47270	2·11552	49423	2·02335	51614	1·93746	42
19	45187	2·21304	47305	2·11392	49459	3·02187	51651	1·93608	41
20	45222	2·21132	47341	2·11233	49495	2·02039	51688	1·93470	40
21	45257	2·20961	47377	2·11075	49532	2·01891	51724	1·93332	39
22	45292	2·20790	47412	2·10916	49568	2·01743	51761	1·93195	38
23	45327	2·20619	47448	2·10758	49604	2·01596	51798	1·93057	37
24	45362	2·20449	47483	2·10600	49640	2·01449	51835	1·92920	36
25	45397	2·20278	47519	2·10442	49677	2·01302	51872	1·92782	35
26	45432	2·20108	47555	2·10284	49713	2·01155	51909	1·92645	34
27	45467	2·19938	47590	2·10126	49749	2·01008	51946	1·92508	33
28	45502	2·19769	47626	2·09969	49786	2·00862	51983	1·92371	32
29	45537	2·19599	47662	2·09811	49822	2·00715	52020	1·92235	31
30	45573	2·19430	47698	2·09654	49858	2·00569	52057	1·92098	30
31	45608	2·19261	47733	2·09498	49894	2·00423	52094	1·91962	29
32	45643	2·19092	47769	2·09341	49931	2·00277	52131	1·91826	28
33	45678	2·18923	47805	2·09184	49967	2·00131	52168	1·91690	27
34	45713	2·18755	47840	2·09028	50004	1·99986	52205	1·91554	26
35	45748	2·18587	47876	2·08872	50040	1·99841	52242	1·91418	25
36	45784	2·18419	47912	2·08716	50076	1·99695	52279	1·91282	24
37	45819	2·18251	47948	2·08560	50113	1·99550	52316	1·91147	23
38	45854	2·18084	47984	2·08405	50149	1·99406	52353	1·91012	22
39	45889	2·17916	48019	2·08250	50185	1·99261	52390	1·90876	21
40	45924	2·17749	48055	2·08094	50222	1·99116	52427	1·90741	20
41	45960	2·17582	48091	2·07939	50258	1·98972	52464	1·90607	19
42	45995	2·17416	48127	2·07785	50295	1·98828	52501	1·90472	18
43	46030	2·17249	48163	2·07630	50331	1·98684	52538	1·90337	17
44	46065	2·17083	48198	2·07476	50368	1·98540	52575	1·90203	16
45	46101	2·16917	48234	2·07321	50404	1·98396	52613	1·90069	15
46	46136	2·16751	48270	2·07167	50441	1·98253	52650	1·89935	14
47	46171	2·16585	48306	2·07014	50477	1·98110	52687	1·89801	13
48	46206	2·16420	48342	2·06860	50514	1·97966	52724	1·89667	12
49	46242	2·16255	48378	2·06706	50550	1·97823	52761	1·89533	11
50	46277	2·16090	48414	2·06553	50587	1·97680	52798	1·89400	10
51	46312	2·15925	48450	2·06400	50623	1·97538	52836	1·89266	9
52	46348	2·15760	48486	2·06247	50660	1·97395	52873	1·89133	8
53	46383	2·15596	48521	2·06094	50696	1·97253	52910	1·89000	7
54	46418	2·15432	48557	2·05942	50733	1·97111	52947	1·88867	6
55	46454	2·15268	48593	2·05790	50769	1·96969	52984	1·88734	5
56	46489	2·15104	48629	2·05637	50806	1·96827	53022	1·88602	4
57	46525	2·14940	48665	2·05485	50843	1·96685	53059	1·88469	3
58	46560	2·14777	48701	2·05333	50879	1·96544	53096	1·88337	2
59	46595	2·14614	48737	2·05182	50916	1·96402	53134	1·88205	1
60	46631	2·14451	48773	2·05030	50953	1·96261	53171	1·88073	0
	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	
′	65°		64°		63°		62°		′

NATURAL TANGENTS AND COTANGENTS.

′	28°		29°		30°		31°		′
	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	
0	53171	1·88073	55431	1·80405	57735	1·73205	60086	1·66428	60
1	53208	1·87941	55469	1·80281	57774	1·73089	60126	1·66318	59
2	53246	1·87809	55507	1·80158	57813	1·72973	60165	1·66209	58
3	53283	1·87677	55545	1·80034	57851	1·72857	60205	1·66099	57
4	53320	1·87546	55583	1·79911	57890	1·72741	60245	1·65990	56
5	53358	1·87415	55621	1·79788	57929	1·72625	60284	1·65881	55
6	53395	1·87283	55659	1·79665	57968	1·72509	60324	1·65772	54
7	53432	1·87152	55697	1·79542	58007	1·72393	60364	1·65663	53
8	53470	1·87021	55736	1·79419	58046	1·72278	60403	1·65554	52
9	53507	1·86891	55774	1·79296	58085	1·72163	60443	1·65445	51
10	53545	1·86760	55812	1·79174	58124	1·72047	60483	1·65337	50
11	53582	1·86630	55850	1·79051	58162	1·71932	60522	1·65228	49
12	53620	1·86499	55888	1·78929	58201	1·71817	60562	1·65120	48
13	53657	1·86369	55926	1·78807	58240	1·71702	60602	1·65011	47
14	53694	1·86239	55964	1·78685	58279	1·71588	60642	1·64903	46
15	53732	1·86109	56003	1·78563	58318	1·71473	60681	1·64795	45
16	53769	1·85979	56041	1·78441	58357	1·71358	60721	1·64687	44
17	53807	1·85850	56079	1·78319	58396	1·71244	60761	1·64579	43
18	53844	1·85720	56117	1·78198	58435	1·71129	60801	1·64471	42
19	53882	1·85591	56156	1·78077	58474	1·71015	60841	1·64363	41
20	53920	1·85462	56194	1·77955	58513	1·70901	60881	1·64256	40
21	53957	1·85333	56232	1·77834	58552	1·70787	60921	1·64148	39
22	53995	1·85204	56270	1·77713	58591	1·70673	60960	1·64041	38
23	54032	1·85075	56309	1·77592	58631	1·70560	61000	1·63934	37
24	54070	1·84946	56347	1·77471	58670	1·70446	61040	1·63826	36
25	54107	1·84818	56385	1·77351	58709	1·70332	61080	1·63719	35
26	54145	1·84689	56424	1·77230	58748	1·70219	61120	1·63612	34
27	54183	1·84561	56462	1·77110	58787	1·70106	61160	1·63505	33
28	54220	1·84433	56500	1·76990	58826	1·69992	61200	1·63398	32
29	54258	1·84305	56539	1·76869	58865	1·69879	61240	1·63292	31
30	54296	1·84177	56577	1·76749	58904	1·69766	61280	1·63185	30
31	54333	1·84049	56616	1·76630	58944	1·69653	61320	1·63079	29
32	54371	1·83922	56654	1·76510	58983	1·69541	61360	1·62972	28
33	54409	1·83794	56693	1·76390	59022	1·69428	61400	1·62866	27
34	54446	1·83667	56731	1·76271	59061	1·69316	61440	1·62760	26
35	54484	1·83540	56769	1·76151	59101	1·69203	61480	1·62654	25
36	54522	1·83413	56808	1·76032	59140	1·69091	61520	1·62548	24
37	54560	1·83286	56846	1·75913	59179	1·68979	61561	1·62442	23
38	54597	1·83159	56885	1·75794	59218	1·68866	61601	1·62336	22
39	54635	1·83033	56923	1·75675	59258	1·68754	61641	1·62230	21
40	54673	1·82906	56962	1·75556	59297	1·68643	61681	1·62125	20
41	54711	1·82780	57000	1·75437	59336	1·68531	61721	1·62019	19
42	54748	1·82654	57039	1·75319	59376	1·68419	61761	1·61914	18
43	54786	1·82528	57078	1·75200	59415	1·68308	61801	1·61808	17
44	54824	1·82402	57116	1·75082	59454	1·68196	61842	1·61703	16
45	54862	1·82276	57155	1·74964	59494	1·68085	61882	1·61598	15
46	54900	1·82150	57193	1·74846	59533	1·67974	61922	1·61493	14
47	54938	1·82025	57232	1·74728	59573	1·67863	61962	1·61388	13
48	54975	1·81899	57271	1·74610	59612	1·67752	62003	1·61283	12
49	55013	1·81774	57309	1·74492	59651	1·67641	62043	1·61179	11
50	55051	1·81649	57348	1·74375	59691	1·67530	62083	1·61074	10
51	55089	1·81524	57386	1·74257	59730	1·67419	62124	1·60970	9
52	55127	1·81399	57425	1·74140	59770	1·67309	62164	1·60865	8
53	55165	1·81274	57464	1·74022	59809	1·67198	62204	1·60761	7
54	55203	1·81150	57503	1·73905	59849	1·67088	62245	1·60657	6
55	55241	1·81025	57541	1·73788	59888	1·66978	62285	1·60553	5
56	55279	1·80901	57580	1·73671	59928	1·66867	62325	1·60449	4
57	55317	1·80777	57619	1·73555	59967	1·66757	62366	1·60345	3
58	55355	1·80653	57657	1·73438	60007	1·66647	62406	1·60241	2
59	55393	1·80529	57696	1·73321	60046	1·66538	62446	1·60137	1
60	55431	1·80405	57735	1·73205	60086	1·66428	62487	1·60033	0
′	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	′
	61°		60°		59°		58°		

NATURAL TANGENTS AND COTANGENTS.

′	32° Tangent.	32° Cotang.	33° Tangent.	33° Cotang.	34° Tangent.	34° Cotang.	35° Tangent.	35° Cotang.	′
0	62487	1·60033	64941	1·53986	67451	1·48256	70021	1·42815	60
1	62527	1·59930	64982	1·53888	67493	1·48163	70064	1·42726	59
2	62568	1·59826	65023	1·53791	67536	1·48070	70107	1·42638	58
3	62608	1·59723	65065	1·53693	67578	1·47977	70151	1·42550	57
4	62649	1·59620	65106	1·53595	67620	1·47885	70194	1·42462	56
5	62689	1·59517	65148	1·53497	67663	1·47792	70238	1·42374	55
6	62730	1·59414	65189	1·53400	67705	1·47699	70281	1·42286	54
7	62770	1·59311	65231	1·53302	67748	1·47607	70325	1·42198	53
8	62811	1·59208	65272	1·53205	67790	1·47514	70368	1·42110	52
9	62852	1·59105	65314	1·53107	67832	1·47422	70412	1·42022	51
10	62892	1·59002	65355	1·53010	67875	1·47330	70455	1·41934	50
11	62933	1·58900	65397	1·52913	67917	1·47238	70499	1·41847	49
12	62973	1·58797	65438	1·52816	67960	1·47146	70542	1·41759	48
13	63014	1·58695	65480	1·52719	68002	1·47053	70586	1·41672	47
14	63055	1·58593	65521	1·52622	68045	1·46962	70629	1·41584	46
15	63095	1·58490	65563	1·52525	68088	1·46870	70673	1·41497	45
16	63136	1·58388	65604	1·52429	68130	1·46778	70717	1·41409	44
17	63177	1·58286	65646	1·52332	68173	1·46686	70760	1·41322	43
18	63217	1·58184	65688	1·52235	68215	1·46595	70804	1·41235	42
19	63258	1·58083	65729	1·52139	68258	1·46503	70848	1·41148	41
20	63299	1·57981	65771	1·52043	68301	1·46411	70891	1·41061	40
21	63340	1·57879	65813	1·51946	68343	1·46320	70935	1·40974	39
22	63380	1·57778	65854	1·51850	68386	1·46229	70979	1·40887	38
23	63421	1·57676	65896	1·51754	68429	1·46137	71023	1·40800	37
24	63462	1·57575	65938	1·51658	68471	1·46046	71066	1·40714	36
25	63503	1·57474	65980	1·51562	68514	1·45955	71110	1·40627	35
26	63544	1·57372	66021	1·51466	98557	1·45864	71154	1·40540	34
27	63584	1·57271	66063	1·51370	68600	1·45773	71198	1·40454	33
28	63625	1·57170	66105	1·51275	68642	1·45682	71242	1·40367	32
29	63666	1·57069	66147	1·51179	68685	1·45592	71285	1·40281	31
30	63707	1·56969	66189	1·51084	68728	1·45501	71329	1·40195	30
31	63748	1·56868	66230	1·50988	68771	1·45410	71373	1·40109	29
32	63789	1·56767	66272	1·50893	68814	1·45320	71417	1·40022	28
33	63830	1·56667	66314	1·50797	68857	1·45229	71461	1·39936	27
34	63871	1·56566	66356	1·50702	68900	1·45139	71505	1·39850	26
35	63912	1·56466	66398	1·50607	68942	1·45049	71549	1·39764	25
36	63953	1·56366	66440	1·50512	68985	1·44958	71593	1·39679	24
37	63994	1·56265	66482	1·50417	69028	1·44868	71637	1·39593	23
38	64035	1·56165	66524	1·50322	69071	1·44778	71681	1·39507	22
39	64076	1·56065	66566	1·50228	69114	1·44688	71725	1·39421	21
40	64117	1·55966	66608	1·50133	69157	1·44598	71769	1·39336	20
41	64158	1·55866	66650	1·50038	69200	1·44508	71813	1·39250	19
42	64199	1·55766	66692	1·49944	69243	1·44418	71857	1·39165	18
43	64240	1·55666	66734	1·49849	69286	1·44329	71901	1·39079	17
44	64281	1·55567	66776	1·49755	69329	1·44239	71946	1·38994	16
45	64322	1·55467	66818	1·49661	69372	1·44149	71990	1·38909	15
46	64363	1·55368	66860	1·49566	69416	1·44060	72034	1·38824	14
47	64404	1·55269	66902	1·49472	69459	1·43970	72078	1·38738	13
48	64446	1·55170	66944	1·49378	69502	1·43881	72122	1·38653	12
49	64487	1·55071	66986	1·49284	69545	1·43792	72166	1·38568	11
50	64528	1·54972	67028	1·49190	69588	1·43703	72211	1·38484	10
51	64569	1·54873	67071	1·49097	69631	1·43614	72255	1·38399	9
52	64610	1·54774	67113	1·49003	69675	1·43525	72299	1·38314	8
53	64652	1·54675	67155	1·48909	69718	1·43436	72344	1·38229	7
54	64693	1·54576	67197	1·48816	69761	1·43347	72388	1·38145	6
55	64734	1·54478	67239	1·48722	69804	1·43258	72432	1·38060	5
56	64775	1·54379	67282	1·48629	69847	1·43169	72477	1·37976	4
57	64817	1·54281	67324	1·48536	69891	1·43080	72521	1·37891	3
58	64858	1·54183	67366	1·48442	69934	1·42992	72565	1·37807	2
59	64899	1·54085	67409	1·48349	69977	1·42903	72610	1·37722	1
60	64941	1·53986	67451	1·48256	70021	1·42815	72654	1·37638	0
′	57° Cotang.	57° Tangent.	56° Cotang.	56° Tangent.	55° Cotang.	55° Tangent.	54° Cotang.	54° Tangent.	′

NATURAL TANGENTS AND COTANGENTS.

′	36°		37°		38°		39°		′
	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	
0	72654	1·37638	75355	1·32704	78129	1·27994	80978	1·23490	60
1	72699	1·37554	75401	1·32624	78175	1·27917	81027	1·23416	59
2	72743	1·37470	75447	1·32544	78222	1·27841	81075	1·23343	58
3	72788	1·37386	75492	1·32464	78269	1·27764	81123	1·23270	57
4	72832	1·37302	75538	1·32384	78316	1·27688	81171	1·23196	56
5	72877	1·37218	75584	1·32304	78363	1·27611	81220	1·23123	55
6	72921	1·37134	75629	1·32224	78410	1·27535	81268	1·23050	54
7	72966	1·37050	75675	1·32144	78457	1·27458	81316	1·22977	53
8	73010	1·36967	75721	1·32064	78504	1·27382	81364	1·22904	52
9	73055	1·36883	75767	1·31984	78551	1·27306	81413	1·22831	51
10	73100	1·36800	75812	1·31904	78598	1·27230	81461	1·22758	50
11	73144	1·36716	75858	1·31825	78645	1·27153	81510	1·22685	49
12	73189	1·36633	75904	1·31745	78692	1·27077	81558	1·22612	48
13	73234	1·36549	75950	1·31666	78739	1·27001	81606	1·22539	47
14	73278	1·36466	75996	1·31586	78786	1·26925	81655	1·22467	46
15	73323	1·36383	76042	1·31507	78834	1·26849	81703	1·22394	45
16	73368	1·36300	76088	1·31427	78881	1·26774	81752	1·22321	44
17	73413	1·36217	76134	1·31348	78928	1·26698	81800	1·22249	43
18	73457	1·36133	76180	1·31269	78975	1·26622	81849	1·22176	42
19	73502	1·36051	76226	1·31190	79022	1·26546	81898	1·22104	41
20	73547	1·35968	76272	1·31110	79070	1·26471	81946	1·22031	40
21	73592	1·35885	76318	1·31031	79117	1·26395	81995	1·21959	39
22	73637	1·35802	76364	1·30952	79164	1·26319	82044	1·21886	38
23	73681	1·35719	76410	1·30873	79212	1·26244	82092	1·21814	37
24	73726	1·35637	76456	1·30795	79259	1·26169	82141	1·21742	36
25	73771	1·35554	76502	1·30716	79306	1·26093	82190	1·21670	35
26	73816	1·35472	76548	1·30637	79354	1·26018	82238	1·21598	34
27	73861	1·35389	76594	1·30558	79401	1·25943	82287	1·21526	33
28	73906	1·35307	76640	1·30480	79449	1·25867	82336	1·21454	32
29	73951	1·35224	76686	1·30401	79496	1·25792	82385	1·21382	31
30	73996	1·35142	76733	1·30323	79544	1·25717	82434	1·21310	30
31	74041	1·35060	76779	1·30244	79591	1·25642	82483	1·21238	29
32	74086	1·34978	76825	1·30166	79639	1·25567	82531	1·21166	28
33	74131	1·34896	76871	1·30087	79686	1·25492	82580	1·21094	27
34	74176	1·34814	76918	1·30009	79734	1·25417	82629	1·21023	26
35	74221	1·34732	76964	1·29931	79781	1·25343	82678	1·20951	25
36	74267	1·34650	77010	1·29853	79829	1·25268	82727	1·20879	24
37	74312	1·34568	77057	1·29775	79877	1·25193	82776	1·20808	23
38	74357	1·34487	77103	1·29696	79924	1·25118	82825	1·20736	22
39	74402	1·34405	77149	1·29618	79972	1·25044	82874	1·20665	21
40	74447	1·34323	77196	1·29541	80020	1·24969	82923	1·20593	20
41	74492	1·34242	77242	1·29463	80067	1·24895	82972	1·20522	19
42	74538	1·34160	77289	1·29385	80115	1·24820	83022	1·20451	18
43	74583	1·34079	77335	1·29307	80163	1·24746	83071	1·20379	17
44	74628	1·33998	77382	1·29229	80211	1·24672	83120	1·20308	16
45	74674	1·33916	77428	1·29152	80258	1·24597	83169	1·20237	15
46	74719	1·33835	77475	1·29074	80306	1·24523	83218	1·20166	14
47	74764	1·33754	77521	1·28997	80354	1·24449	83268	1·20095	13
48	74810	1·33673	77568	1·28919	80402	1·24375	83317	1·20024	12
49	74855	1·33592	77615	1·28842	80450	1·24301	83366	1·19953	11
50	74900	1·33511	77661	1·28764	80498	1·24227	83415	1·19882	10
51	74946	1·33430	77708	1·28687	80546	1·24153	83465	1·19811	9
52	74991	1·33349	77754	1·28610	80594	1·24079	83514	1·19740	8
53	75037	1·33268	77801	1·28533	80642	1·24005	83564	1·19669	7
54	75082	1·33187	77848	1·28456	80690	1·23931	83613	1·19599	6
55	75128	1·33107	77895	1·28379	80738	1·23858	83662	1·19528	5
56	75173	1·33026	77941	1·28302	80786	1·23784	83712	1·19457	4
57	75219	1·32946	77988	1·28225	80834	1·23710	83761	1·19387	3
58	75264	1·32865	78035	1·28148	80882	1·23637	83811	1·19316	2
59	75310	1·32785	78082	1·28071	80930	1·23563	83860	1·19246	1
60	75355	1·32704	78129	1·27994	80978	1·23490	83910	1·19175	0
′	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	′
	53°		52°		51°		50°		

NATURAL TANGENTS AND COTANGENTS.

′	40°		41°		42°		43°		′
	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	
0	83910	1·19175	86929	1·15037	90040	1·11061	93252	1·07237	60
1	83960	1·19105	86980	1·14969	90093	1·10996	93306	1·07174	59
2	84009	1·19035	87031	1·14902	90146	1·10931	93360	1·07112	58
3	84059	1·18964	87082	1·14834	90199	1·10867	93415	1·07049	57
4	84108	1·18894	87133	1·14767	90251	1·10802	93469	1·06987	56
5	84158	1·18824	87184	1·14699	90304	1·10737	93524	1·06925	55
6	84208	1·18754	87236	1·14632	90357	1·10672	93578	1·06862	54
7	84258	1·18684	87287	1·14565	90410	1·10607	93633	1·06800	53
8	84307	1·18614	87338	1·14498	90463	1·10543	93688	1·06738	52
9	84357	1·18544	87389	1·14430	90516	1·10478	93742	1·06676	51
10	84407	1·18474	87441	1·14363	90569	1·10414	93797	1·06613	50
11	84457	1·18404	87492	1·14296	90621	1·10349	93852	1·06551	49
12	84507	1·18334	87543	1·14229	90674	1·10285	93906	1·06489	48
13	84556	1·18264	87595	1·14162	90727	1·10220	93961	1·06427	47
14	84606	1·18194	87646	1·14095	90781	1·10156	94016	1·06365	46
15	84656	1·18125	87698	1·14028	90834	1·10091	94071	1·06303	45
16	84706	1·18055	87749	1·13961	90887	1·10027	94125	1·06241	44
17	84756	1·17986	87801	1·13894	90940	1·09963	94180	1·06179	43
18	84806	1·17916	87852	1·13828	90993	1·09899	94235	1·06117	42
19	84856	1·17846	87904	1·13761	91046	1·09834	94290	1·06056	41
20	84906	1·17777	87955	1·13694	91099	1·09770	94345	1·05994	40
21	84956	1·17708	88007	1·13627	91153	1·09706	94400	1·05932	39
22	85006	1·17638	88059	1·13561	91206	1·09642	94455	1·05870	38
23	85057	1·17569	88110	1·13494	91259	1·09578	94510	1·05809	37
24	85107	1·17500	88162	1·13428	91313	1·09514	94565	1·05747	36
25	85157	1·17430	88214	1·13361	91366	1·09450	94620	1·05685	35
26	85207	1·17361	88265	1·13295	91419	1·09386	94676	1·05624	34
27	85257	1·17292	88317	1·13228	91473	1·09322	94731	1·05562	33
28	85307	1·17223	88369	1·13162	91526	1·09258	94786	1·05501	32
29	85358	1·17154	88421	1·13096	91580	1·09195	94841	1·05439	31
30	85408	1·17085	88473	1·13029	91633	1·09131	94896	1·05378	30
31	85458	1·17016	88524	1·12963	91687	1·09067	94952	1·05317	29
32	85509	1·16947	88576	1·12897	91740	1·09003	95007	1·05255	28
33	85559	1·16878	88628	1·12831	91794	1·08940	95062	1·05194	27
34	85609	1·16809	88680	1·12765	91847	1·08876	95118	1·05133	26
35	85660	1·16741	88732	1·12699	91901	1·08813	95173	1·05072	25
36	85710	1·16672	88784	1·12633	91955	1·08749	95229	1·05010	24
37	85761	1·16603	88836	1·12567	92008	1·08686	95284	1·04949	23
38	85811	1·16535	88888	1·12501	92062	1·08622	95340	1·04888	22
39	85862	1·16466	88940	1·12435	92116	1·08559	95395	1·04827	21
40	85912	1·16398	88992	1·12369	92170	1·08496	95451	1·04766	20
41	85963	1·16329	89045	1·12303	92223	1·08432	95506	1·04705	19
42	86014	1·16261	89097	1·12238	92277	1·08369	95562	1·04644	18
43	86064	1·16192	89149	1·12172	92331	1·08306	95618	1·04583	17
44	86115	1·16124	89201	1·12106	92385	1·08243	95673	1·04522	16
45	86166	1·16056	89253	1·12041	92439	1·08179	95729	1·04461	15
46	86216	1·15987	89306	1·11975	92493	1·08116	95785	1·04401	14
47	86267	1·15919	89358	1·11909	92547	1·08053	95841	1·04340	13
48	86318	1·15851	89410	1·11844	92601	1·07990	95897	1·04279	12
49	86368	1·15783	89463	1·11778	92655	1·07927	95952	1·04218	11
50	86419	1·15715	89515	1·11713	92709	1·07864	96008	1·04158	10
51	86470	1·15647	89567	1·11648	92763	1·07801	96064	1·04097	9
52	86521	1·15579	89620	1·11582	92817	1·07738	96120	1·04036	8
53	86572	1·15511	89672	1·11517	92872	1·07676	96176	1·03976	7
54	86623	1·15443	89725	1·11452	92926	1·07613	96232	1·03915	6
55	86674	1·15375	89777	1·11387	92980	1·07550	96288	1·03855	5
56	86725	1·15308	89830	1·11321	93034	1·07487	96344	1·03794	4
57	86776	1·15240	89883	1·11256	93088	1·07425	96400	1·03734	3
58	86827	1·15172	89935	1·11191	93143	1·07362	96457	1·03674	2
59	86878	1·15104	89988	1·11126	93197	1·07299	96513	1·03613	1
60	86929	1·15037	90040	1·11061	93252	1·07237	96569	1·03553	0
′	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	Cotang.	Tangent.	′
	49°		48°		47°		46°		

NATURAL TANGENTS AND COTANGENTS.

′	44° Tangent.	44° Cotang.	′	′	44° Tangent.	44° Cotang.	′
0	96569	1·03553	60	31	98327	1·01702	29
1	96625	1·03493	59	32	98384	1·01642	28
2	96681	1·03433	58	33	98441	1·01583	27
3	96738	1·03372	57	34	98499	1·01524	26
4	96794	1·03312	56	35	98556	1·01465	25
5	96850	1·03252	55	36	98613	1·01406	24
6	96907	1·03192	54	37	98671	1·01347	23
7	96963	1·03132	53	38	98728	1·01288	22
8	97020	1·03072	52	39	98786	1·01229	21
9	97076	1·03012	51	40	98843	1·01170	20
10	97133	1·02952	50	41	98901	1·01112	19
11	97189	1·02892	49	42	98958	1·01053	18
12	97246	1·02832	48	43	99016	1·00994	17
13	97302	1·02772	47	44	99073	1·00935	16
14	97359	1·02713	46	45	99131	1·00876	15
15	97416	1·02653	45	46	99189	1·00818	14
16	97472	1·02593	44	47	99247	1·00759	13
17	97529	1·02533	43	48	99304	1·00701	12
18	97586	1·02474	42	49	99362	1·00642	11
19	97643	1·02414	41	50	99420	1·00583	10
20	97700	1·02355	40	51	99478	1·00525	9
21	97756	1·02295	39	52	99536	1·00467	8
22	97813	1·02236	38	53	99594	1·00408	7
23	97870	1·02176	37	54	99652	1·00350	6
24	97927	1·02117	36	55	99710	1·00291	5
25	97984	1·02057	35	56	99768	1·00233	4
26	98041	1·01998	34	57	99826	1·00175	3
27	98098	1 01939	33	58	99884	1·00116	2
28	98155	1·01879	32	59	99942	1·00058	1
29	98213	1·01820	31	60	Unit.	Unit.	0
30	98270	1·01761	30				
′	Cotang. 45°	Tangent. 45°	′	′	Cotang. 45°	Tangent. 45°	′

TABLE OF CONSTANTS.

Base of Napier's system of logarithms = ...ϵ = 2·718281828459

Mod. of common syst. of logarithms = ... com. log. ϵ = M = 0·4342944819032

Ratio of circumference to diameter of a circle =..........π = 3·141592653590

log. π = 0·497149872694

π^2 = 9·869604401089............ $\sqrt{\pi}$ = 1·772453850906

Arc of same length as radius = ...180° ÷ π = 10800′ ÷ π = 648000″ ÷ π

180° ÷ π = 57°·2957795130,.............................log. = 1·758122632409

10800′ ÷ π = 3437′·7467707849,.........................log. = 3·536273882793

648000″ ÷ π = 206264″·8062470964, ...log. = 5·314425133176

Tropical year = 365d. 5h. 48m. 47s. ·588 = 365d. ·242217456, log. = 2·5625810

Sidereal year = 365d. 6h. 9m. 10s. ·742 = 365d. ·256374332, log. = 2·5625978

24h. sol. t.=24h. 3m. 56s. ·555335 sid. t.=24h.×1·00273791, log. 1·002=0·0011874

24h. sid. t.=24h.−(3m.55s.·90944) sol. t.=24h.×0·9972696, log. 0·997=9·9988126

British imperial gallon = 277·274 cubic inches,...............log. = 2·4429091

Length of sec. pend., in inches, at London, 39·13929; Paris, 39·1285; New York, 39·1285.

French *metre* = 3·2808992 English *feet* = 39·3707904 *inches*.

1 cubic inch of water (bar. 30 inches, Fahr. therm. 62°) = 252·458 Troy grains.

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